August, 2005

Comprehensive Examination **Department of Mathematics**

ALGEBRA

Part I: do three of the following problems. Make sure that all your answers are justified.

1. Let G be a finite group and let x be an element of G of order n. Suppose there exists $g \in G$ such that $gxg^{-1} = x^t$ for some positive integer t.

(a) Show that t and n are relatively prime.

(b) Let m be the order of t mod n, i.e., the smallest positive integer such that $t^m \equiv 1$ mod$n$, and let $X =< x >$, be the cyclic subgroup of G generated by x. Show that $g \in N_G(X)$ and $g^m \in C_G(X)$.

(c) Show that m divides $|N_G(X)/C_G(X)|$ and use this fact to conclude that m divides the order of the conjugacy class of x in G.

2. Let R_1 and R_2 be commutative rings with identity and let I_1 and I_2 be ideals of R_1 and R_2 respectively.

(a) Show that $I_1 \times I_2$ is an ideal of $R_1 \times R_2$. Moreover, show that every ideal of $R_1 \times R_2$ is of the form $I_1 \times I_2$ where I_1 is an ideal of R_1 and I_2 is an ideal of R_2.

(b) Show that $R_1 \times R_2 / I_1 \times I_2 \cong R_1/I_1 \times R_2/I_2$.

(c) Show that every prime ideal P of $R_1 \times R_2$ is either of the form $P_1 \times R_2$ or of the form $R_1 \times P_2$ where P_1 and P_2 are prime ideals of R_1 and R_2 respectively.

3. Let $A \in M_n(\mathbb{Q})$ be an $n \times n$ matrix with rational entries. Suppose there exists a prime p such that $A^p = I_n$, the $n \times n$ identity matrix.

(a) Determine all possible values of the minimal polynomial of A.

(b) Show that either A has an eigenvalue 1 or $p-1$ divides n.

4. Let $F \subset R$ where F is a field and R a commutative ring.

(a) Suppose R is an integral domain and that $dim_F R$ is finite. Show that R is a field.

(b) Give an example $F \subset R$ such that $dim_F R$ is finite but R is not a field.

(c) Give an example $F \subset R$ such that R is an integral domain but R is not a field.

Part II: Do two of the following problems

1. Let G be a group of order 105.

(a) Show that G has a normal subgroup of order 5 or order 7.

(b) Show that G has a normal subgroup of order 35.

2. Let R be an integral domain and let M be an R-module. We call $m \in M$ torsion if there exists an $r \in R$, $r \neq 0$, such that $rm = 0$. We call M torsion-free if M has no torsion elements other than 0.

(a) Let $M_{tor} = \{m \in M : \ m \text{ torsion}\}$. Show that M_{tor} is a submodule of M and that M/M_{tor} is torsion-free.

(b) Suppose R is a principal ideal domain and M is a finitely generated R-module. Show that there exists a submodule $M' \subset M$ such that $M = M_{tor} \oplus M'$.

3. Let $F = \mathbb{Q}(\sqrt{2}, \sqrt{3}, i)$.

(a) Find $[F : \mathbb{Q}]$.

(b) Show that F is Galois over $\mathbb{Q}$ and determine the Galois group of F over $\mathbb{Q}$.

(c) Determine the number of extensions of $\mathbb{Q}$ of degree 4 contained in F. How many of them are Galois over $\mathbb{Q}$ and why?