August, 2002

Comprehensive Examination **Department of Mathematics**

ALGEBRA

Part I: Do three of the following problems. Make sure that all your answers are justified.

1. Let G be a finite group and K a cyclic normal subgroup of G.

i) Show that every subgroup of K is normal in G.

ii) Give an example of a finite group G, strictly containing a normal subgroup K, such that K contains a subgroup not normal in G.

2. Let R be a commutative ring, and let $f(x) = a_n x^n + ... + a_0$ be a polynomial in $R[x]$, for $a_1, \ldots, a_n \in R$. Show that $f(x)$ is nilpotent if and only if each of the $a_1, \ldots, a_n$ is nilpotent. (Recall that an element r of a commutative ring is nilpotent when $r^\ell = 0$ for some positive integer ℓ.)

3. Let K be a field, and let $f(x)$ be a monic polynomial of degree n with coefficients in K. Show that there exists at least one, and at most finitely many, similarity classes of $n \times n$ matrices whose characteristic polynomial is $f(x)$. (Recall that two $n \times n$ matrices A and B are similar if there exists an invertible $n \times n$ matrix C such that $B = C^{-1}AC$.)

4. i) Let $f(x) \in \mathbf{Q}[x]$ be an irreducible polynomial of odd degree n. Suppose the splitting field of $f(x)$ over $\mathbf{Q}$ has degree n. Show that all roots of $f(x)$ are real.

ii) Does the same result hold if n is even? Prove, or give a counterexample.

Part II: Do two of the following problems. Make sure that all your answers are justified.

1. Let p and q be distinct primes, with $p < q$. Show that every group of order p^2q^2 has a nontrivial proper normal subgroup.

2. Let R be a commutative ring, M an R-module, and I an ideal of R.

i) Show that

$$IM = \left\{ \sum_{i=1}^{n} a_i m_i \ : \ a_i \in I, \ m_i \in M \right\}$$

is an R-submodule of M.

ii) Show that M/IM has a natural R/I-module structure.

iii) Show that if M is a free R-module of rank n, then M/IM is a free R/I-module of rank n.

3. Let p be a prime, and let F be the splitting field of $x^p - 2$ over $\mathbf{Q}$.

i) Show that $[F : \mathbf{Q}] = p(p-1)$.

ii) Let $G = Gal(F/\mathbf{Q})$. Show that G has a normal subgroup K of order p and a cyclic subgroup H of order $p-1$. Also show that $G = HK$.

iii) Show that H is not normal in G. Conclude that G is not abelian.