Algebra Exam - Summer/Fall, 2000

Part I

Answer 3 questions from this part. (If more than 3 are answered, only the first 3 appearing in the blue book will be considered.)

1. Let G be a finite group, H a subgroup of G, and N a normal subgroup of G.

a) Prove that HN is a subgroup of G and that the index $[G : HN]$ is a divisor of both $[G : H]$ and $[G : N]$.

b) Show that if $[G : H]$ and $[G : N]$ are relatively prime then $G = HN$.

2. Let A be a 3×3 matrix over a field F such that $A^2 = A$. Find all polynomials which could be the characteristic polynomial of A as well as all possible Jordan canonical forms for A.

3. Let R be a commutative ring with unity. Prove that

a) Every maximal ideal is a prime ideal.

b) Every prime ideal of finite index is a maximal ideal.

c) If R is a principal ideal domain, then every nonzero prime ideal is a maximal ideal.

4. Let K be a finite extension of a field F.

a) Prove that K is an algebraic extension of F.

b) Let $a \in K$, and let $p(x)$ be a monic polynomial over F of minimal degree such that $p(a) = 0$. Prove each of the following:

i) $p(x)$ is the unique monic polynomial over F of minimal degree such that $p(a) = 0$.

ii) $p(x)$ is irreducible over F.

iii) $p(x)$ divides every other polynomial $f(x)$ over F of which a is a root.

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Part II

Answer 2 questions from this part. (Only the first two answers appearing in the blue book will be considered.)

5. a) Show that the rings $Q[x]/(x^2)$, $Q[x]/(x^2-1)$ and $Q[x]/(x^2+1)$ are not isomorphic to each other.

b) Find a monic polynomial $p(x) \in Q[x]$, such that $p(x) \neq x^2+1$ but that $Q[x]/(p(x)) \cong Q[x]/(x^2+1)$.

6. Let F_q be a finite field with $q = p^n$ elements, p prime. Let F_p denote the prime field of F_q. Finally, let $\sigma : F_q \to F_q$ be the mapping which takes every element to its p^{th} power. I.e., $\sigma(x) = x^p$, for all $x \in F_q$.

a) Show that σ is a field automorphism of F_q which fixes F_p pointwise. I.e., for $x \in F_p$, $\sigma(x) = x$.

b) Show that F_q is a Galois extension of F_p and that $Gal(F_q,F_p) = < \sigma >$.

7. Let g be an element of a finite group G, and let S be the conjugacy class of g. That is, $S = \{y \in G \mid y = x^{-1}gx \text{ for some } x \in G\}$.

a) Prove that $|S|$ is a divisor of $|G|$.

b) Use the result in a) to prove that if G is a finite p-group, then G has a non-trivial center.

c) Use the result in b) to prove that every group of order p^2 is abelian.