August, 1999

Comprehensive Examination **Department of Mathematics**

ALGEBRA

PART I: Do three of the following problems.

1. Let G be a group. Define $\Delta(G) = \{g \in G \mid [G : \mathbb{C}_G(g)] < \infty\}$. (Here, $\mathbb{C}_G(g) = \{x \in G \mid xg = gx\}$ denotes the centralizer of g.)

   (a) Show that $\Delta(G)$ is a subgroup of G and that $\Delta(G)$ is characteristic; i.e., $\varphi(\Delta(G)) \subseteq \Delta(G)$ holds for all automorphisms φ of G.

   (b) Show that $\Delta(G)$ contains the center of G and every finite normal subgroup of G.

   (c) Determine $\Delta(G)$ for the infinite dihedral group $G = D_\infty = \langle x, y \mid y^2 = 1, yxy = x^{-1}\rangle$.

2. Let R be a commutative ring with 1, and let $e \in R$ be an idempotent ($e^2 = e$). Prove:

   (a) $e' = 1 - e$ is an idempotent of R;

   (b) the ideal eR is a ring with multiplicative identity e;

   (c) R is the direct sum of eR and $e'R$.

3. Prove that two diagonalizable matrices A, B (over some field) are simultaneously diagonalizable (diagonalized by conjugation by the same invertible matrix) if and only if $AB = BA$.

4. Give an explicit example or explain why it is impossible:

   (a) A field with 49 elements.

   (b) An infinite field in which every non-zero element is a root of unity.

   (c) A finite field in which not every non-zero element is a root of unity.

   (d) A finite, algebraically closed field.

   (e) An infinite field F whose multiplicative group $F^* = F \setminus \{0\}$ is cyclic.

PART II: Do two of the following problems.

1. Let p, q be square-free, relatively prime integers, and let $F = \mathbb{Q}(\sqrt{p} + \sqrt{q})$.
   (a) Show $\sqrt{p}$ is in F.
   (b) Show F is normal over $\mathbb{Q}$.
   (c) Find the Galois group of F over $\mathbb{Q}$ and describe its action.
2. Let R be the subring of $\mathbb{Q}(x)$ given by $R = \{f(x)/g(x) \mid g(1) \neq 0\}$.
   (a) Find a maximal ideal of R.
   (b) Show that R is a local ring; i.e., R has a unique maximal ideal.
   (c) Explicitly determine all the ideals of R.
3. Prove:
   (a) The additive group $(\mathbb{Q}, +)$ of rationals is not the direct sum of two proper subgroups.
   (b) The quotient of additive groups $\mathbb{Q}/\mathbb{Z}$ is not an infinite direct sum of cyclic subgroups.