# Numerical Analysis Qualifying Written Exam (August 2024)

# Part I: do 3 of 4

**1. Interpolation.**
Let $f : [a, b] \to \mathbb{R}$. We wish to interpolate $f(x)$ using $n + 1$ points $x_i$, $i = 0, \dots, n$, i.e., we seek a polynomial $p(x)$ such that $p(x_i) = f(x_i)$ $i = 0, \dots, n$.
**(a)** What are the smoothness conditions on $f(x)$, and the degree of $p(x)$ so that the interpolating polynomial is unique?
**(b)** Prove the uniqueness using the hypotheses you describe in part (a).
**(c)** What is the interpolation error? i.e., $f(x) - p(x)$, for $x \in [a, b]$.
**(d)** Let us assume that the points are equidistant, i.e., that $x_i = \frac{i(b-a)}{n}$, $i = 0, \dots, n$. Prove or disprove the following statement:
Let $p_n$ the interpolating polynomial using $n + 1$ points. Then, for every $x \in [a, b]$, $\lim_{n\to\infty} |p_n(x) - f(x)| = 0$.

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**2. Numerical Integration.**
We wish to approximate the numerical value of the integral $\int_a^b f(x)dx$ using the evaluation of the funtion at $n$ points, $x_i$, $i = 1, \dots, n$..
**(a)** Give the expression for the general Newton-Cotes rule using $n$ points.
**(b)** Give a formula for the error obtained with the formula provided in part (a).
**(c)** Let $f(x) = p(x)$ be a polynomial. Given the formula from part (a), for what degree of $p(x)$ is the formula exact?.
**(d)** Let $\alpha_i$ be the weights for the Newton-Cotes rule. Assume that the points $x_i$ are equidistant. Prove that

$$\sum_{i=1}^{n} \alpha_i = n.$$

**(e)** Describe a numerical method using $n$ points to evaluate integrals of the form $\int_a^b \omega(x)f(x)dx$ for some weight function $\omega(x)$ so that the method is exact for $f(x)$ being a polynomial of degree $2n - 1$.

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**3. Multistep methods; zero-stability and convergence.**
For the ODE initial value problem $u'(t) = f(u(t))$, $u(0) = u_0$, $t \in [0, T]$, consider the following multistep methods:

(i) $U^{n+2} - 3U^{n+1} + 2U^n = -kf(U^n)$
(ii) $U^{n+2} - \frac{4}{3}U^{n+1} + \frac{1}{3}U^n = \frac{2}{3}kf(U^{n+2})$
(iii) $U^{n+2} = U^{n+1} + \frac{3}{2}kf(U^{n+1}) - \frac{1}{2}kf(U^n)$.

**(a)** Work out the local truncation error for each of the methods, and state the resulting order of the methods. Also state which of the schemes are consistent.
**(b)** Define zero-stability for a general linear $r$-step method of the form $\sum_{j=0}^{r} \alpha_j U^{n+j} = k\sum_{j=0}^{r} \beta_j f(U^{n+j})$. Then prove which of the three methods above are zero-stable.
**(c)** Provide the definition of convergence of a numerical method. Then state the general relationship between consistency, convergence, and zero-stability. Explain what this implies about the usefulness of each of the three methods above.
**(d)** Of the schemes above which are convergent, argue (via a rigorous argument) which method(s) allow one to solve the test problem $u'(t) = -10^6\, u(t)$, $u(0) = 1$ with non-tiny time steps.

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**4. Absolute stability and stiff problems.**
**(a)** For the ODE initial value problem $u'(t) = f(u(t))$, $u(0) = u_0$, $t \in [0, T]$, formulate (i) Heun's method (aka the explicit trapezoidal rule, aka RK2) and (ii) the Crank-Nicolson method (aka the implicit trapezoidal rule).
**(b)** Calculate/derive the order of accuracy for each of these two methods.
**(c)** How is the region of absolute stability of a general Runge-Kutta method defined? Calculate and sketch the regions of absolute stability for each of the two methods above.
**(d)** Explain what a stiff problem is. Then, via the regions of absolute stability found above, explain how well-suited (or not) each of the methods above is for stiff problems.
**(e)** Provide the cost per step (optimally efficient implementation) of each of the two methods, measured in evaluations of $f$ and solves with $f$. Then, give (describe or provide pseudo-code) a possible way to implement the Crank-Nicolson method (i.e., how can the implicit solves be conducted).

# Part II: do 2 of 3

**1. Roots of nonlinear functions (1d).** Consider the function

$$f(x) = e^x \cos\left(\frac{\pi x}{2}\right), \quad x \in [-1, 1].$$

**(a)** Show that there exists at least one $x \in [-1, 1]$ such that it is a fixed point, i.e., $x = f(x)$.
**(b)** Consider the standard fixed point iteration $x_{k+1} = f(x_k)$. Explain why will this method converge or not converge to the fixed point.
**(c)** Describe Newton's method to find such a fixed point. Indicate the conditions needed for its convergence and whether they are satisfied in this case.

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**2. Newton's method in 2d.**
Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$F(x) = \begin{bmatrix} x_1 x_2 \\ x_1 x_2 \end{bmatrix}.$$

**(a)** Can you identify five different roots for $F(x)$?
**(b)** Describe Newton's method in 2 dimensions so that it can be applied here to find $x$ such that $F(x) = 0$.
**(c)** Describe conditions on the initial vector $x_0$ and on a general $F$ for Newton's method to converege, and discuss when these conditions apply to the case of $F$ in this exercise.

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**3. A-stability and L-stability.**
For the ODE initial value problem $u'(t) = f(u(t))$, $u(0) = u_0$, $t \in [0, T]$, consider the following numerical methods:
(i) $U^{n+1} = U^n + \frac{1}{2}kf(U^{n+1}) + \frac{1}{2}kf(U^n)$
(ii) $U^{n+2} - \frac{4}{3}U^{n+1} + \frac{1}{3}U^n = \frac{2}{3}kf(U^{n+2})$.
**(a)** Provide the definition of A-stability for Runge-Kutta methods and for linear multistep methods. Then prove that both schemes are A-stable. [You may make use of the following fact: the curve $\frac{3}{2} - 2e^{i\theta} + \frac{1}{2}e^{2i\theta}$, $\theta \in \mathbb{R}$ does not penetrate the left half plane.]
**(b)** Calculate (or prove) the order of accuracy of each of the methods.
**(c)** Provide a suitable definition of L-stability for Runge-Kutta methods and for linear multistep methods. Then, prove that one of the above methods is L-stable, while the other is not L-stable.
**(d)** Provide a (class of) problem(s) for which the lack of L-stability may be of concern, and describe in which way the lack of L-stability may manifest.