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Phase Transition for the hard-core Stochastic Ising Model

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in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

by
Yan Lyansky
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ABSTRACT

Phase Transition for the hard-core Stochastic Ising Model

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Temple University, January, 2002

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Examining a nearest neighbor anti-ferromagnetic stochastic Ising model, we prove that there is phase transition if and only if the model is ergodic. We also prove the same holds for the hard-core stochastic Ising model.

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TABLE OF CONTENTS

INTRODUCTION

The Ising model is a $\{0,1\}$ spin model on a d-dimensional lattice. A configuration, η, is an element of $\{0,1\}^{Z^d}$. A potential $\{J_R\}$ is a set of real numbers indexed by subsets of Z^d such that $\Sigma_{R\subset Z^d}|J_R| < \infty$. A finite volume Gibbs state on T with boundry condition ζ is a measure on $\{0,1\}^T$, $|T| < \infty$, that takes the form

$\nu(\eta) = \frac{1}{Z_T^\zeta} exp(\Sigma_{R\cap T=\emptyset} J_R \chi_R(\eta^\zeta))$

$\chi_R(\eta) = \Pi_{x\in R}(2\eta(x)-1)$, Z_T^ζ is the normalization constant, and

$$\eta^\zeta(x) = \begin{cases} \eta(x) & \text{if x} \in \text{T} \\ \zeta(x) & \text{otherwise} \end{cases}$$

A Gibbs state is any limit of $\nu(\eta)$ as $T \uparrow Z^d$. For the potential given by

$$(0.1) \qquad J_R = \begin{cases} \beta H & \text{for } x \in Z^d \\ \beta J(y-x) & \text{for } x, y \in Z^d \\ 0 & \text{for } |R| \geq 3 \end{cases}$$

the model is said to be ferromagnetic if $J_R > 0$ where $\beta = \frac{1}{T}$, and T is the absolute temperature. The Ising model was first studied by Ising in 1925. He proved that there is no phase transition in the 1-dimensional case. He also mistakenly conjectured that there was no phase transition in any dimension. Onsager[12] in 1944 proved there exists phase transition for sufficiently small temperature in the 2-dimensional H=0 case. Up until this time phase transition was proved by the non-differentiability of free energy with respect to H. Free energy equals $-lim_{T\uparrow Z^d}\beta^{-1}ln(Z_T^\zeta)$. Dobrushin[1] in 1968 proved phase transition for the ferromagnetic, anti-ferromagnetic and hard-core Ising models by proving the existence of multiple Gibbs states. In 1972, Lebowitz and Martin-Lof[9] proved that free energy is differentiable if and only if there exists a unique Gibbs state for the ferromagnetic case. However, in 1992 Klein and Yang[8] proved that free energy is differentiable with respect to the external field even though there exists two Gibbs states for a 2-dimensional antiferromagnetic Ising model.

This gave rise to the stochastic Ising model, which is defined as a spin system with strictly positive rates, $c(x,\eta)$, relative to the potential if c(x,η) exp[$\Sigma_{x\in R} J_R \chi_R(\eta)$] does not depend on the coordinate η(x). This model was first studied by Glauber[6] in 1963. Let R represents the set of all reversible measures, and G the set of all Gibbs states. R=G was proved by Spitzer[14] 1971 for finite T, and generalized by Dobrushin[2] in 1971 and Logan[11] in 1974 for general T. In 1974 Holly[7] proved a system is ergodic if and only if $|G|$=1. Additionally, a survey of interacting particle systems emphasizing the role of the stochastic Ising model is given by Durrett[4] 1981.

In section 1 we define a Markov process, Markov semigroup and Markov generator. Using the Hille-Yosida Theorem we have a unique relationship between Markov processes and Markov generators. We go on to conclude there exists a reationship between a spin system with given flip rates, $c(x,\eta)$, and a Markov process. The ferromagnetic stochastic Ising model is analyzed in the last two parts of this section, the potential for such a spin system is given in (0.1). This potential gives rise to a process on $\{0,1\}^{Z^d}$ evolving over time. Hence for every subset T of Z^d we can examine the number of Gibbs states on T and phase transition of this model. This thesis is broken into three main parts. First we discuss the ferromagnetic stochastic Ising model. The main result of this section is due to Dubrushin[1] in 1968; we have phase transition if and only if the model is ergodic. This is proved in the second part of Theorem 1.4.10.

In section 2 we examine the antiferromagnetic stochastic Ising model with potential given by

$$J_R=\begin{cases} \beta H & \text{for } x \in Z^d \\ \beta J(|y-x|) & \text{for } x,y \in Z^d \\ 0 & \text{for } |R| \geq 3 \end{cases}$$

where $J(y-x) < 0$ if $|y-x|$ is an odd, and $J(y-x) > 0$ if $|y-x|$ is an even First examining the case where H=0 and H=$(-1)^{|x|}$ in both cases we conclude this model is equivalent to the ferromagnetic stochastic Ising model. Then assuming H$\neq$0 Griffiths inequality no longer holds, hence we find this model to be completely different from the ferromagnetic case. Nevertheless, we conclude that we have phase transition if and only if the model is ergodic for the antiferromagnetic stochastic Ising model. This is proved in Theorem 2.3.13 part one.

In section 3 we examine the hard-core stochastic Ising model. This model has a potential defined by

$$J_R = \begin{cases} \beta H & \text{if } |R| = 1, \\ \beta J(|y-x|) & \text{if } |R| = 2,\ x, y \in R,\ |x-y| = 1, \\ 0 & \text{if } |R| > 2 \end{cases}$$

Investigating the nearest neighbor hard-core model where J(y-x)=0 for $|y - x| \geq 2$, J(y-x)=-∞ if $\eta(x) = \eta(y) = 1$, and J(y-x)=-1 otherwise, configurations with two neighboring 1's are not allowed. Using the ideas developed in sections 1 and 2 we create a Markov generator, Markov semigroup, Markov process associated with this spin system. Extending the stochastic Ising model and a Gibbs state to the hard-core case, we conclude a Gibbs state with respect to a given potential on $T \subset Z^d$ $|T| < \infty$ is given by

$$\nu(\eta) = \begin{cases} 0 & \text{if } \eta(x) = 1 \text{ and } \eta(y) = 1 \text{ where } |y-x| = 1 \\ \frac{1}{z} exp(\Sigma_{R \subset T} J_R \chi_R(\eta)) & \text{otherwise} \end{cases}$$

This section is concluded with a summary of our results namely the hard-core stochastic Ising model is ergodic if and only if phase transition occurs, and for β sufficiently small we have no phase transition. Both results are listed in Theorem 3.7.

CHAPTER 1

Ferromagnetic Stochastic Ising Model

1.1 Markov Processes and Generators

Let Z^d denote a d-dimensional lattice, and let $\Lambda = \{A$: A is a finite subset of $Z^d\}$. First we have to define some basic terminology. A configuration, η, is an element of $X = \{0,1\}^{Z^d}$. Next we define some notations. Let $D[0,\infty)$ be the set of all functions, $\eta_.$, on $[0,\infty)$ with values in X, with $\|\eta_a\| = \Sigma_{i\in Z^d} \frac{\eta_a(x_i)}{2^i}$ which are right continuous and have left limits. For $s \in [0,\infty)$, the mapping π_s from $D[0,\infty)$ to X is defined by $\pi_s(\eta_.) = \eta_s$. Let F be the smallest σ algebra on $D[0,\infty)$ relative to which all the mappings π_s are measurable. Let
$\pi_A = \{\pi_y : y \in A\}$, $F = \sigma\{\pi_{[0,\infty)}\}$
$F_t = \sigma\{\pi_{[0,t]}\}$

Definition 1.1.1: A Markov process on X is a collection $\{P^\eta, \eta \in X\}$ of probability measures on $D[0,\infty)$ with the following properties

a)$P^\eta[\zeta \in D[0,\infty) : \zeta_0 = \eta] = 1$ for all $\eta \in X$

b)The mapping $\eta \to P^{\eta}(A)$ from X to [0,1] is measurable for every $A \in F$

c) $P^{\eta}[\eta_{s+.} \in A|F_s] = P^{\eta_s}(A)$ a.s.(P^{η}) for every $\eta \in X$ and $A \in F$.

C(X) denotes the collection of continuous functions on X with $||f||=sup_{\eta \in X}|f(\eta)|$ for $f \in C(X)$ Next, we define the linear operator S(t), $S(t)f(\eta) = E^{\eta}f(\eta_t)$.

Proposition 1.1.1: Suppose $\{P^{\eta}, \eta \in X\}$ is a Markov process, and $S(t)f \in C(X)$ for every $t \geq 0$ and $f \in C(X)$. Then the collection of linear operators S(t), t≥0 on C(X) has the following properties:

1. S(0)=I, the identity operator on C(X)
2. The mapping $t \to S(t)f$ from $[0,\infty)$ to C(X) is right continuous on C(X) for every f∈C(X).
3. S(t+s)f=S(t)S(s)f ∀f∈C(X) and all s,t≥ 0.
4. S(t)1=1 $\forall t \geq 0$
5. S(t)f≥0 for all nonnegative f∈C(X)

Proof: 1) is equivalent to a) in the above definition.

2) η_t is right continuous and f is continuous therefore $f(\eta_t)$ is right continuous take

$$lim_{h\to 0^+}S(t+h)f(\eta) - S(t)f(\eta) =$$
$$lim_{h\to 0^+}E^{\eta}(f(\eta_{t+h}) - f(\eta_t))$$

which is equals 0 for simple functions by extending this we can take the limit inside the expectation to conclude that $S(t)f(\eta)$ is right continuous.

3)

$$S(t+s)f(\eta) = E^{\eta}f(\eta_{t+s})$$

$$= E^{\eta}[E[f(\eta_{t+s})|F_t]]$$

$= E^{\eta}[E^{\eta_t}[f(\eta_s)]]$

$= E^{\eta}[S(s)f](\eta_t)$

$= S(t)S(s)f(\eta).$

Hence S(t+s)f=S(t)S(s)f.

4) follows from the definition

5) again follows from the definition

Definition 1.1.2: A family $\{S(t), t \geq 0\}$ of linear operators on C(X) is called a Markov semigroup if it satisfies conditions 1-5 above.

The next theorem explains that each Markov semigroup corresponds to a Markov process, thus instead of constructing a process we can look to create a semigroup.

Theorem 1.1.1: Suppose $\{S(t), t \geq 0\}$ is a Markov semigroup on C(X). Then there exists a unique Markov process $\{P^{\eta}, \eta \in X\}$ such that $S(t)f(\eta) = E^{\eta}f(\eta_t)$ $\forall$ f∈C(X),$\eta \in X$ and $t \geq 0$

The proof can be seen in Dynkin[3] Chapter 1.

Definition 1.1.3: Let $\wp$ represent the set of all probability measures on X, with weak convergence. Suppose $\{S(t), t \geq 0\}$ is a Markov semigroup on C(X). Given $\mu \in \wp, \mu S(t) \in \wp$ is given by $\int f d[\mu S(t)] = \int S(t) f d\mu \; \forall f \in C(X)$

Definition 1.1.4: A $\mu \in \wp$ is said to be invariant for the process with Markov semigroup $\{S(t), t \geq 0\}$ if $\mu S(t) = \mu$ for all $t \geq 0$. This class will be denoted as $\Im$.

Definition 1.1.5: A Markov process with semigroup $\{S(t), t \geq 0\}$ is said to be ergodic if

1. The set of all invariant measures is a singleton, $\Im = \nu$
2. $lim_{t \to \infty} \mu S(t) = \nu \; \forall \mu \in \wp$

Definition 1.1.6: A linear operator Ω on C(X) with domain $D(\Omega)$ is said to be a Markov pregenerator if it satisfies the following conditions:

1) $1 \in D(\Omega)$ and $\Omega 1 = 0$

2) $D(\Omega)$ is dense in C(X).

3) If $f \in D(\Omega)$, $\lambda \geq 0$ and $f - \lambda \Omega f = g$, then
$min_{\zeta \in X} f(\zeta) \geq min_{\zeta \in X} g(\zeta)$

Proposition 1.1.2: Suppose that the linear operator Ω on C(X) satisfies the following property: if $f \in D(\Omega)$ and $f(\eta) = min_{\zeta \in X} f(\zeta)$, then $\Omega f(\eta) \geq 0$. Then Ω satisfies property 3) of the above definition.

Proof: Suppose $f \in D(\Omega)$, $\lambda \geq 0$, and $f - \lambda \Omega f = g$. Let η be any point at which f attains its minimum. Such a point exists by the compactness of X and the continuity of f. Then

$$min_{\zeta \in X} f(\zeta) = f(\eta) \geq f(\eta) - \lambda \Omega f(\eta) = g(\eta) \geq min_{\zeta \in X} g(\zeta)$$

Definition 1.1.7: A linear operator Ω on C(X) is said to be closed if its graph is a closed subset of C(X)$\times$ C(X). A linear operator, $\bar{\Omega}$, is called the closure of Ω if $\bar{\Omega}$ is the smallest closed extension of Ω.

Definition 1.1.8: A Markov generator is a closed Markov pregenerator Ω which satisfies

$R(I - \lambda\Omega) = C(X)$; where R represents the range
for sufficiently small positive λ.

Proposition 1.1.3: A bounded everywhere defined Markov pregenerator is a Markov generator.

Proof: A bounded operator is automatically closed. To check that a bounded operator Ω satisfies

$R(I - \lambda\Omega) = C(X)$

for all sufficiently small positive λ we need to solve f-$\lambda\Omega f$=g for $g \in C(X)$ and $0 \leq \lambda \leq ||\Omega||^{-1}$. Thus we let
$f = \Sigma_{n=0}^{\infty} \lambda^n \Omega^n g$

Theorem 1.1.2(Hille-Yosida): There is a 1-1 correspondence between Markov generators on C(X) and Markov semigroups on C(X). This correspondence is given by:

1)$D(\Omega)$=$f \in C(X) : lim_{t \downarrow 0} \frac{S(t)f-f}{t}$ exists and $\Omega f = lim_{t \downarrow 0} \frac{S(t)f-f}{t}$

2)$S(t)f = lim_{n \to \infty}(I - \frac{t}{n}\Omega)^{-n} f$ for $f \in C(X)$ and $t \geq 0$

In addition,
3) if $f \in D(\Omega)$, it follows that S(t)f$\in D(\Omega)$ and (d/dt)S(t)f=$\Omega S(t)f = S(t)\Omega f$

4) for $g \in C(X)$ and $\lambda \geq 0$, the solution to $f - \lambda\Omega f = g$ is given by
$f = \int_0^\infty e^{-t} S(\lambda t) g dt$

A proof of this Theorem can be found in Dynkin[3] Chapter 1.

Ω is called the generator of S(t), and S(t) is the semigroup generated by Ω.

1.2 Spin System

We will now construct a Markov generator for a speific model which will represent a specific Markov process. The local dynamics of our system are described by a collection of transition measures $c_T(d\zeta, \eta)$. For a spin system, the transition mechanism for a non-negative function $c(x, \eta)$, $x \in Z^d$, $\eta \in X$, thus the process η_t with state space X will satisfy

$P^\eta[\eta_t(x) \neq \eta(x)] = c(x, \eta)t + o(t)$
as t$\downarrow$ 0 for each $x \in Z^d, \eta \in X$.

We will also restrict our process to only change 1 coordinate for each transition of time, or more formally:

$P^\eta[\eta_t(x) \neq \eta(x), \eta_t(y) \neq \eta(y)] = \text{o(t)}$
as $t \downarrow 0 x \neq y$

$c_T(x, \eta)$ is related to $c_T(d\zeta, \eta)$, be the transition measure associated with the configuration η changing only at the point $x \in Z^d$. For each $\eta \in X$ and finite $T \subset Z^d$, $c_T(d\zeta, \eta)$ is assumed to be a finite positive measure on $\{0,1\}^T$. We will also assume the mapping $\eta \to c_T(d\zeta, \eta)$ is continuous from X to the space of finite measures on $\{0,1\}^T$ with the topology of weak convergence. Let $c_T = sup\{c_T(\{0,1\}^T, \eta) : \eta \in X\}$

Let $\eta^\zeta(x) = \begin{cases} \eta(x) & \text{if x} \in \text{T} \\ \zeta(x) & \text{otherwise} \end{cases}$

Proposition 1.2.1: Assume that $sup_{x \in Z^d} \Sigma_{x \in T} c_T \leq \infty$

1) For $f \in D(X)$, the series $\Omega f(\eta) = \Sigma_T \int_{\{0,1\}^T} c_T(d\zeta, \eta)[f(\eta^\zeta) - f(\eta)]$ con-

verges uniformly and defines a function in C(X), and

$||\Omega f|| \leq (sup_{x\in Z^d}\Sigma_{x\in T}c_T)\ |||f|||$, where $|||f||| = \Sigma_{x\in Z^d}||\Delta_f(x)||$, $\Delta_f(x) = sup_\eta|f(\eta) - f(\eta_x)|$

2) Ω is a Markov pregenerator.

Proof: $\int_{\{0,1\}^T} c_T(d\zeta,\eta)[f(\eta^\zeta) - f(\eta)]$
is in C(X) for each T and each f$\in C(X)$. By regarding η^ζ as the result of changing the coordinates of η corresponding to sites in T one at a time, it is clear that
$|f(\eta^\zeta) - f(\eta)| \leq \Sigma_{x\in T}\Delta_f(x)$, $\Delta_f(x) = sup_\eta|f(\eta) - f(\eta_x)|$

therefore

$$|| \int c_T(\eta, d\zeta)[f(\eta^\zeta) - f(\eta)]\ || \leq c_T\Sigma_{x\in T}||\Delta_f(x)||$$

$$\Sigma_T\ || \int c_T(\eta, d\zeta)[f(\eta^\zeta) - f(\eta)]|| \leq (sup_x\Sigma_{x\in T}c_T)\ |||f|||$$

for any $f \in D(X)$. Hence the series defining Ωf converges uniformly. Since the summands are continuous, it follows that $\Omega f \in C(X)$.

To prove 2) we must simply show property 3) of Definition 1.1.6. Suppose $f \in D(X)$ and $f(\eta) = min(f(\zeta) : \zeta \in X)$. Then $f(\zeta) \geq f(\eta)$ for all $\zeta \in X$, so $\Omega f(\eta) \geq 0$.

Thus simply giving a transition function, c_T defines the Markov pregenerator it is necessary to show that R(I-$\lambda\Omega$) is dense in C(X) for all sufficiently small $\lambda > 0$. To prove this we approximate Ω by a sequence of bounded pregenerators Ω^n, since bounded pregenerators are generators we conclude

$$R(I - \lambda\Omega^n) = C(X)$$

for each n and each $\lambda \geq 0$. Therefore given a g$\in D(X)$, there are $f_n \in C(X)$ so that $f_n - \lambda\Omega^n f_n = g$. Thus if $g_n = f_n - \lambda\Omega f_n$ and it will follow that

$$||g_n - g|| = \lambda||(\Omega - \Omega^n)f_n|| \to 0$$

$R(I - \lambda\Omega)$ is dense is a consequence of the fact that $g_n \in R(I - \lambda\Omega)$ for each n, and that D(X) is dense. Therefore, Ω is a Markov generator, which is uniquely associated with a Markov semigroup, which is then associated with a Markov process.

1.3 Stochastic Ising Model

Definition 1.3.1: A potential, J_R, $R \in \Lambda$ is a collection of real numbers indexed by finite subsets of Z^d such that $\Sigma_{x \in R}|J_R| < \infty$, $\forall$ x$\in Z^d$.

Definition 1.3.2: Given a potential J_R, a spin system with strictly positive rates, $c(x, \eta)$, is called a Stocastic Ising model relative to the potential if c(x,η) exp$[\Sigma_{x \in R} J_R \chi_R(\eta)]$ does not depend on the coordinate η(x), where $\chi_R(\eta) = \prod_{x \in R}[2\eta(x) - 1]$

This has been used as a very good model for the magnetism of iron see Simon[13] Chapter I Preliminaries.

Definition 1.3.3: Take $|S| < \infty$, a Gibbs state relative to the potential J_R is the unique probability measure on the space $\{0, 1\}^S$

(1.3.3.1) $\nu(\eta) = Cexp[\Sigma_R J_R \chi_R(\eta)]$, C is the normalization constant.

For general S we use the following,

Definition 1.3.4: A probability measure ν on $\{0, 1\}^S$ is said to be a Gibbs state relative to the potential J_R provided that for all x $\in$ S, the conditional probability at a point
$\rho_x(\zeta) = \nu(\eta : \eta(x) = \zeta(x) | \eta(u) = \zeta(u) \forall u \neq x)$ is given by

(1.3.4.1) $\frac{1}{1+exp[-2\Sigma_{x \in R} J_R \chi_R(\zeta)]}$

Definition 1.3.5: A spin system with rates is reversible if $c(x,\eta)\nu(\eta) = c(x,\eta_x)\nu(\eta_x)$ $\forall x \in Z^d$ and $\eta \in X$.

In layman's terms a system is reversible if it looks the same evolving from start to finish as from finish to start. A real world example would be a song that is identical played forwards or backwards. Given $\{J_R\}$, let G(S) be the set of all Gibbs States on $\{0,1\}^S$ relative to $\{J_R\}$. For simplicity, from now on we may refer to G(X) as G, where X=$\{0,1\}^{Z^d}$.

Definition 1.3.6: A potential $\{J_R\}$ is said to exhibit phase transition if G(X), the set of all Gibbs states, contains more than 1 element.

Our first Theorem shows that our two definitions do coincide.

Theorem 1.3.1: Suppose $S \subset Z^d$ is finite then Definition 1.3.3 is equivalent to 1.3.4.

Proof: Suppose 1.3.3, then

$\nu(\eta : \eta(x) = \zeta(x) | \eta(u) = \zeta(u) \forall u \neq x) =$

$\frac{\nu(\zeta)}{\nu(\zeta)+\nu(\zeta_x)} =$

$\frac{exp[\Sigma_R J_R \chi_R(\zeta)]}{exp[\Sigma_R J_R \chi_R(\zeta)] + exp[\Sigma_R J_R \chi_R(\zeta_x)]} =$

$\frac{1}{1+exp[-2\Sigma_{x\in R} J_R \chi_R(\zeta)]}$

since $\chi_R(\eta_x) = -\chi_R(\eta)$ if $x \in R$ and $\chi_R(\eta)$ if $x \notin R$

For the converse suppose ν is a Gibbs state as in 1.3.4. Then ν satisfies 1.3.3 for possibly some other potential, say J_R^* since S is finite, but then we must have $\Sigma_{x\in R} J_R \chi_R(\eta) = \Sigma_{x\in R} J_R^* \chi_R(\eta)$ for all x. Since χ_R are linearly independant we conclude $J_R = J_R^*$ $\forall R \neq \emptyset$. Changing $J_\emptyset$ is just changing the normalization constant.

Theorem 1.3.2: Suppose that ν is a probability measure on X such that for all $x \in Z^d$, there is a version of the conditional probability, $\rho_x(\zeta)$, given in Definition 1.3.4 which can be written in the form

(1.3.2.1) $\rho_X(\zeta) = \frac{1}{1+exp[-2\Sigma_R J_R^x \chi_R(\zeta)]}$

for some family $\{J_R^x\}$ which satisfies $\Sigma_R |J_R^x| < \infty$ for each $x \in Z^d$. Then ν is a Gibbs state relative to some potential $\{J_R\}$

Proof: We simply need to show $J_R^x = 0$ if $x \notin R$ and $J_R^x = J_R^x$ if $x, y \in R$ by comparing Definition 1.3.4 to the above statement, since then we can define $J_R^x = J_R$ for $x \in R$. By definition $\rho_x(\zeta) + \rho_x(\zeta_x) = 1$ $\forall x \in Z^d$ and $\zeta \in X$. The configuration outside R is fixed, the leftmost term describes the probability of a configuration with value of $\zeta(x)$ at x, the other term is the probability of a configuration with value $\zeta_x(x)$ at x. Since these two are compliments the above equality holds. Therefore

(1.3.2.2) $\Sigma_R J_R^x \chi_R(\zeta) = -\Sigma_R J_R^x \chi_R(\zeta_x)$

since $\frac{1}{1+exp(-2\Sigma_{x\in R} J_R \chi_R(\zeta)} + \frac{1}{1+exp(-2\Sigma_{x\in R} J_R \chi_R(\zeta_x)} = 1$

clearing denominators we get

$1 + exp(-2\Sigma_{x\in R} J_R \chi_R(\zeta) + 1 + exp(-2\Sigma_{x\in R} J_R \chi_R(\zeta_x) =$

$(1 + exp(-2\Sigma_{x\in R} J_R \chi_R(\zeta)))(1 + exp(-2\Sigma_{x\in R} J_R \chi_R(\zeta_x))$

Which implies $1 = exp(-2\Sigma_{x\in R} J_R \chi_R(\zeta)) \times exp(-2\Sigma_{x\in R} J_R \chi_R(\zeta_x))$ which is what we need. So that $J_R^x = 0$ since χ_R are linearly independant $\forall\ x \in Z^d$ choose x outside R then $\Sigma_R J_R^x \chi_R(\zeta) = -\Sigma_R J_R^x \chi_R(\zeta)$ which implies $J_R^x = 0$ To finish the proof we will look at

$[\frac{1}{\rho_x(\zeta)-1}][\frac{1}{\rho_x(\zeta_x)-1}]$

which is symetric in x and y. Let T be a finite set of Z^d that ν assigns positive probability, then

$$[\frac{1}{\nu(\eta:\eta(x)=\zeta(x)|\eta=\zeta \text{ on } T\cap\{y\})} - 1] \times [\frac{1}{\nu(\eta:\eta(y)=\zeta(y)|\eta=\zeta_x \text{ on} T\cap\{x\})} - 1] =$$

$$\frac{\nu(\eta:\eta(x)\neq\zeta(x),\eta(y)\neq\zeta(y),\eta=\zeta \text{ on } T)}{\nu(\eta:\eta=\zeta \text{ on } T\cap\{x,y\})}$$

which is symetric in x and y, and converges to 1 as T increases to $Z^d\{x,y\}$. Next we conclude that

$$\Sigma_{x\in R} J_R^x \chi_R(\zeta) + \Sigma_{x\in R} J_R^y \chi_R(\zeta_x) =$$
$$\Sigma_{x,y\in R}(J_R^x - J_R^y)\chi_R(\zeta) + \Sigma_{x\in R,y\notin R} J_R^x \chi_R(\zeta) + \Sigma_{y\in R,x\notin R} J_R^y \chi_R(\zeta)$$

is symetric in x and y. Since the sum of the second and the third terms on the right is symetric it follows that the first must also, this implies that $\Sigma_{x,y\in R}(J_R^x - J_R^y)\chi_R(\zeta) = 0$ which implies $J_R^x = J_R^y$.

(1.3.2.3) Let $\nu_{T,\zeta}(\eta) = c(T,\zeta)exp[\Sigma_{R\cap T\neq\emptyset} J_R\chi_R(\eta^\zeta)]$, where $c(T,\zeta)$ is the normalizing constant

Theorem 1.3.3:

1.) $T_2 \supset T_1$ implies $G(T_1) \supset G(T_2)$

2.) If $\nu \in G$ then for finite $S \supset T$ and $\zeta \in \{0,1\}^{S/T}$, then $\nu(.|\eta(u) = \zeta(u) \forall u \notin T) = \nu_{T,\zeta}(.)$

3.) $G = \cap_T G(T)$

4.) G is nonempty, convex, and compact

Proof: To prove 1), suppose $T_1 \subset T_2$ and take $\zeta \in \{0,1\}^{S/T_2}$
For $\gamma \in \{0,1\}^{S/T_1}$ such that $\zeta = \gamma$ on S/T_2.

(1.3.3.1) $\nu_{T_2,\zeta}(.|\eta = \zeta \text{ on } T_2/T_1) = \nu_{T_1,\zeta}(.)$

are measures on $\{0,1\}^{T_1}$ as can be seen from Definition 1.3.4. Therefore

(1.3.3.2) $\nu_{T_2,\zeta} = \Sigma_{\gamma:\gamma=\zeta on S/T_2} \nu_{T_2,\zeta}(\eta : \eta = \gamma \text{ on } T_2/T_1)\ \nu_{T_1,\gamma}$

which exhibits $\nu_{T_2,\zeta}$ as a convex combination of elements of $G(T_1)$. Therefore $G(T_2) \subset G(T_1)$.

For 2) take (1.3.3.1) with $T_2 = T$ and $T_1 = \{x\}$ where $x \in T$ to write

$\nu_{T,\zeta}(\eta : \eta(x) = \zeta(x) | \eta(u) = \zeta(u) \forall u \in T/\{x\}) = \nu_{\{x\},\zeta}(\zeta(x)) =$

(1.3.3.3) $\frac{exp[\Sigma_{x\in R} J_R \chi_R(\zeta)]}{exp[\Sigma_{x\in R} J_R \chi_R(\zeta)] + exp[\Sigma_{x\in R} J_R \chi_R(\zeta_x)]} =$
$\frac{1}{[1+exp[-2\Sigma_{x\in R} J_R \chi_R(\zeta)]]}$

for $\zeta \in X$ Comparing this to Definition 1.3.3, it suffices to show that $\nu_{T,\zeta}$ is the only probability measure on $\{0,1\}^T$ whose one point conditional probabilities are given by (1.3.3.3), this follows from Theorem 1.3.3.

For 3) We need to show $G \subset \cap_T G(T)$. Since $\nu \in G$ this implies

$\nu(\eta : \eta(x) = \zeta(x) | \eta(u) = \zeta(u)\ \forall u \neq x) = \frac{1}{1+exp[-2\Sigma_{x\in R} J_R \chi_R(\zeta)]}$

Therefore by the Theorem of Total Probability

$\nu = \Sigma \nu(\eta : \eta(x) = \zeta(x) | \eta(u) = \zeta(u)\ \forall u \neq x) \times \Pr(\nu(\eta(u) = \zeta(u)\ \forall u \neq x))$

which expresses ν as a linear combination of elements in G(T).

4) Since G(T) are each closed convex we conclude the intersection is convex and non-empty, and since X is compact the set of all probability measures on X is a compact set, since $\cap_T G(T)$ is a closed subset of a compact set, we conclude G is compact.

Theorem 1.3.4: Suppose that S $\subset Z^d$ is finite, and that $\{J_R, R \subset S\}$ is a potential which is ferromagnetic (i.e. $J_R \geq 0\ \forall R$). Let ν be a corresponding Gibbs state then,

(1.3.4.1) $\int \chi_A d\nu \geq 0\ \forall A \subset S$ and

(1.3.4.2) $\frac{\partial}{\partial J_B} \int \chi_A d\nu = \int \chi_A \chi_B d\nu - \int \chi_A d\nu \int \chi_B d\nu \geq 0$ for all $A, B \subset S$

Proof: The proof of (1.3.4.2) uses (1.3.4.1) so we must prove (1.3.4.1) first. In order to do so, write

$$\int \chi_A d\nu = K \Sigma_\eta \chi_A(\eta) exp[\Sigma_R J_R \chi_R(\eta)]$$

$$= K \Sigma_\eta \chi_A(\eta) \Sigma_{n=0}^{\infty} \frac{1}{n!} [\Sigma_R J_R \chi_R(\eta)]^n$$

$$= K \Sigma_{n=0}^{\infty} \frac{1}{n!} \Sigma_{R_1,\ldots R_n} [\Pi_{k=1}^{n} J_{R_k}] \Sigma_\eta \chi_A(\eta) \Pi_{k=1}^{n} \chi_{R_k}(\eta)$$

To see that this is a sum of nonnegative terms, it suffices to note that

$\chi_A(\eta) \Pi_{k=1}^{n} \chi_{R_k}(\eta) = \chi_B(\eta)$ where $B = \{x \in S :$ x is in an odd number of the sets $A, R_1, R_2, \ldots, R_n\}$ and that for any $B \subset S$ $\Sigma_\eta \chi_B(\eta) = \begin{cases} 2^{|S|} & \text{if } B = \emptyset \\ 0 & \text{if } B \neq \emptyset \end{cases}$

Turning to (1.3.4.2) use the explicit expression for K to write

$$\int \chi_A d\nu = \frac{\Sigma_\eta \chi_A(\eta) exp[\Sigma_R J_R \chi_R(\eta)]}{\Sigma_\eta exp[\Sigma_R J_R \chi_R(\eta)]}$$

$$\int \chi_A d\nu = \frac{\Sigma_\eta \chi_A(\eta) exp[\Sigma_R J_R \chi_R(\eta)]}{\Sigma_\eta exp[\Sigma_R J_R \chi_R(\eta)]}$$

Therefore the equality in (1.3.4.2) is the result of a simple differentiation. To check the inequality, write

$\int \chi_A \chi_B d\nu - \int \chi_A d\nu \int \chi_B d\nu =$

$K^2 \Sigma_{\eta,\zeta}[\chi_A(\eta)\chi_B(\eta) - \chi_A(\eta)\chi_B(\zeta)] exp[\Sigma_R J_R [\chi_R(\eta) + \chi_R(\zeta)]]$

Let C be the symetric difference $A \nabla B$ and let $\gamma \in X$ be defined by

$$\gamma(x) = \begin{cases} 1 & \text{if } \eta(x) = \zeta(x) \\ 0 & \text{if } \eta(x) \neq \zeta(x) \end{cases}$$

$\chi_A(\eta)\chi_B(\eta) = \chi_C(\eta)$

$\chi_A(\eta)\chi_B(\zeta) = \chi_B(\gamma)\chi_C(\eta)$

$\chi_R(\eta) + \chi_R(\zeta) = \chi_R(\eta)[1 + \chi_R(\gamma)]$

Making these substitutions above yields

$\int \chi_A \chi_B d\nu - \int \chi_A d\nu \int \chi_B d\nu =$

$K^2 \Sigma_{\eta,\zeta}[\chi_C(\eta)[1 - \chi_B(\gamma)] exp[\Sigma_R J_R \chi_R(\eta)[1 + \chi_R(\gamma)]$

For fixed γ, we can defined a new potential by $J_R^\gamma = J_R[1+\chi_R(\gamma)]$ which is again ferromagnetic, by (1.3.4.1) applied to this potential we see that $\Sigma_\eta \chi_C(\eta) exp[\Sigma_R J_R^\gamma \chi_R(\eta)]$ which is again ferromagnetic. Therefore $\Sigma_\eta \chi_C(\eta) exp[\Sigma_R J_R^\gamma \chi_R(\eta)] \geq 0$, thus (1.3.4.2) follows from the above equation by first summing on η and then on γ

Definition 1.3.7: A probability measure μ on X is said to have positive correlations if $\int fgd\mu \geq \int fd\mu \int gd\mu \ \forall f,g \in M$

We will say
(1.3.5.11) $\eta \leq \zeta$ if $\eta(x) \leq \zeta(x)$ $\forall x$ and $(\eta \vee \zeta)_i = max(\eta_i, \zeta_i), (\eta \wedge \zeta)_i = min(\eta_i, \zeta_i)$. Denote M=$\{f : f \in C(X)$ and f is increasing $\}$. We will refer to the construction of 2 or more processes on a common probability space as a coupling.

Theorem 1.3.5: Let K=$\{(\eta, \zeta \in X \times X : \eta \leq \zeta\}$ Suppose when $\eta \leq \zeta$
$c_1(x,\eta) \leq c_2(x,\zeta)$ if $\eta(x) = \zeta(x) = 0$ and
$c_2(x,\eta) \leq c_1(x,\zeta)$ if $\eta(x) = \zeta(x) = 1$, then $\forall(\eta,\zeta) \in K$ and $t \geq 0$
$P^{(\eta,\zeta)}[(\eta_t, \zeta_t) \in K] = 1$.
Proof: Let A be the set of all functions f in $C(X \times X)$ such that $f \geq 0$ and $f = 0$ on K. For $\lambda \geq 0$ and $f \in A$, define $h \in D(\bar{\Omega})$ by $h - \lambda\bar{\Omega}h = f$. Let $(\eta,\zeta) \in K$ be a point where h achieves its maximum on K. We will show that $\bar{\Omega}h(\eta,\zeta) \leq 0$, so that $h(\eta,\zeta) \leq f(\eta,\zeta) = 0$. Since $h \geq 0$, it will follow that $h \in A$ as well. To show $\bar{\Omega}h(\eta,\zeta) \leq 0$, we will check that each of the terms in the sum defining $\bar{\Omega}h$ is nonpositive. Consider the following three cases:

a)$\eta(x) \neq \zeta(x)$ in which case $(\eta,\zeta) \in K$ implies that $(\eta_x,\zeta) \in K$ and $(\eta,\zeta_x) \in K$, so that $h(\eta_x,\zeta) \leq h(\eta,\zeta)$ and $h(\eta,\zeta_x) \leq h(\eta,\zeta)$.

b)$\eta(x) = \zeta(x) = 0$, in which case $(\eta,\zeta) \in K$ implies that $(\eta_x,\zeta_x) \in K$ and $(\eta,\zeta_x) \in K$, so that $h(\eta_x,\zeta_x) \leq h(\eta,\zeta)$ and $h(\eta,\zeta_x) \leq h(\eta,\zeta)$. However, in this case $(\eta_x,\zeta) \notin K$, so we need to have $c_1(x,\eta) = min(c_1(x,\eta), c_2(x,\zeta))$ which is part of our assumption.

c)$\eta(x) = \zeta(x) = 1$ which is analogous to b), using the second part of the assumption.

Theorem 1.3.6: (FKG Inequality) Let μ_1, μ_2 be probability measures on X. Suppose that μ_1 and μ_2 assign strictly positive probabilities to each point of

X. If $\mu_1(\eta \wedge \zeta) \geq \mu_1(\eta)\mu_2(\zeta)$ $\forall \eta, \zeta \in$ X, then $\mu_1 \leq \mu_2$.

Proof: Now we couple the Markov chains as follows. Consider the product space $\chi \times \chi$. Define a flip rate for a Markov Chain on $\chi \times \chi$ by

$W_{(\eta,\zeta),(\eta^i,\zeta)} = 1$ if $\eta_i = 0, \zeta_i = 1,$

$W_{(\eta,\zeta),(\eta,\zeta^i)} = \frac{\mu_2(\zeta^i)}{\mu_2(\zeta)}$ if $\eta_i = 0, \zeta_i = 1$

$W_{(\eta,\zeta),(\eta^i,\zeta^i)} = 1$ if $\eta_i = \zeta_i = 0$

$W_{(\eta,\zeta),(\eta^i,\zeta)} = \frac{\mu_2(\zeta^i)}{\mu_2(\zeta)}$ if $\eta_i = \zeta_i = 1$

$W_{(\eta,\zeta),(\eta^i,\zeta)} = \frac{\mu_1(\eta^i)}{\mu_1(\eta)} - \frac{\mu_2(\zeta^i)}{\mu_2(\zeta)}$ if $\eta_i = \zeta_i = 1$

$W_{(\eta,\zeta),(\xi,\gamma)} = 0$ for any other case $(\xi, \gamma) \neq (\eta, \zeta)$

By assumption 1) $\frac{\mu_1(\eta^i)}{\mu_1(\eta)} - \frac{\mu_2(\zeta^i)}{\mu_2(\zeta)} \geq 0$ Therefore, W is a well-defined flip rate.

Let (α_t, β_t) be the Markov Chain on $\chi \times \chi$ with flip rate W.

Let S(t) be the semigroup on $C(\chi \times \chi)$ with generate W.

Let $L_1 = \{f \in C(\chi \times \chi) : f(\alpha, \beta) = f(\alpha)$ depends only on the first coordinate $\}$ By definition of W, it can be checked directly that $\Sigma_\xi \Sigma_\gamma W_{(\eta,\zeta),(\xi,\gamma)} f(\zeta) \in L_1$ iterating this we get $\Sigma_\xi \Sigma_\gamma W^n_{(\eta,\zeta),(\xi,\gamma)} f(\zeta) \in L_1$ $\forall n = 0, 1, 2....$

Therefore $S(t) = e^{tW} : L_1 \to L_1$ by Hille-Yosida Theorem.

Similarly $S(t) : L_2 \to L_2$, where L_2 is the space of functions in $C(\chi \times \chi)$ depending on the second variable only.

We shall show that $\alpha_t = \eta_t$ and $\beta_t = \zeta_t$ in distributions given $\alpha_0 = \eta_0$ and $\beta_0 = \zeta_0$. To do this it is sufficient to show the flip rate of α_t is q and the flip rate of β_t is r. Since the proofs are similiar, we will compute the flip rates of α_t only. The flip rate of α_t from η to η^i is

$\frac{d}{dt} P[\alpha_t = \eta^i | \alpha_0 = \eta](0) =$

$\frac{d}{dt}\Sigma_\gamma\Sigma_\zeta P[\alpha_t = \eta^i, \beta_t = \gamma|\alpha_0 = \eta, \beta_0 = \zeta](0) \times \frac{P[\alpha_0=\eta,\beta_0=\zeta]}{P[\alpha_0=\eta]} =$

$\Sigma_\zeta\Sigma_\gamma W_{(\eta,\zeta),(\eta^i,\gamma)} P[\alpha_0 = \eta, \beta_0 = \zeta]/P[\alpha_0 = \eta] = q_{(\eta,\eta_i)}$ by the definition of W and q

Let A=$(\eta, \zeta) \in \chi \times \chi : \eta \leq \zeta$ By * $(\alpha_t, \beta_t) \in A \; \forall t \geq 0$ if $(\alpha_0, \beta_0) \in A$. Let $f \in M$ then

$f(\alpha) - f(\beta) \leq 0$ if $(\alpha, \beta) \in A$

Therefore,

$E[f(\alpha_t) - f(\beta_t)|\alpha_0 = \alpha, \beta_0 = \beta] \leq 0 \; \forall t, (\alpha, \beta) \in A$

This implies

$E[f(\alpha_t)|\alpha_0 = \alpha] = E[f(\alpha_t)|\alpha_0\alpha, \beta_0 = \beta] \leq E[f(\beta_t)|\alpha_0 = \alpha, \beta_0 = \beta] =$
$E[f(\beta_t)|\beta_0 = \beta]$

Here we have used $S(t) : L_1 \to L_1$. Therefore we have $(S_1(t)f)(\alpha) \leq (S_2(t)f)(\beta)$ $\forall(\alpha, \beta) \in A, f \in M$ where $\{S_1(t)\}$ and $\{S_2(t)\}$ are the semigroups for $\{\alpha_t\}$ and $\{\beta_t\}$ respectively. By the Ergodic Theorem for finite state Markov chains passing to limit as t$\to \infty$ we have $\int f d\mu_1 \leq \int f d\mu_2$.

1.4 Ergodicity

From now on let

$$J_R=\begin{cases} \beta H & \text{for } x \in Z^d \\ \beta J(y-x) & \text{for } x, y \in Z^d \\ 0 & \text{for } |R| \geq 3 \end{cases}$$

Theorem 1.4.1: Suppose that the potential above is given with $\beta \geq 0$ and $J(x) \geq 0$. Then $\zeta_1 \leq \zeta_2$ implies that $\nu_{T,\zeta_1} \leq \nu_{T,\zeta_2}$ for any finite $T \subset S$.

Proof: By the FKG inequality which is proven earlier, it suffices to check that for $\eta_1, \eta_2 \in \{0,1\}^T$
$\Sigma_{R\cap T\neq\emptyset} J_R[\chi_R(\eta_1^{\zeta_1} \wedge \eta_2^{\zeta_1}) + \chi_R(\eta_1^{\zeta_2} \wedge \eta_2^{\zeta_2})] \geq \Sigma_{R\cap T\neq\emptyset} J_R[\chi_R(\eta_1^{\zeta_1}) + \chi_R(\eta_2^{\zeta_2})]$ whenever $\zeta_1 \leq \zeta_2$. Using the special form for J_R which we have assumed, this can be rewritten as the statement that the expression

$$2\beta H \Sigma_{x\in T}[(\eta_1 \wedge \eta_2)(x) + (\eta_1 \vee \eta_2)(x) - \eta_1(x) - \eta_2(x)]$$
$$+ 2\beta \Sigma_{x,y\in T; x\neq y} J(y-x)[(\eta_1 \wedge \eta_2)(x)(\eta_1 \wedge \eta_2)(y) + (\eta_1 \vee \eta_2)(x)(\eta_1 \vee \eta_2)(y)$$
$$- \eta_1(x)\eta_1(y) - \eta_2(x)\eta_2(y)$$
$$+ 4\beta \Sigma_{x\in T, y\notin T} J(y-x)[(\eta_1 \wedge \eta_2)(x)\zeta_1(y) + (\eta_1 \vee \eta_2)(x)\zeta_2(y) - \eta_1(x)\zeta_1(y) - \eta_2(x)\zeta_2(y)$$

is nonnegative. The terms in the first sum are all zero so the sign of the H is irrelevant in verifying the non-negativity of this expression. The terms in brackets in the second sum is zero unless $\eta_1(x) = \eta_2(y) = 0$ and $\eta_2(x) = \eta_1(y) = 1$ or $\eta_1(x) = \eta_2(y) = 1$ and $\eta_2(x) = \eta_1(y) = 0$ in which case it is equal to 1. The term in brackets in the third sum is zero unless $\eta_1(x) = 1$ and $\eta_2(x) = 0$, in which case it is equal to $\zeta_2(y) - \zeta_1(y)$. So, since $\beta \geq 0$ and $J(y-x) \geq 0$ the required sums are nonnegative whenever $\zeta_1 \leq \zeta_2$.

Let $\nu_{\underline{T}}$ denote the Gibbs state on T taking the value $\zeta \equiv 0$ outside T, and $\nu_{\bar{T}}$ is the Gibbs state on T taking the value $\zeta \equiv 1$ outside T

Corollary 1.4.1: Under the assumptions of Theorem 1.4.1, $T_1 \subset T_2$ implies that $\nu_{\underline{T}_1} \leq \nu_{\underline{T}_2}$ and $\nu_{\bar{T}_1} \geq \nu_{\bar{T}_2}$

Proof: By Theorem 1.4.1 $\nu_{T_1,\zeta} \leq \nu_{\underline{T}_1}\ \forall\zeta$. Therefore $\nu_{\bar{T}_1} = \Sigma_{\gamma:\gamma\equiv 1 on T_2/T_1} \nu^{\text{-}}\nu_{T_2}$ which clearly implies that $\nu_{\bar{T}_2} \leq \nu_{\bar{T}_1}$ The opposite statement holds true analogously.

Corollary 1.4.2: Under the assumptions of Theorem 1.4.1 we have

1)$\nu_{\text{-}} = lim_{T\uparrow S}\nu_{\underline{T}}$ exists

2)$\nu^{\text{-}} = lim_{T\uparrow S}\nu_{\bar{T}}$ exists

3) $\nu_{\text{-}} \leq \nu \leq \nu^{\text{-}}\ \forall\nu \in G$

4) phase transition occurs if and only if $\nu_{\text{-}} \neq \nu^{\text{-}}$

5) phase transition occurs if and only if $\nu_{\text{-}}(\eta : \eta(x) = 1) \neq \nu^{\text{-}}(\eta : \eta(x) = 1)$

Proof: 1) and 2) exist since Corollary 1.4.1 implies monotonicity.

3) By Theorem 1.4.1 we have $\nu_{\underline{T}} \leq \nu_{T,\zeta} \leq \nu_{\bar{T}}$

for any Gibbs state ν_T is a convex combination of $\nu_{T,\zeta}$ which are less than or equal to $\nu_{\bar{T}}$. Therefore $\nu \leq \nu_{\bar{T}}$ Likewise we conclude $\nu \geq \nu_{\underline{T}}$

4) This follows from 3) and the definition of phase transition.

5) follows from 4) since $\nu_{\text{-}} \neq \nu^{\text{-}}$

Theorem 1.4.2: In addition to the assumptions of Theorem 1.4.1, suppose that H=0. Then 1)$\nu_{\text{-}}(\eta : \eta(x) = 1) + \nu^{\text{-}}(\eta : \eta(x) = 1) = 1$

2)$\nu^{\text{-}}(\eta : \eta(x) = 1)$ is an increasing function of β

3)There is a critical value $0 \leq \beta_c \leq \infty$ such that there is no phase transition if $\beta \leq \beta_c$ and there is phase transitionif $\beta \geq \beta_c$

4)β_c is a decreasing function of the numbers J(x).

Proof: 1) Since H=0, $\nu_{\bar{T}}$ is obtained from $\nu_{\mathcal{T}}$ by interchanging the roles of 0 and 1. Since

$\nu_{\mathcal{T}}(\eta : \eta(x) = 1) + \nu_{\mathcal{T}}(\eta : \eta(x) = 0) = 1$ we conclude

$\nu_{\mathcal{T}}(\eta : \eta(x) = 1) + \nu^{\mathcal{T}}(\eta : \eta(x) = 1) = 1$

Let $T \uparrow \infty$ we conclude $\nu_{\_}(\eta : \eta(x) = 1) + \nu^{\text{-}}(\eta : \eta(x) = 1) = 1$

2)Note that if $\zeta \equiv 1$ then $\chi_R(\eta^\zeta) = \chi_{R\cap T}(\eta)$ so that 1.3.2.3 takes on the form 1.3.3. We apply 1.3.4.2 with A=$\{x\}$ to conclude that

$\nu_{\bar{T}}(\eta : \eta(x) = 1) - \nu_{\bar{T}}(\eta : \eta(x) = 0) = 2\nu_{\bar{T}}(\eta : \eta(x) = 1) - 1$

is an increasing function of β for each x$\in T$

3) Follows directly from 1) and 2) and the definition of phase transition

4)Follows from (1.3.4.2)

Definition 1.4.1: A spin system with rates c(x,η) is attractive whenever $\eta \leq \zeta$ we have

$c(x,\eta) \leq c(x,\zeta)$ if $\eta(x) = \zeta(x) = 0$

and $c(x,\eta) \geq c(x,\zeta)$ if $\eta(x) = \zeta(x) = 1$

if $x = (x_1, x_2, ..., x_d)$ then let $|x| = |x_1| + |x_2| + ... + |x_d|$ and let δ_0 denote the configuration $\eta(x) \equiv 0$ δ_1 denotes the configuration $\eta(x) \equiv 1$

Theorem 1.4.3: Suppose c(x, η) is attractive, and let S(t) be the semigroup for the spin system, then the following hold

1.$\delta_0 S(t) \leq \delta_0 S(s)$ for $0 \leq t \leq s$

2.$\delta_1 S(t) \leq \delta_1 S(s)$ for $0 \leq t \leq s$

3.$\delta_0 S(t) \leq \mu S(t) \leq \delta_1 S(t)$ for $t \geq 0$ and $\mu \in \wp$

4.$\nu_{\_} = lim_{t\to\infty}\delta_0 S(t)$ and $\nu^{\text{-}} = lim_{t\to\infty}\delta_1 S(t)$ exist

5. if $\mu \in \wp, t_n \to \infty$ and $\nu = lim_{n\to\infty}\mu S(t_n)$ then $\nu_{\_} \leq \nu \leq \nu^{\text{-}}$

6. $\nu_{\_}, \nu^{\text{-}} \in \Im_e$

Proof:

1) By definition, $\delta_0 \leq \delta_0 S(t-s)$ for $0 \leq s \leq t$. Therefore using the semigroup property, $\delta_0 S(s) \leq \delta_0 S(t-s)S(s) = \delta_0 S(t)$

2) Same as 1.

3) Note by monotonicity $\delta_0 \le \mu \le \delta_1 \forall \mu \in \wp$ therefore $\delta_0 S(t) \le \mu S(t) \le \delta_1 S(t)$

4) Follows from 1), 2) and the compactness of $\wp$ in the topology of weak convergence, and the fact that M has the following property:

$\int f d\mu_1 = \int f d\mu_2$

for all $f \in M$ and some $\mu_1, \mu_2 \in \wp$ implies that $\mu_1 = \mu_2$

From now on we can interpret the following,

$\nu_{\text{-}} = lim_{t\to\infty} \delta_0 S(t)$

$\nu^{\text{-}} = lim_{t\to\infty} \delta_1 S(t)$

which is consistent with our present definition.

Definition 1.4.2: $c_i^n(x,\eta) = \begin{cases} c(x,\eta^i) & \text{if } x \in S_n \\ 0 & \text{if } \notin S_n \text{ and } \eta(x) = i \\ M(x) & \text{if } x \notin S_n \text{ and } \eta(x) \ne i \end{cases}$ where $M(x) = sup_\eta c(x,\eta)$, with $\eta^i = \eta(u)\ \forall u \in S$ and $\eta^i(u) = i$ for $u \notin S$ S_n is a square centered at the origin with sides of length n.

Theorem 1.4.4: Suppose c(x,η) is attractive, then $c_i^n(x,\eta)$ is attractive for each i and n. If $\mu_0, \mu, \mu_1 \in \wp$ satisfy $\mu_0 \le \mu \le \mu_1$, then $\mu_0 S(t) \le \mu S(t) \le \mu_1 S(t) \forall t \ge 0$.

Proof: Since $\eta \le \zeta$ implies $\eta^i \le \zeta^i$ the attractiveness of $c_i^n(x,\eta)$ follows from that of c(x,η). To prove the Thm. it suffices to check that $c_0^n \le c(x,\eta) \le c_1^n$ if $\eta(x) = 0$ and $c_0^n \ge c(x,\eta) \ge c_1^n$ if $\eta(x) = 1$. This is true for x$\in Z^d$, since $\eta^0 \le \eta \le \eta^1$ and for x$\notin Z^d$ since 0$\le c(x,\eta) \le M(x)$

Theorem 1.4.5: For an attractive spin system, the following three statements are equivalent

1) The process is ergodic

2) $\Im$ is a singleton

3) $\nu_{\text{-}} = \nu^{\text{-}}$

Proof: 1) implies 2) from the Definition of Ergodicity
2) implies 3) follows from Theorem 1.4.3 (part 6)
3) implies 1) If $\mu \in \wp$ then the set of probability measures $\{\mu S(t), t \geq 0\}$ is relatively compact. By Theorem 1.4.3 (part 5) all subsequential limits of this family are as $t \to \infty$ are equal to the common value of $\nu_{\_}$ and ν^{-} therefore $lim_{t\to\infty}\mu S(t)$ exists and equals that common value. Therefore the process is ergodic

Theorem 1.4.6: Suppose that ν is a probability measure on X and that $c(x,\eta)$ are the rates for a spin system. Then ν is reversible for the spin system if and only if

(1.4.6.1) $\int c(x,\eta)[f(\eta_x) - f(\eta)]d\nu = 0$

$\forall x \in Z^d$ and $f \in C(X)$. If the rates are strictly positive then this is equivalent to the statement that ν has the following conditional probabilities:

(1.4.6.2) $\nu(\eta : \eta(x) = \zeta(x) | \eta(u) = \zeta(u)\ \forall u \neq x) = \frac{c(x,\zeta_x)}{c(x,\zeta)+c(x,\zeta_x)}$

Proof: If (1.4.6.1) holds for all f∈C(X), then it can be applied to the function $f(\eta_x)g(\eta)$ for f,g∈D(X) to obtain

$\int c(x,\eta)f(\eta)g(\eta_x)d\nu = \int c(x,\eta)f(\eta_x)g(\eta)d\nu$

or equivalently

$\int c(x,\eta)f(\eta)[g(\eta_x) - g(\eta)]d\nu = \int c(x,\eta)g(\eta)[f(\eta_x) - f(\eta)]d\nu$

summing on x we get that ν is reversible for the spin system.

To prove the converse, assume that ν is reversible.

For a finite subset T of Z^d and an x∈T, let $f(\eta) = \Pi_{y\in T}\eta(y)$ and $g(\eta) = f(\eta_x)$. Then

$g(\eta)\Omega f(\eta) = f(\eta_x)\Sigma_{y\in T}c(y,\eta)[f(\eta_x) - f(\eta)] = c(x,\eta)f(\eta_x)$ and

$f(\eta)\Omega g(\eta) = f(\eta)\Sigma_{y\in T}c(y,\eta)[g(\eta_x) - g(\eta)] = c(x,\eta)f(\eta)$

so that (1.4.6.1) holds for that f by Proposition . By linearity, it holds for all f∈D since D is dense in C(X) (1.4.6.1) holds for all f∈C(X). Now assume that $c(x,\eta) > 0$ $\forall x \in S$ and $\eta \in X$. Fix an $x \in Z^d$ and let $c_A(\eta)$ and $c_B(\eta)$ be the unique functions on X which do not depend on $\eta(x)$ such that then (1.4.6.2) can be rewritten as the statement that

$\int \eta(x) f(\eta) d\nu = \int \frac{c_A(\eta)}{c_A(\eta)+c_B(\eta)} f(\eta) d\nu$ for all $f \in C(X)$ which do not depend on $\eta(x)$.

Since $c_A(\eta) + c_B(\eta)$ does not depend on $\eta(x)$ and is strictly positive, this is equivalent to the statement that

$\int \eta(x) g(\eta)[c_A(\eta) + c_B(\eta)] d\nu = \int c_A(\eta) g(\eta) d\nu$ for all g∈C(X) which do not depend on $\eta(x)$.

But this can be rewritten as

$\int g(\eta)(\eta(x)c_A(\eta) - [1-\eta(x)]c_A(\eta))d\nu = 0$ or

(1.4.6.3) $\int c(x,\eta) g(\eta)[2\eta(x) - 1] d\nu = 0$

On the other hand, if f∈C(X) is written as

$f(\eta) = f_A(\eta)[1-\eta(x)] + f_B(\eta)\eta(x)$ where f_A and f_B do not depend on $\eta(x)$ then

$f(\eta_x) - f(\eta) = [f_A(\eta) - f_B(\eta)][2\eta(x) - 1]$ so that (1.4.6.1) can be rewritten as

$\int c(x,\eta)[f_A(\eta) - f_B(\eta)][2\eta(x) - 1] d\nu = 0$

Theorem 1.4.7: Suppose that c(x,η) is strictly positive, and that for each x , c(x,η) depends on only finitely many coordinates. If the spin system is reversible with respect to some probability measure ν, then it is a Stochastic Ising model relative to some potential $\{J_R\}$. Proof: By Theorem 1.4.6 ν has conditional probabilities given by (1.4.6.2). By the finite dependence assumption on the rates and Theorem 1.4.2, ν is a Gibbs state relative to some potential (note any finite state measure that never equals zero can be written in the form defined as a Gibbs state). Using (1.4.6.2) and Definition 1.3.4 we see that:

$\frac{c(x,\zeta_x)}{c(x,\zeta)+c(x,\zeta_x)} = \frac{1}{1+exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)]}$

which implies

$\frac{c(x,\zeta)}{c(x,\zeta_x)} = exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)]$

using the multiplicative property of χ we conclude

$c(x,\zeta)exp[\Sigma_{x\in R}J_R\chi_R(\zeta)] = c(x,\zeta_x)exp[\Sigma_{x\in R}J_R\chi_R(\zeta_x)]$

which implies that our spin system is a Stochatic Ising model (independent of the coordinate ζ_x

Theorem 1.4.8: Suppose that $c(x,\eta)$ are the rates for a Stochastic Ising model relative to the potential $\{J_R\}$. Then G=R where G denotes the set of all Gibbs states relative to the same potential.

Proof: By the Theorem 1.4.6 and Definition 1.3.3 it suffices to show that for a Stochastic Ising model,

$\frac{c(x,\zeta_x}{c(x,\zeta)+c(x,\zeta_x)} = \frac{1}{1+exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)]}$

but this is shown just as in the proof of Theorem 1.4.7. Since we have a stochastic Ising model we have

$c(x,\zeta)exp[\Sigma_{x\in R}J_R\chi_R(\zeta)] = c(x,\zeta_x)exp[\Sigma_{x\in R}J_R\chi_R(\zeta_x)]$

therefore

$\frac{c(x,\zeta)}{c(x,\zeta_x)} = exp[-2\Sigma_{x\in R} J_R \chi_R(\zeta)]$

and we conclude $\frac{c(x,\zeta_x)}{c(x,\zeta)+c(x,\zeta_x)} = \frac{1}{1+exp[-2\Sigma_{x\in R} J_R \chi_R(\zeta)]}$

Theorem 1.4.9: Suppose our space is Z^2, $J_R = \beta$ if R=$\{x, y\}$ with $|y - x| = 1$ and $J_R = 0$ otherwise. For sufficiently large positive β, this potential exhibits phase transition.

Proof: For $n \geq 1$ define ν_n as the unique Gibbs state on T with $\zeta \equiv 1$ and $T = [-n, n]^2 \subset Z^2$. It suffices to show that

(1.4.9.1) $lim_{\beta\to\infty}\nu_n(\eta : \eta(0) = 0) = 0$ uniformly in n, since phase transition will occur for any β such that $lim_{n\to\infty}\nu_n(\eta : \eta(0) = 0) < \frac{1}{2}$. To visualize the proof it is important to visualize a configuration $\eta \in \{0, 1\}^T$ in a certain way. Write + for 1 and - for 0 and agree to draw horizontal and vertical lines of unit length between adjacent sites which have opposite signs. An illustration with a particular configuration is given below. Let $B(\eta)$ be the union of all these vertical and horizontal lines. Note that the configuration can be reconstructed from $B(\eta)$ if the boundary is fixed. Also $B(\eta)$ is a disjoint union of contours, where a contour is a closed non self-intersecting polygonal curve. The length of all the contours which make up $B(\eta)$ will be denoted by $|B(\eta)|$. With this notation we can proceed to prove (1.4.9.1). If $\eta(0) = 0$ then 0 is surrounded by at least one contour γ. Let Γ be the set of contours surrounding 0. Then

(1.4.9.2) $\nu_n(\eta : \eta(0) = 0) = \Sigma_{\gamma\in\Gamma}\nu_n(\eta : \gamma \in B(\eta))$

so we need to estimate $\nu_n(\eta : \gamma \in B(\eta))$ for fixed $\gamma \in \Gamma$. To do so use the Definition of Gibbs state to write

(1.4.9.3) $\nu_n(\eta : \gamma \in B(\eta)) = \frac{\Sigma_{\eta:\gamma\in B(\eta)} exp[-2\beta|B(\eta)|]}{\Sigma_\eta exp[-2\beta|B(\eta)|]}$

if η is such that $\gamma \in B(\eta)$, define $\tilde{\eta}$ by $\tilde{\eta}(x) = \begin{cases} 1 - \eta(x) & \text{if } \gamma \text{ surrounds x} \\ \eta(x) & \text{otherwise} \end{cases}$

Then $B(\tilde{\eta})$ is obtained from $B(\eta)$ by removing γ, so that $|B(\eta)| = |\gamma| + |B(\tilde{\eta})|$. Therefore by (1.4.9.3) $\nu_n(\eta : \gamma \in B(\eta)) = exp[-2\beta|\gamma|]\frac{\Sigma_{\eta:\gamma\in B(\eta)} exp[-2\beta|B(\tilde{\eta})|]}{\Sigma_\eta exp[-2\beta|B(\eta)|]}$

Since the map $\eta \to \tilde{\eta}$ is 1-1, each term in the numerator on the right side also appears in the denominator. therefore we can conclude that $\nu_n(\eta : \gamma \in B(\eta)) \leq exp[-2\beta|\gamma|]$ using this in (1.4.9.2) gives

(1.4.9.4) $\nu(\eta : \eta(0) = 0) \leq \Sigma_{k=4}^{\infty} e^{-2\beta k} N(k, n)$ where N(k,n) is the number of contours $\gamma \in \Gamma$ of length k.

But $N(k, n) \leq k3^k$ for all n, since each contour $\gamma \in \Gamma$ of length k must cross the positive horizontal axis at least at one of k places, and such a contour can be continued at each point in at most three ways. Thus we obtain the estimate

(1.4.9.5)$\nu_n(\eta : \eta(0) = 0) \leq \Sigma_{k=4}^{\infty} k3^k e^{-2\beta k}$

from which (1.4.9.1) follows by DCT.

Theorem 1.4.10: Consider a stochastic Ising model relative to the potential $\{J_R\}$, and let G be the corresponding Gibbs states. Then $G \subset \Im$. In particular, if the stochastic Ising model is ergodic, then there is no phase transition for that potential.

Proof:$R \subset \Im$ follows from the definition and by the previous Theorem R=G, therefore $G \subset \Im$. If the process is ergodic, then $\Im$ is a singleton, therefore G is a singleton as well, so $\{J_R\}$ shows no phase transition.

Theorem 1.4.11: Consider an attractive stochastic Ising model relative to the potential J_R, then $\nu_{\text{-}}, \nu^{\text{-}} \in G$.

Proof: Let S_n defined as before increase to Z^d, and let $c_i^n(x, \eta)$ be the rates for the approximating spin system. by checking $c(x, \eta)\nu(\eta) = c(x, \eta_x)\nu(\eta_x)$, we see that $\nu_{S_n,\zeta}$ is invariant for $c_1^n(x, \eta)$ if $\zeta \equiv 1$ and for $c_0^n(x, \eta)$ if $\zeta \equiv 0$. By the convergence Theorem of finite state Markov chains, $\nu_{\text{-}}^n$ and $\nu^{\text{-}n}$ are equal to $\nu_{S_n,\zeta}$ with $\zeta \equiv 0$ and $\zeta \equiv 1$. Therefore $\nu_-^n, \nu^{-n} \in G(S_n)$ so that $\nu^-, \nu_- \in$ G by the above Theorem.

CHAPTER 2

Antiferromagnetic Stochastic Ising Model

2.1 H=0 case

In general, the potential of an antiferromagnetic Ising model is of the form

$$J_R = \begin{cases} \beta H & \text{for } x \in Z^d \\ \beta J(y-x) & \text{for } x, y \in Z^d \\ 0 & \text{for } |R| \geq 3 \end{cases}$$

where $J(y-x) < 0$ if $|y-x|$ is an odd taxicab distance from the origin, and $J(y-x) > 0$ if $|y-x|$ is an even boxcar distance from the origin. As we did in Chapter 1 we will simply investigate the nearest neighbor model, thus J(y-x)=0 if $|y-x| > 1$. Since J(y-x) is negative this system, by looking at the Hamiltonian, $\Sigma_R J_R \chi_R(\eta)$, we conclude that +1 spins are attracted to 0 spins and 0 spins are attracted to +1 spins since these states require less energy. Let A(x) represent the configuration, η, where $\eta(x) = 1$ if x is an even taxicab distance from the origin and $\eta(x) = 0$ if x is an odd taxicab distance from the origin. Let B(x) represent the complimentary configuration, η, where $\eta(x) = 0$ if x is an even taxicab distance from the origin and $\eta(x) = 1$ if x is an odd taxicab distance from the origin. These states have the same energy as $\eta(x) \equiv 1$ and $\eta(x) \equiv 0$ in the ferromagnetic model. To relate this model to the ferromagnetic one, we will define a new ordering for configurations on $\{0,1\}^{Z^d}$. We will say $\eta \preceq \zeta$ if $\tilde{\eta} \leq \tilde{\zeta}$ using 1.3.5.11, and $\nu \preceq \mu$ if $\int f d\nu(\tilde{\eta}) \leq \int f d\mu(\tilde{\eta})$ $\forall$ f increasing

$$\text{where } \tilde{\gamma}(x) = \begin{cases} \gamma(x) & \text{if } |x| \text{ is even using the taxicab norm} \\ 1-\gamma(x) & \text{if } |x| \text{ is odd using the taxicab norm} \end{cases}$$

We introduced an ordering for the ferromagnetic model at 1.3.5.11, everything up until that point except Theorem 1.3.4, which we must modify, is independent of J_R thus is true for general J_R. Therefore everything in Chapter 1 up until 1.3.5.11 holds for the antiferromagnetic model discussed here except Theorem 1.3.4. We will modify every Theorem and Definition in Chapter 1 after 1.3.5.11 to show that analogous results hold here as well.

Theorem 2.1.1 Suppose that S$\in Z^d$ is finite, and that $\{J_R, R \subset S\}$ is a potential with $J_R \leq 0$ $\forall R$ and H=0. Let ν be the corresponding Gibbs state then,

(2.1.1.1) $\int \chi_A d\nu \geq 0$ $\forall A \subset S$ such that A has an even number of points which are an odd taxi-cab distance from the origin, and

(2.1.1.2) $\frac{\partial}{\partial J_R} \int \chi_A d\nu = \int \chi_A \chi_B d\nu - \int \chi_A d\nu \int \chi_B d\nu \geq 0$ for all $A, B \subset S$ both A, and B have an even number of points which are an odd taxi-cab distance from the origin

Proof: The proof of (2.1.1.2) uses (2.1.1.1) so we must prove (2.1.1.1) first. In order to do so, write

$\int \chi_A d\nu = K\Sigma_{\tilde{\eta}}\chi_A(\tilde{\eta})exp[\Sigma_R J_R \chi_R(\tilde{\eta})]$

$= K\Sigma_{\tilde{\eta}}\chi_A(\tilde{\eta})\Sigma_{n=0}^{\infty}\frac{1}{n!}[\Sigma_R J_R \chi_R(\tilde{\eta})]^n$

Where $\tilde{\eta}$ is a configuration on Z^d, I wrote $\tilde{\eta}$ instead of η to emphasize that these are configurations on the same space as before, but with different equilibrium states (A and B as opposed to 1 and 0). If we try to analyze the difference between the ferromagnetic case we come up with an interesting revelation. for any set $R \subset S$ $\chi_R(\tilde{\eta}) = -1 \times \chi_R(\eta)$. Therefore comparing to the ferromagnetic case to the antiferromagnetic case we see that every configuration is summed whether we sum over η or $\tilde{\eta}$ (i.e. there is a 1-1 correspondence between the configurations). Therefore $\Sigma_R J_R \chi_R(\tilde{\eta}) = \Sigma_R \tilde{J}_R \chi_R(\eta)$ where $\tilde{J}_R = -J_R$ the ferromagnetic analog to our antiferromagnetic model. This now simplifies to the proof of Theorem 1.3.4 in Chapter 1. I will write the rest of the proof below:

$= K\Sigma_{n=0}^{\infty}\frac{1}{n!}\Sigma_{R_1,\dots R_n}[\Pi_{k=1}^{n} J_{R_k}]\Sigma_{\eta}\chi_A(\eta)\Pi_{k=1}^{n}\chi_{R_k}(\eta)$

To see that this is a sum of nonnegative terms, it suffices to note that

$\chi_A(\eta)\Pi_{k=1}^{n}\chi_{R_k}(\eta) = \chi_B(\eta)$ where $B = \{x \in S :$ x is in an odd number of the sets $A, R_1, R_2, \dots, R_n\}$ and that for any $B \subset S$ $\Sigma_\eta \chi_B(\eta) = \begin{cases} 2^{|S|} & \text{if } B = \emptyset \\ 0 & \text{if } B \neq \emptyset \end{cases}$

Therefore we have a sum of positive terms, and the sum is positive. Turning to (2.1.1.2) use the explicit expression for K to write

$\int \chi_A d\nu = \frac{\Sigma_{\tilde{\eta}}\chi_A(\tilde{\eta})exp[\Sigma_R J_R \chi_R(\tilde{\eta})]}{\Sigma_{\tilde{\eta}} exp[\Sigma_R J_R \chi_R(\tilde{\eta})]}$

Therefore the equality in (2.1.1.2) is the result of a simple differentiation. To check the inequality, write

$$\int \chi_A \chi_B d\nu - \int \chi_A d\nu \int \chi_B d\nu =$$

$$K^2 \Sigma_{\bar{\eta},\bar{\zeta}} [\chi_A(\bar{\eta})\chi_B(\bar{\eta}) - \chi_A(\bar{\eta})\chi_B(\bar{\zeta})] exp[\Sigma_R \bar{J}_R[\chi_R(\bar{\eta}) + \chi_R(\bar{\zeta})]]$$

Notice that $\chi_A(\bar{\eta}) = \chi_A(\eta)$ $\forall \bar{\eta}$ since A has an even number of points an odd distance away from the origin (using the taxi-cab metric). Likewise, $\chi_B(\bar{\eta}) = \chi_B(\eta)$. We must simply check the $\bar{J}_R[\chi_R(\bar{\eta})] + \chi_R(\bar{\zeta})$. Notice we only concern ourselves with $|R| = 2$ since $\bar{J}_R = 0$ otherwise. However, once again this term is equal to the analogous ferromagnetic term in Chap 1 Theorem 1.3.4 because $|R| = 2$, $\chi_R(\bar{\eta}) = -1 \times \chi_R(\eta)$ $\forall \eta$ $\bar{\eta}$ is η flipped at every point odd distance from the origin, and $\bar{J}_R = -1 \times J_R$ therefore the product of the two will be the same as the product in the ferromagnetic case. Thus we have proved Theorem 1.3.4 for the antiferromagnetic case.

Theorem 2.1.5: Let K=$\{(\eta, \zeta \in X \times X : \eta \preceq \zeta\}$ Suppose $\eta \preceq \zeta$
$c_1(x, \eta) \leq c_2(x, \zeta)$if $\eta(x) = \zeta(x) = B(x)$ and
$c_1(x, \eta) \geq c_2(x, \zeta)$ if $\eta(x) = \zeta(x) = A(x)$, then $\forall (\eta, \zeta) \in K$ and $t \geq 0$
$P^{(\eta,\zeta)}[(\eta_t, \zeta_t) \in K] = 1$.

Proof: Let E be the set of all functions f in $C(X \times X)$ such that $f \geq 0$ and $f = 0$ on K. For $\lambda \geq 0$ and $f \in A$, define $h \in D(\bar{\Omega})$ by $h - \lambda \bar{\Omega} h = f$. Let $(\eta, \zeta) \in K$ be a point where h achieves its maximum on K. We will show that $\bar{\Omega} h(\eta, \zeta) \leq 0$, so that $h(\eta, \zeta) \leq f(\eta, \zeta) = 0$. Since $h \geq 0$, it will follow that $h \in E$ as well. To show $\bar{\Omega} h(\eta, \zeta) \leq 0$, we will check that each of the terms in the sum defining $\bar{\Omega} h$ is nonpositive. Consider the following three cases:

a)$\eta(x) \neq \zeta(x)$ in which case $(\eta, \zeta) \in K$ implies that $(\eta_x, \zeta) \in K$ and $(\eta, \zeta_x) \in K$, so that $h(\eta_x, \zeta) \leq h(\eta, \zeta)$ and $h(\eta, \zeta_x) \leq h(\eta, \zeta)$.

b)$\eta(x) = \zeta(x) = 0$, in which case $(\eta, \zeta) \in K$ implies that $(\eta_x, \zeta_x) \in K$ and $(\eta, \zeta_x) \in K$, so that $h(\eta_x, \zeta_x) \leq h(\eta, \zeta)$ and $h(\eta, \zeta_x) \leq h(\eta, \zeta)$. However, in this case $(\eta_x, \zeta) \notin K$, so we need to have $c_1(x, \eta) = min(c_1(x, \eta), c_2(x, \zeta))$ which is part of our assumption.

c)$\eta(x) = \zeta(x) = 1$ which is analagous to b), using the second part of the assumption.

2.2 H=$(-1)^{|x|}$ case

There is a relationship between Gibbs states for the ferromagnetic and antiferromagnetic Ising models. To see this we simply take the mapping $\bar{\eta}(x)$ introduced in Section 2.1. This mapping takes our Hamiltonian $\Sigma J_R \chi_R(\eta)$ and maps it into the Hamiltonian $\Sigma J_R^* \chi_R(\bar{\eta})$, $\bar{\eta}$ is with $J_R^* = -J_R$ $|R| = 2$ and $J_R^* = (-1)^{|x|} J_R$ if $R = \{x\}$thus we have a bijection from the ferromagnetic case to the antiferromagnetic case. All the proofs in Chapter 1 follow using this bijection, we first relate the antiferromagnetic case to the ferromagnetic one then use the proof from Chapter 1 then we use our bijection again to go back to the antiferromagnetic case. We will go through some of the proofs using the procedure quoted, then we will simply restate the remaining theorems and definitions so they apply to the antiferromagnetic case.

Theorem 2.2.1: Suppose that the potential is given with $\beta \geq 0$ and $J(x) \leq 0$. Then $\zeta_1 \preceq \zeta_2$ implies that $\nu_{T,\zeta_1} \preceq \nu_{T,\zeta_2}$ for any finite $T \subset S$.
Proof: By the FKG inequality (1.3.6) which is proven earlier, it suffices to check that for $\eta_1, \eta_2 \in \{0,1\}^T$
$\Sigma_{R\cap T\neq\emptyset} J_R[\chi_R(\eta_1^{\zeta_1} \wedge \eta_2^{\zeta_1}) + \chi_R(\eta_1^{\zeta_2} \wedge \eta_2^{\zeta_2})] \geq \Sigma_{R\cap T\neq\emptyset} J_R[\chi_R(\eta_1^{\zeta_1}) + \chi_R(\eta_2^{\zeta_2})]$ whenever $\zeta_1 \preceq \zeta_2$. Using the special form for $J_R$which we have assumed and the bijextion above, this can be rewritten as the statement that the expression

$$2\beta H \Sigma_{x\in T}(-1)^{|x|}[\bar{\eta_1} \wedge \bar{\eta_2}(x) + (\bar{\eta_1} \vee \bar{\eta_2})(x) - \bar{\eta_1}(x) - \bar{\eta_2}(x)]$$
$$+ 2\beta \Sigma_{x,y\in T; x\neq y} J^*(y-x)[(\bar{\eta_1} \wedge \bar{\eta_2})(x)(\bar{\eta_1} \wedge \bar{\eta_2})(y) + (\bar{\eta_1} \vee \bar{\eta_2})(x)(\bar{\eta_1} \vee \bar{\eta_2})(y)$$
$$- \bar{\eta_1}(x)\bar{\eta_1}(y) - \bar{\eta_2}(x)\bar{\eta_2}(y)$$
$$+ 4\beta \Sigma_{x\in T, y\notin T} J^*(y-x)[(\bar{\eta_1} \wedge \bar{\eta_2})(x)\bar{\zeta_1}(y) + (\bar{\eta_1} \vee \bar{\eta_2})(x)\bar{\zeta_2}(y) - \bar{\eta_1}(x)\bar{\zeta_1}(y) - \bar{\eta_2}(x)\bar{\zeta_2}(y)$$

is nonnegative. The first term is zero and the rest are nonnegative by the proof in Chapter 1

Corollary 2.2.1: Under the assumptions of Theorem 2.2.1, $T_1 \subset T_2$ implies that $\nu_{\underline{T}_1} \preceq \nu_{\underline{T}_2}$ and $\nu_{\bar{T}_1} \succeq \nu_{\bar{T}_2}$

Proof: By Theorem 2.2.1 $\nu_{T_1,\zeta} \preceq \nu_{\underline{T}_1}$ $\forall\zeta$. Therefore $\nu_{\bar{T}_1} = \Sigma_{\gamma:\gamma\equiv 1 on T_2/T_1} \nu^{-}\nu_{T_2}$ which clearly implies that $\nu_{\bar{T}_2} \preceq \nu_{\bar{T}_1}$ The opposite statement holds true analogously.

Corollary 2.2.2: Under the assumptions of Theorem 2.2.1 we have

1)$\nu_{-} = lim_{T\uparrow S}\nu_{\underline{T}}$ exists

2)$\nu^{-} = lim_{T\uparrow S}\nu_{\bar{T}}$ exists

3) $\nu_{-} \preceq \nu \preceq \nu^{-}$ $\forall \nu \in G$

4) phase transition occurs if and only if $\nu_{-} \neq \nu^{-}$

5) phase transition occurs if and only if $\nu_{-}(\eta : \eta(x) = A(x)) \neq \nu^{-}(\eta : \eta(x) = A(x))$

Proof: 1) and 2) exist since Corollary 2.2.1 implies monotonicity.

3) By Theorem we have $\nu_{\underline{T}} \preceq \nu_{T,\zeta} \preceq \nu_{\bar{T}}$

for any Gibbs state ν_T is a convex combination of $\nu_{T,\zeta}$ which are less than or equal to $\nu_{\bar{T}}$. Therefore $\nu \preceq \nu_{\bar{T}}$ Likewise we conclude $\nu \succeq \nu_{\underline{T}}$

4) This follows from 3) and the definition of phase transition.

5) follows from 4) since $\nu_{-} \neq \nu^{-}$

Theorem 2.2.2: In addition to the assumptions of Theorem 2.2,1, suppose that H=0. Then 1)$\nu_{-}(\eta : \eta(x) = A(x)) + \nu^{-}(\eta : \eta(x) = A(x)) = 1$

2)$\nu^{-}(\eta : \eta(x) = A(x))$ is an increasing function of β

3)There is a critical value $0 \leq \beta_c \leq \infty$ such that there is no phase transition if $\beta \leq \beta_c$ and there is phase transition if $\beta \geq \beta_c$

4)β_c is a decreasing function of the numbers J(x).

Proof: 1) Since H=0, $\nu_{\bar{T}}$ is obtained from ν_T by interchanging the roles of 0 and 1. Since

$\nu_T(\eta : \eta(x) = A(x)) + \nu_T(\eta : \eta(x) = B(x)) = 1$ we conclude

$\nu_T(\eta : \eta(x) = A(x)) + \nu^T(\eta : \eta(x) = A(x)) = 1$

Let $T \uparrow \infty$ we conclude $\nu_-(\eta : \eta(x) = A(x)) + \nu^-(\eta : \eta(x) = A(x)) = 1$

2)Note that if $\zeta \equiv 1$ then $\chi_R(\eta^\zeta) = \chi_{R\cap T}(\eta)$ so that 1.3.2.3 takes on the form 1.3.3. We apply 2.1.1.2 with A=$\{x\}$ to conclude that

$\nu_{\bar{T}}(\eta : \eta(x) = A(x)) - \nu_{\bar{T}}(\eta : \eta(x) = B(x)) = 2\nu_{\bar{T}}(\eta : \eta(x) = A(x)) - 1$

is an increasing function of β for each x$\in T$

3) Follows directly from 1) and 2) and the definition of phase transition

4) Follows from 2.1.1.2

Definition 2.2.2: A spin system with rates c(x,η) is attractive whenever $\eta \preceq \zeta$ we have

$c(x, \eta) \leq c(x, \zeta)$ if $\eta(x) = \zeta(x) = A$

and $c(x, \eta) \geq c(x, \zeta)$ if $\eta(x) = \zeta(x) = B$

Theorem 2.2.3: Suppose c(x, η) is attractive, and let S(t) be the semigroup for the spin system, then the following hold

1.$\delta_B S(t) \preceq \delta_B S(s)$ for $0 \leq t \leq s$

2.$\delta_A S(t) \preceq \delta_A S(s)$ for $0 \leq t \leq s$

3.$\delta_B S(t) \preceq \mu S(t) \preceq \delta_A S(t)$ for $t \geq 0$ and $\mu \in \wp$

4.$\nu_- = lim_{t\to\infty} \delta_B S(t)$ and $\nu^- = lim_{t\to\infty} \delta_A S(t)$ exist

5. if $\mu \in \wp, t_n \to \infty$ and $\nu = lim_{n\to\infty} \mu S(t_n)$ then $\nu_- \preceq \nu \preceq \nu^-$

6. $\nu_-, \nu^- \in \Im_e$

Proof: 1) By definition, $\delta_B \preceq \delta_B S(t-s)$ for $0 \leq s \leq t$. Therefore using the semigroup property, $\delta_B S(s) \preceq \delta_B S(t-s) S(s) = \delta_B S(t)$

2) Same as 1.

3) Note by monotonicity $\delta_B \preceq \mu \preceq \delta_A \forall \mu \in \wp$ therefore $\delta_B S(t) \preceq \mu S(t) \leq \delta_A S(t)$

.

4) Follows from 1), 2) and the compactness of $\wp$ in the topology of weak convergence, and the fact that M has the following property:
$\int f d\mu_1 = \int f d\mu_2$
for all $f \in M$ and some $\mu_1, \mu_2 \in \wp$ implies that $\mu_1 = \mu_2$
Theorem 2.2.4: Suppose c(x,η) is attractive, then $c_i^n(x,\eta)$ is attractive for each i and n. If $\mu_A, \mu, \mu_B \in \wp$ satisfy $\mu_B \preceq \mu \preceq \mu_A$, then $\mu_B S(t) \preceq \mu S(t) \preceq \mu_A S(t) \forall t \geq 0$.
Proof: Since $\eta \preceq \zeta$ implies $\eta^i \preceq \zeta^i$ the attractiveness of $c_i^n(x,\eta)$ follows from that of c(x,η). To prove the theorem it suffices to check that $c_B^n \leq c(x,\eta) \leq c_A^n$ if $\eta(x) = B(x)$ and $c_B^n \geq c(x,\eta) \geq c_A^n$ if $\eta(x) = A(x)$. This is true for x$\in$ S, since $\eta^B \preceq \eta \preceq \eta^A$ and for x$\notin$ S since 0$\leq c(x,\eta) \leq M(x)$

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Theorem 2.2.6: Suppose that ν is a probability measure on X and that $c(x,\eta)$ are the rates for a spin system. Then ν is reversible for the spin system if and only if

(2.2.6.1) $\int c(x,\eta)[f(\eta_x) - f(\eta)]d\nu = 0$

$\forall x \in S$ and $f \in C(X)$. If the rates are strictly positive then this is equivalent to the statement that ν has the following conditional probabilities:
(2.2.6.2) $\nu(\eta : \eta(x) = \zeta(x) | \eta(u) = \zeta(u)\ \forall u \neq x) = \frac{c(x,\zeta_x)}{c(x,\zeta)+c(x,\zeta_x)}$
Proof: If (2.2.6.1) holds for all f$\in$C(X), then it can be applied to the function $f(\eta_x)g(\eta)$ for f,g$\in$D(X) to obtain
$\int c(x,\eta)f(\eta)g(\eta_x)d\nu = \int c(x,\eta)f(\eta_x)g(\eta)d\nu$

or equivalently

$$\int c(x,\eta)f(\eta)[g(\eta_x) - g(\eta)]d\nu = \int c(x,\eta)g(\eta)[f(\eta_x) - f(\eta)]d\nu$$

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summing on x we get that ν is reversible for the spin system.

To prove the converse, assume that ν is reversible.

For a finite subset T of Z^d and an x∈T, let $f(\eta) = \Pi_{y\in T}\eta(y)$ and $g(\eta) = f(\eta_x)$. Then

$g(\eta)\Omega f(\eta) = f(\eta_x)\Sigma_{y\in T}c(y,\eta)[f(\eta_x) - f(\eta)] = c(x,\eta)f(\eta_x)$ and

$f(\eta)\Omega g(\eta) = f(\eta)\Sigma_{y\in T}c(y,\eta)[g(\eta_x) - g(\eta)] = c(x,\eta)f(\eta)$ so that (2.2.6.1) holds for that f by Proposition . By linearity, it holds for all f∈D since D is dense in C(X) (2.4.6.1) holds for all f∈C(X). Now assume that $c(x,\eta) > 0\ \forall x \in Z^d$ and $\eta \in X$. Fix an $x \in Z^d$ and let $c_A(\eta)$ and $c_B(\eta)$ be the unique functions on X which do not depend on $\eta(x)$ such that

then (2.2.6.2) can be rewritten as the statement that $\int \eta(x)f(\eta)d\nu = \int \frac{c_A(\eta)}{c_A(\eta)+c_B(\eta)}f(\eta)d\nu$ for all $f \in C(X)$ which do not depend on $\eta(x)$. Since $c_A(\eta) + c_B(\eta)$ does not depend on $\eta(x)$ and is strictly positive, this is equivalent to the statement that $\int \eta(x)g(\eta)[c_A(\eta)+c_B(\eta)]d\nu = \int c_A(\eta)g(\eta)d\nu$ for all g∈C(X) which do not depend on $\eta(x)$. But this can be rewitten as $\int g(\eta)(\eta(x)c_A(\eta) - [1-\eta(x)]c_A(\eta))d\nu = 0$ or $\int c(x,\eta)g(\eta)[2\eta(x) - 1]d\nu = 0$ On the other hand, if f∈C(X) is written as $f(\eta) = f_A(\eta)[1-\eta(x)] + f_B(\eta)\eta(x)$ where f_A and f_B do not depend on $\eta(x)$ then $f(\eta_x) - f(\eta) = [f_A(\eta) - f_B(\eta)][2\eta(x) - 1]$

so that (2.2.6.1) can be rewritten as $\int c(x,\eta)[f_A(\eta) - f_B(\eta)][2\eta(x) - 1]d\nu = 0$

Theorem 2.2.7: Suppose that c(x,η) is strictly positive, and that for each x , c(x,η) depends on only finitely many coordinates. If the spin system is reversible with respect to some probability measure ν, then it is a stochastic Ising model relative to some potential $\{J_R\}$. Proof: By Theorem 2.2.6 ν has conditional probabilities given by (2.2.6.2). ν is a Gibbs state relative to some potential (note any finite state measure that never equals zero can be written in the form defined as a Gibbs state). Using (2.2.6.2) and Definition 1.3.4 we see that:

$$\frac{c(x,\zeta_x)}{c(x,\zeta)+c(x,\zeta_x)} = \frac{1}{1+exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)]}$$

which implies

$$\frac{c(x,\zeta)}{c(x,\zeta_x)} = exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)]$$

using the multiplicative property of χ we conclude

$c(x,\zeta)exp[\Sigma_{x\in R}J_R\chi_R(\zeta)] = c(x,\zeta_x)exp[\Sigma_{x\in R}J_R\chi_R(\zeta_x)]$
which implies that our spin system is a Stochastic Ising model (independent of the coordinate ζ_x

Theorem 2.2.8: Suppose that $c(x,\eta)$ are the rates for a Stochastic Ising model relative to the potential $\{J_R\}$. Then G=R where G denotes the set of all Gibbs states relative to the same potential.

Proof: By the Theorem 2.2.6 and Definition 1.3.3 it suffices to show that for a stochastic Ising model, $\frac{c(x,\zeta_x)}{c(x,\zeta)+c(x,\zeta_x)} = \frac{1}{1+exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)]}$ but this is shown just as in the proof of Theorem 2.2.7. Since we have a Stochastic Ising model we have
$c(x,\zeta)exp[\Sigma_{x\in R}J_R\chi_R(\zeta)] = c(x,\zeta_x)exp[\Sigma_{x\in R}J_R\chi_R(\zeta_x)]$
therefore
$\frac{c(x,\zeta)}{c(x,\zeta_x)} = exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)]$
and we conclude $\frac{c(x,\zeta_x)}{c(x,\zeta)+c(x,\zeta_x)} = \frac{1}{1+exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)]}$

Theorem 2.2.9: Suppose our space is Z^2, $J_R = -\beta$ if R=$\{x,y\}$ with $|y-x| = 1$ and $J_R = 0$ otherwise. For sufficiently large positive β, this potential exhibits phase transition.

Proof: For $n \geq 1$ define ν_n as the unique Gibbs state on T with $\zeta \equiv 1$ and $T = [-n,n]^2 \subset Z^2$. It suffices to show that
(2.2.9.1) $lim_{\beta\to\infty}\nu_n(\eta : \eta(0) = B) = 0$
uniformly in n, since phase transition will occur for any β such that
$lim_{n\to\infty}\nu_n(\eta : \eta(0) = B) < \frac{1}{2}$. To visualize the proof it is important to visualize a configuration $\eta \in \{0,1\}^T$ in a certain way. Write + for 1 and - for 0 and agree to draw horizontal and vertical lines of unit legnth between adjacent sites which have opposite signs. An illustration with a particular configuration is given below. Let $B(\eta)$ be the union of all these vertical and horizontal lines. Note that the configuration can be reconstructed from $B(\eta)$ if the boundary is fixed. Also $B(\eta)$ is a disjoint union of contours, where a contour is a closed non self-intersecting polygonal curve. The length of all the contours which

make up $B(\eta)$ will be denoted by $|B(\eta)|$. With this notation we can proceed to prove (2.2.9.1). If $\eta(0) = 0$ then 0 is surrounded by at least one contour γ. Let Γ be the set of contours surrounding 0. Then

(2.2.9.2) $\nu_n(\eta : \eta(0) = 0) = \Sigma_{\gamma \in \Gamma} \nu_n(\eta : \gamma \in B(\eta))$

so we need to estimate $\nu_n(\eta : \gamma \in B(\eta))$ for fixed $\gamma \in \Gamma$. To do so use the Definition of Gibbs state to write

(2.2.9.3) $\nu_n(\eta : \gamma \in B(\eta)) = \frac{\Sigma_{\eta : \gamma \in B(\eta)} exp[-2\beta |B(\eta)|]}{\Sigma_\eta exp[-2\beta |B(\eta)|]}$

if η is such that $\gamma \in B(\eta)$, define $\bar{\eta}$ by $\bar{\eta}(x) = \begin{cases} 1 - \eta(x) & \text{if } \gamma \text{ surrounds x} \\ \eta(x) & \text{otherwise} \end{cases}$

Then $B(\bar{\eta})$ is obtained from $B(\eta)$ by removing γ, so that $|B(\eta)| = |\gamma| + |B(\bar{\eta})|$. Therefore by (2.2.9.3) $\nu_n(\eta : \gamma \in B(\eta)) = exp[-2\beta|\gamma|] \frac{\Sigma_{\eta : \gamma \in B(\eta)} exp[-2\beta |B(\bar{\eta})|]}{\Sigma_\eta exp[-2\beta |B(\eta)|]}$

Since the map $\eta \to \bar{\eta}$ is 1-1, each term in the numerator on the right side also appears in the denomenator. therefore we can conclude that $\nu_n(\eta : \gamma \in B(\eta)) \leq exp[-2\beta|\gamma|]$ using this in (2.2.9.2) gives

(2.2.9.4) $\nu(\eta : \eta(0) = 0) \leq \Sigma_{k=4}^{\infty} e^{-2\beta k} N(k, n)$ where N(k,n) is the number of contours $\gamma \in \Gamma$ of length k.

But $N(k, n) \leq k3^k$ for all n, since each contour $\gamma \in \Gamma$ of length k must cross the positive horizontal axis at least at one of k places, and such a contour can be continued at each point in at most three ways. Thus we obtain the estimate

(2.2.9.5)$\nu_n(\eta : \eta(0) = 0) \leq \Sigma_{k=4}^{\infty} k3^k e^{-2\beta k}$ from which (2.2.9.1) follws by DCT.

Theorem 2.2.10: Consider a stochastic Ising model relative to the potential

$\{J_R\}$, and let G be the corresponding Gibbs states. Then $G \subset \Im$. In particular, if the stochastic Ising model is ergodic, then there is no phase transition for that potential.

Proof: $R \subset \Im$ follows from the defenition and by the previous Theorem R=G, therefore $G \subset \Im$. If the process is ergodic, then $\Im$ is a singleton, therefore G is a singleton as well, so $\{J_R\}$ shows no phase transition.

Theorem 2.2.11: Consider an attractive stochastic Ising model relative to the potential J_R, then $\nu_-, \nu^- \in G$.

Proof: Let S_n defined before increase to Z^d, and let $c_i^n(x,\eta)$ be the rates for the approximating spin system. by checking $c(x,\eta)\nu(\eta) = c(x,\eta_x)\nu(\eta_x)$, we see that $\nu_{S_n,\zeta}$ is invariant for $c_A^n(x,\eta)$ if $\zeta \equiv A$ and for $c_B^n(x,\eta)$ if $\zeta \equiv B$. By the convergence Theorem of finite state Markov chains, ν_-^n and ν^{-n} are equal to $\nu_{S_n,\zeta}$ with $\zeta \equiv$B and $\zeta \equiv A$. Therefore $\nu_-^n, \nu^{-n} \in G(S_n)$ so that $\nu^-, \nu_- \in$ G by the above Theorem.

2.3 Antiferromagnetic model with H$\neq$0

Assuming H$\neq$ 0 some theorems from the previous section hold but several do not. We will go through the Theorems that hold for this case individually and then prove phase transition for this specific case (this was not done in Secion 2). In section 1 the first theorem (Grifith) does not hold, unless $J_{\{x\}} = (-1)^{|x|}$, however the other theorem from section does hold namely.

Theorem 2.3.1: Let K=$\{(\eta, \zeta \in X \times X : \eta \preceq \zeta\}$ Suppose $\eta \preceq \zeta$
$c_1(x, \eta) \leq c_2(x, \zeta)$if $\eta(x) = \zeta(x) = B(x)$ and
$c_1(x, \eta) \geq c_2(x, \zeta)$ if $\eta(x) = \zeta(x) = A(x)$, then $\forall(\eta, \zeta) \in K$ and $t \geq 0$
$P^{(\eta,\zeta)}[(\eta_t, \zeta_t) \in K] = 1$.
Proof: Let E be the set of all functions f in $C(X \times X)$ such that $f \geq 0$ and $f = 0$ on K. For $\lambda \geq 0$ and $f \in A$, define $h \in D(\bar{\Omega})$ by $h - \lambda\bar{\Omega}h = f$. Let $(\eta, \zeta) \in K$ be a point where h achieves its maximum on K. We will show that $\bar{\Omega}h(\eta, \zeta) \leq 0$, so that $h(\eta, \zeta) \leq f(\eta, \zeta) = 0$. Since $h \geq 0$, it will follow that $h \in E$ as well. To show $\bar{\Omega}h(\eta, \zeta) \leq 0$, we will check that each of the terms in the sum defining $\bar{\Omega}h$ is nonpositive. Consider the following three cases:

a)$\eta(x) \neq \zeta(x)$ in which case $(\eta, \zeta) \in K$ implies that $(\eta_x, \zeta) \in K$ and $(\eta, \zeta_x) \in K$, so that $h(\eta_x, \zeta) \leq h(\eta, \zeta)$ and $h(\eta, \zeta_x) \leq h(\eta, \zeta)$.

b)$\eta(x) = \zeta(x) = 0$, in which case $(\eta, \zeta) \in K$ implies that $(\eta_x, \zeta_x) \in K$ and $(\eta, \zeta_x) \in K$, so that $h(\eta_x, \zeta_x) \leq h(\eta, \zeta)$ and $h(\eta, \zeta_x) \leq h(\eta, \zeta)$. However, in this case $(\eta_x, \zeta) \notin K$, so we need to have $c_1(x, \eta) = min(c_1(x, \eta), c_2(x, \zeta))$ which is part of our assumption.

c)$\eta(x) = \zeta(x) = 1$ which is analagous to b), using the second part of the assumption.

Theorem 2.3.2: Suppose that the potential is given with $\beta \geq 0$ and $J(x) \leq 0$. Then $\zeta_1 \preceq \zeta_2$ implies that $\nu_{T,\zeta_1} \preceq \nu_{T,\zeta_2}$ for any finite $T \subset S$.

Proof: By the FKG inequality (1.3.6) which is proven earlier, it suffices to check that for $\eta_1, \eta_2 \in \{0,1\}^T$
$\Sigma_{R\cap T\neq\emptyset} J_R[\chi_R(\eta_1^{\zeta_1} \wedge \eta_2^{\zeta_1}) + \chi_R(\eta_1^{\zeta_2} \wedge \eta_2^{\zeta_2})] \geq \Sigma_{R\cap T\neq\emptyset} J_R[\chi_R(\eta_1^{\zeta_1}) + \chi_R(\eta_2^{\zeta_2})]$ whenever $\zeta_1 \preceq \zeta_2$. Using the special form for J_R which we have assumed and the bijection above, this can be rewritten as the statement that the expression

$$2\beta H \Sigma_{x\in T}(-1)^{|x|}[\tilde{\eta}_1 \wedge \tilde{\eta}_2(x) + (\tilde{\eta}_1 \vee \tilde{\eta}_2)(x) - \tilde{\eta}_1(x) - \tilde{\eta}_2(x)]$$
$$+ 2\beta \Sigma_{x,y\in T x\neq y} J^*(y-x)[(\tilde{\eta}_1 \wedge \tilde{\eta}_2)(x)(\tilde{\eta}_1 \wedge \tilde{\eta}_2)(y) + (\tilde{\eta}_1 \vee \tilde{\eta}_2)(x)(\tilde{\eta}_1 \vee \tilde{\eta}_2)(y)$$
$$- \tilde{\eta}_1(x)\tilde{\eta}_1(y) - \tilde{\eta}_2(x)\tilde{\eta}_2(y)$$
$$+ 4\beta \Sigma_{x\in T, y\notin T} J^*(y-x)[(\tilde{\eta}_1 \wedge \tilde{\eta}_2)(x)\tilde{\zeta}_1(y) + (\tilde{\eta}_1 \vee \tilde{\eta}_2)(x)\tilde{\zeta}_2(y) - \tilde{\eta}_1(x)\tilde{\zeta}_1(y) - \tilde{\eta}_2(x)\tilde{\zeta}_2(y)$$

is nonnegative. The first term is zero eventhough H $\neq 0$ which is very special. This allows us to have our results for this section, in the previous sections at some point we set H=0, hence the first line was trivially 0. The rest are nonnegative by the proof in Chapter 1

Corollary 2.3.1: Under the assumptions of Theorem 2.3.2, $T_1 \subset T_2$ implies that $\nu_{\underline{T_1}} \preceq \nu_{\underline{T_2}}$ and $\nu_{\bar{T}_1} \succeq \nu_{\bar{T}_2}$

Proof: By Theorem 2.3.2 $\nu_{T_1,\zeta} \preceq \nu_{\underline{T_1}}$ $\forall \zeta$. Therefore $\nu_{\bar{T}_1} = \Sigma_{\gamma:\gamma\equiv 1 on T_2/T_1} \nu^{\cdot} \nu_{T_2}$ which clearly implies that $\nu_{\bar{T}_2} \preceq \nu_{\bar{T}_1}$ The opposite statement holds true analogously.

Corollary 2.3.2: Under the assumptions of Theorem 2.3.2 we have

1)$\nu_{\_} = lim_{T\uparrow S} \nu_{\underline{T}}$ exists

2)$\nu^{-} = lim_{T\uparrow S} \nu_{\bar{T}}$ exists

3) $\nu_{\_} \preceq \nu \preceq \nu^{-}$ $\forall \nu \in G$

4) phase transition occurs if and only if $\nu_{\_} \neq \nu^{-}$

5) phase transition occurs if and only if $\nu_{\_}(\eta : \eta(x) = A(x)) \neq \nu^{-}(\eta : \eta(x) =$

$A(x))$

Proof: 1) and 2) exist since Corollary 2.3.1 implies monotonicity.

3) By Theorem we have $\nu_{\mathcal{I}} \preceq \nu_{T,\zeta} \preceq \nu_{\bar{T}}$

for any Gibbs state ν_T is a convex combination of $\nu_{T,\zeta}$ which are less than or equal to $\nu_{\bar{T}}$. Therefore $\nu \preceq \nu_{\bar{T}}$ Likewise we conclude $\nu \succeq \nu_{\mathcal{I}}$

4) This follows from 3) and the definition of phase transition.

5) follows from 4) since $\nu_{\_} \neq \nu^{\text{-}}$

Eventhough H$\neq$ 0 the model is still attractive for

$c(x, \eta) = exp(\Sigma_R J_R \chi_R(\eta))$

we check if $\eta \preceq \zeta$ (or equivalently $\tilde{\eta} \leq \tilde{\zeta}$) implies $c(x, \tilde{\eta}) \leq c(x, \tilde{\zeta})$ if $\tilde{\eta(x)} = \tilde{\zeta(x)} = -(-1)^{|x|}$ and $c(x, \tilde{\eta}) \geq c(x, \tilde{\zeta})$ if $\tilde{\eta(x)} = \tilde{\zeta(x)} = (-1)^{|x|}$.

We look at all cases where $|R|$=2 (for $|R| = 1$ the desired result quickly follows). If $\tilde{\eta(x)} = \tilde{\eta(y)} = \tilde{\zeta(x)} = 1$ then either $\tilde{\zeta(y)} = 1$ in which case we get the same term in both expressions, or $\tilde{\zeta(y)} = -1$ hence the latter term is larger, thus by our initial assumption $\eta \preceq \zeta$ we conclude $c(x, \eta) \leq c(x, \zeta)$. All other cases are identical.

Theorem 2.3.3: Suppose c(x, η) is attractive, and let S(t) be the semigroup for the spin system, then the following hold

1.$\delta_B S(t) \preceq \delta_B S(s)$ for $0 \leq t \leq s$

2.$\delta_A S(t) \preceq \delta_A S(s)$ for $0 \leq t \leq s$

3.$\delta_B S(t) \preceq \mu S(t) \preceq \delta_A S(t)$ for $t \geq 0$ and $\mu \in \wp$

4.$\nu_{\_} = lim_{t \to \infty} \delta_B S(t)$ and $\nu^{\text{-}} = lim_{t \to \infty} \delta_A S(t)$ exist

5. if $\mu \in \wp, t_n \to \infty$ and $\nu = lim_{n \to \infty} \mu S(t_n)$ then $\nu_{\_} \preceq \nu \preceq \nu^{\text{-}}$

6. $\nu_{\_}, \nu^{\text{-}} \in \Im_e$

Proof: 1) By definition, $\delta_B \preceq \delta_B S(t-s)$ for $0 \leq s \leq t$. Therefore using the semigroup property, $\delta_B S(s) \preceq \delta_B S(t-s) S(s) = \delta_B S(t)$

2) Same as 1.

3) Note by monotonicity $\delta_B \preceq \mu \preceq \delta_A \forall \mu \in \wp$ therefore $\delta_B S(t) \preceq \mu S(t) \leq \delta_A S(t)$

4) Follows from 1), 2) and the compactness of $\wp$ in the topology of weak convergence, and the fact that M has the following property:
$\int f d\mu_1 = \int f d\mu_2$
for all $f \in M$ and some $\mu_1, \mu_2 \in \wp$ implies that $\mu_1 = \mu_2$

Theorem 2.3.4: Suppose c(x,η) is attractive, then $c_i^n(x, \eta)$ is attractive for each i and n. If $\mu_A, \mu, \mu_B \in \wp$ satisfy $\mu_B \preceq \mu \preceq \mu_A$, then $\mu_B S(t) \preceq \mu S(t) \preceq \mu_A S(t) \forall t \geq 0$.
Proof: Since $\eta \preceq \zeta$ implies $\eta^i \preceq \zeta^i$ the attractiveness of $c_i^n(x, \eta)$ follows from that of c(x,η). To prove the theorem it suffices to check that $c_B^n \leq c(x, \eta) \leq c_A^n$ if $\eta(x) = B(x)$ and $c_B^n \geq c(x, \eta) \geq c_A^n$ if $\eta(x) = A(x)$. This is true for x$\in$ S, since $\eta^B \preceq \eta \preceq \eta^A$ and for x$\notin$ S since 0$\leq c(x, \eta) \leq M(x)$

Theorem 2.3.6: Suppose that ν is a probability measure on X and that $c(x, \eta)$ are the rates for a spin system. Then ν is reversible for the spin system if and only if

(2.3.6.1) $\int c(x, \eta)[f(\eta_x) - f(\eta)] d\nu = 0$

$\forall x \in S$ and $f \in C(X)$. If the rates are strictly positive then this is equivalent to the statement that ν has the following conditional probabilities:

(2.3.6.2) $\nu(\eta : \eta(x) = \zeta(x) | \eta(u) = \zeta(u)\ \forall u \neq x) = \frac{c(x,\zeta_x)}{c(x,\zeta)+c(x,\zeta_x)}$

Proof: If (2.3.6.1) holds for all f$\in$C(X), then it can be applied to the function $f(\eta_x)g(\eta)$ for f,g$\in$D(X) to obtain
$\int c(x, \eta) f(\eta) g(\eta_x) d\nu = \int c(x, \eta) f(\eta_x) g(\eta) d\nu$

or equivalently

$\int c(x,\eta)f(\eta)[g(\eta_x)-g(\eta)]d\nu = \int c(x,\eta)g(\eta)[f(\eta_x)-f(\eta)]d\nu$

summing on x we get that ν is reversible for the spin system.
To prove the converse, assume that ν is reversible.
For a finite subset T of Z^d and an x∈T, let $f(\eta)=\Pi_{y\in T}\eta(y)$ and $g(\eta)=f(\eta_x)$.
Then
$g(\eta)\Omega f(\eta)=f(\eta_x)\Sigma_{y\in T}c(y,\eta)[f(\eta_x)-f(\eta)]=c(x,\eta)f(\eta_x)$ and
$f(\eta)\Omega g(\eta)=f(\eta)\Sigma_{y\in T}c(y,\eta)[g(\eta_x)-g(\eta)]=c(x,\eta)f(\eta)$ so that (2.3.6.1) holds for that f by Proposition . By linearity, it holds for all f∈D since D is dense in C(X) (2.4.6.1) holds for all f∈C(X). Now assume that $c(x,\eta)>0\ \forall x\in Z^d$ and $\eta\in X$. Fix an $x\in Z^d$ and let $c_A(\eta)$ and $c_B(\eta)$ be the unique functions on X which do not depend on $\eta(x)$ such that

then (2.2.6.2) can be rewritten as the statement that $\int \eta(x)f(\eta)d\nu = \int \frac{c_A(\eta)}{c_A(\eta)+c_B(\eta)}f(\eta)d\nu$ for all $f\in C(X)$ which do not depend on $\eta(x)$. Since $c_A(\eta)+c_B(\eta)$ does not depend on $\eta(x)$ and is strictly positive, this is equivalent to the statement that $\int \eta(x)g(\eta)[c_A(\eta)+c_B(\eta)]d\nu=\int c_A(\eta)g(\eta)d\nu$ for all g∈C(X) which do not depend on $\eta(x)$. But this can be rewitten as $\int g(\eta)(\eta(x)c_A(\eta)-[1-\eta(x)]c_A(\eta))d\nu=0$ or (1.4.6.3) $\int c(x,\eta)g(\eta)[2\eta(x)-1]d\nu=0$

On the other hand, if f∈C(X) is written as $f(\eta)=f_A(\eta)[1-\eta(x)]+f_B(\eta)\eta(x)$ where f_A and f_B do not depend on $\eta(x)$ then $f(\eta_x)-f(\eta)=[f_A(\eta)-f_B(\eta)][2\eta(x)-1]$ so that (2.3.6.1) can be rewritten as $\int c(x,\eta)[f_A(\eta)-f_B(\eta)][2\eta(x)-1]d\nu=0$

Theorem 2.3.7: Suppose that c(x,η) is strictly positive, and that for each x , c(x,η) depends on only finitely many coordinates. If the spin system is reversible with respect to some probability measure ν, then it is a stochastic Ising model relative to some potential $\{J_R\}$. Proof: By Theorem 2.3.6 ν has conditional probabilities given by (2.2.6.2). ν is a Gibbs state relative to some potential (note any finite state measure that never equals zero can be written in the form defined as a Gibbs state). Using (2.3.6.2) and Definition 1.3.4 we see that:

$\frac{c(x,\zeta_x)}{c(x,\zeta)+c(x,\zeta_x)} = \frac{1}{1+exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)]}$

which implies

$\frac{c(x,\zeta)}{c(x,\zeta_x)} = exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)]$

using the multiplicative property of χ we conclude

$c(x,\zeta)exp[\Sigma_{x\in R}J_R\chi_R(\zeta)] = c(x,\zeta_x)exp[\Sigma_{x\in R}J_R\chi_R(\zeta_x)]$

which implies that our spin system is a Stochastic Ising model (independent of the coordinate ζ_x

Theorem 2.3.8: Suppose that $c(x,\eta)$ are the rates for a Stochastic Ising model relative to the potential $\{J_R\}$. Then G=R where G denotes the set of all Gibbs states relative to the same potential.

Proof: By the Theorem 2.3.6 and Definition 1.3.3 it suffices to show that for a Stochastic Ising model, $\frac{c(x,\zeta_x)}{c(x,\zeta)+c(x,\zeta_x)} = \frac{1}{1+exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)]}$ but this is shown just as in the proof of Theorem 2.3.7. Since we have a Stochastic Ising model we have

$c(x,\zeta)exp[\Sigma_{x\in R}J_R\chi_R(\zeta)] = c(x,\zeta_x)exp[\Sigma_{x\in R}J_R\chi_R(\zeta_x)]$

therefore

$\frac{c(x,\zeta)}{c(x,\zeta_x)} = exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)]$

and we conclude $\frac{c(x,\zeta_x)}{c(x,\zeta)+c(x,\zeta_x)} = \frac{1}{1+exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)]}$

Theorem 2.3.10: Consider a stochastic Ising model relative to the potential $\{J_R\}$, and let G be the corresponding Gibbs states. Then $G \subset \Im$. In particular, if the stochastic Ising model is ergodic, then there is no phase transition for that potential.

Proof: $R \subset \Im$ follows from the definition and by the previous Theorem R=G, therefore $G \subset \Im$. If the process is ergodic, then $\Im$ is a singleton, therefore G is a singleton as well, so $\{J_R\}$ shows no phase transition.

Theorem 2.3.11: Consider an attractive stochastic Ising model relative to the potential J_R, then $\nu_{\text{-}}, \nu^{\text{-}} \in G$.

Proof: Let S_n defined before increase to Z^d, and let $c_i^n(x,\eta)$ be the rates for the approximating spin system. by checking $c(x,\eta)\nu(\eta) = c(x,\eta_x)\nu(\eta_x)$, we see that $\nu_{S_n,\zeta}$ is invariant for $c_A^n(x,\eta)$ if $\zeta \equiv A$ and for $c_B^n(x,\eta)$ if $\zeta \equiv B$. By the convergence Theorem of finite state Markov chains, ν_-^n and ν^{-n} are equal to $\nu_{S_n,\zeta}$ with $\zeta \equiv$B and $\zeta \equiv A$. Therefore $\nu_-^n, \nu^{-n} \in G(S_n)$ so that $\nu^-, \nu_- \in$ G by the above Theorem.

Theorem 2.3.12: With J_R as given in section 2 the Stochastic Ising model has two Gibbs states for $d > |\tilde{H}|, \beta > (1 - \tilde{H}/d)^{-1}c_d$ where $\tilde{H} = \frac{H+8}{4} - d$; d= dimension

Proof: We have $\nu(\eta) = \frac{1}{Z}exp(\Sigma_R J_R \chi_R(\eta))$

We can rewrite this as

$\nu(\eta) = \frac{1}{Z}exp(\beta[(H+8)\Sigma_{i=1}^{|V|}\eta(i) - 2\Sigma_{i\in V|i-j|=1}\Sigma_{j\in V}\eta(i)\eta(j) - 4\Sigma_{i\in V|i-t|=1}\Sigma_{t\notin V}\eta(i)\eta(t)])$, $V \subset Z^d$

Now let us examine a configuration, say η, on Z^d. If $\eta(i) = 0$ we draw a unit square around i and shade it black, and if $\eta(i) = 1$ we draw a unit square around i and shade it white. Within our set V $\cup\partial$V, we will shade any two edges that have the same color on both sides. The sum of all the sides for a given configuration is defined as $\Gamma(\eta(1), \eta(2), ...\eta(|V|)/\eta(t))$ We will also shade any two edges that have different colors on both sides, the sum of these sides will be called $\tilde{\Gamma}(\eta(1), \eta(2), ...\eta(|V|)/\eta(t))$. Notice that $\Gamma + \tilde{\Gamma}$ =constant since either 2 neighboring sites are the same or different, thus the sum is equal to the total number of sides. If H+8=2d we conclude

$$\nu(\eta) = \frac{exp(\beta\Gamma(\eta(1),\eta(2),...\eta(|V|)/\eta(t)))}{\Sigma_{\eta(1)\in\{0,1\}...\eta(|V|)\in\{0,1\}}exp(\beta\Gamma(\eta(1),\eta(2),...\eta(|V|)/\eta(t)))}$$

this follows by combining the first and second terms above and noticing the last term is equal to $4\Sigma_{i\in V,|i-t|=1}\Sigma_{t\notin V}\eta(i)(1-\eta(t)))$, $V \subset Z^d$ hence we also conclude

$$\nu(\eta) = \frac{exp(\beta(\bar{H}\Sigma_{i=1}^{|V|}\eta(i)+2\bar{\Gamma}(\eta(1),\eta(2),...\eta(|V|)/\eta(t)))}{\Sigma_{\eta(1)\in\{0,1\}...\eta(|V|)\in\{0,1\}}exp(\beta\bar{\Gamma}(\eta(1),\eta(2),...\eta(|V|)/\eta(t)))}$$

by using the formula above and using the fact that $\Gamma + \bar{\Gamma} = c$ Next we introduce a transformation assuming u(i) is the site directly below i

$$T_G\ \tilde{\eta(i)} = \begin{cases} \eta(i) & \text{if i is outside G} \\ \eta(u(i)) & \text{if i and u(i) are inside G} \\ 1-\eta(u(i)) & \text{if i lies inside G and u(i) lies outside G} \end{cases}$$

This T_G annihilates the contour G, does not change anything outside G, and raises all the contours lying inside G by one. Therefore we conclude

$\bar{\Gamma}(\tilde{\eta(1)}, \tilde{\eta(2)}, ...\eta(|\tilde{V}|)/\eta(t)) = \bar{\Gamma}(\eta(1), \eta(2), ...\eta(|V|) - |G|\ |\Sigma_{i=1}^{|V|}\tilde{\eta(i)} - \Sigma_{i=1}^{|V|}\eta(i)| <$ $\frac{|G|_{hor}}{2}$ where $|G|_{hor}$ is the number of horizontal sides of the contour G, clearly the same holds for $|G|_{vert}$ so we conclude that

$$Pr(G) = \frac{\Sigma_{(\eta(1),\eta(2),....\eta(|V|))\in B(G)}exp(\beta(\bar{H}\Sigma_{i=1}^{|V|}\eta(i)+2\bar{\Gamma}(\eta(1),\eta(2),...\eta(|V|)/\eta(t))}{\Sigma_{(\eta(1),\eta(2),....\eta(|V|))\in\{0,1\}^{|V|}}exp(\beta(\bar{H}\Sigma_{i=1}^{|V|}\eta(i)+2\bar{\Gamma}(\eta(1),\eta(2),...\eta(|V|)/\eta(t))}$$

$$\leq \frac{\Sigma_{(\eta(1),\eta(2),....\eta(|V|))\in B(G)}exp(\beta(\bar{H}\Sigma_{i=1}^{|V|}\eta(i)+2\bar{\Gamma}(\eta(1),\eta(2),...\eta(|V|)/\eta(t))}{\Sigma_{(\eta(1),\eta(2),....\eta(|V|))\in T_{B(G)}}exp(\beta(\bar{H}\Sigma_{i=1}^{|V|}\eta(i)+2\bar{\Gamma}(\eta(1),\eta(2),...\eta(|V|)/\eta(t))}$$

$$\leq exp(-\beta|G|/2 - \beta|H + 8||G|_{hor}/8)$$

then we conclude that

$$Pr(G) \leq exp(-\beta|G|/2 - \beta|H + 8||G|/8)$$

Assume we are in Z^2 if a point $x = (x_1, x_2) \in Z^2$ lies inside the contour G and this contour contains a side of a square with center at $x^0 = (x_1^0, x_2^0)$ inside G then the length of the contour G must be no less than $2(|x_1^0 - x_1| + |x_2^0 - x_2 + 2|)$. In addition there are at most 3^{m-1} possible paths for G, of length m through a given side. Let $\pi(x)$ be the probability that x lies inside at least one contour created by the configuration $(\eta(1), \eta(2), ..., \eta(|V|))$. Summing over all G that contain x and pass along a side of the square x^0, these conditions will occur r times for a contour of length r, which is always a multiple of 2 and each square has four sides we conclude:

$$\pi(x) \leq \Sigma_{x \in V} \Sigma_{k \geq \frac{|x_1^0 - x_1| + |x_2^0 - x_2 + 2|}{2}} \frac{1}{3 \times 2m} exp(\frac{-\beta m}{2} - \frac{\beta m |H+8|}{8} + 2m ln3)$$
$$= \frac{2}{3} \Sigma_{m=1}^{\infty} m^{-1}(m^2 + (m-1)^2) exp(\frac{-\beta m}{2} - \frac{\beta m |H+8|}{8} + 2m ln3)$$
$$\leq \frac{4}{3} \frac{exp(\frac{-\beta}{2} - \frac{\beta |H+8|}{8} + \beta 2 ln3)}{(1 - exp(\frac{-\beta}{2} - \frac{\beta |H+8|}{8} + \beta 2 ln3))^2}$$

Assume $c_d > 0$ is the solution to $\frac{4}{3} \frac{exp(c_d + \beta 2 ln3)}{(1 - exp(c_d + \beta 2 ln3))^2} = \frac{1}{2}$

Thus $\pi(x) \leq \gamma < \frac{1}{2}$ for all x$\in Z^d$.

Next assume $\eta(t) \equiv A(t)$

$$Pr(\eta(i) = B(t)) \leq \gamma < \frac{1}{2} \; \forall i \in Z^d$$

Analogously using $\eta(t) \equiv B(t)$ we conclude a Gibbs state exists such that

$$Pr(\bar{\eta}(i) = B(t)) \geq 1 - \gamma > \frac{1}{2} \; \forall i \in Z^d.$$

Thus we have two Gibbs states.

Theorem 2.3.13: Given a stochastic Ising model with J_R as before we have the following:

1. The model is ergodic if and only if there is no phase transition.
2. If $d > |\tilde{H}|, \beta > (1 - \tilde{H}/d)^{-1} c_d$ where $\tilde{H} = \frac{H+8}{4} - d$ then there is phase transition, which implies the model is not ergodic
3. If $0 < \beta <<$ then there is no phase transition hence the model is ergodic.

Proof: To prove 1) we first use Theorem 2.3.10 to show that if the model is ergodic then there is no phase transition. To go the other direction we use Corollary 2.2.2 part 4 to prove there is only 1 Gibbs state, hence it is ergodic.

2) is proved using 1) and Theorem 2.3.12

3) Suppose we have a family of probability measures on $\{0,1\}$, say $\{\mu_j^s : j \in Z^d, s \in X\}$. Let $||\mu_j^s - \mu_j^t||$ be the total variation between the two configurations s and t.

$\rho_{i,j} := \frac{1}{2} sup_{s,t}$ s=t except at i$||\mu_j^s - \mu_j^t||$

Dobrushkins Uniqueness Theorem states suppose $sup_j(\Sigma_{i \neq j} \rho_{i,j}) < 1$, then there exists at most one Gibbs state. Since we already know at least one Gibbs state exists we conclude that it is unique. As before we rewrite our Hamiltonian as $-\mu\Sigma_i x_i + \frac{1}{2}\Sigma_{i \in V, i \neq j}\Sigma_{j \in V} U(t_i - t_j) + \Sigma_{t \notin V}\Sigma_{i \in V} U(t_i - t)$ using this form we can express $\mu_j^s(\sigma) = \frac{1}{Z^s} exp(-\beta(\Sigma_{k \neq j} \sigma s_k U(j-k) - \mu\sigma) =$
$\frac{1}{Z^s} - \beta\Sigma_{k \neq j, k \neq i} \sigma s_k U(j-k) - \beta\mu\sigma$
where $\sigma = 0$ or 1.
Analogously $\mu_j^t = \frac{1}{Z^t} exp(-\beta(\Sigma_{k \neq j} \sigma t_k U(j-k) - \mu\sigma) =$
$\frac{1}{Z^t} - \beta\Sigma_{k \neq j, k \neq i} \sigma s_k U(j-k) - \beta\mu\sigma - \beta\sigma U(i-j)$

Now we can assume $s_i = 0$ and $t_i = 1$.
(2.3.13.1) $sup_s\{||\mu_j^t - \mu_j^s||\} =$
$sup_{s,t}$s=t except at i$\{|\frac{1}{Z^s} - \frac{1}{Z^t}| + |(\frac{1}{Z^s} - \frac{exp(-\beta U(i-j))}{Z^t} exp(-\beta\Sigma_{j \neq k, k \neq i} s_k U(j-k) - \beta\mu)|$

Finding the sup of the above is equivalent to finding the $inf_s \beta[\Sigma_{k/neqj, k \neq i} s_k U(j-k) + \mu] \geq$
$\beta inf_s[-\Sigma_{k \neq j, k \neq i} s_k |U(j-k)| - |\mu|] \geq$
$-\beta\Sigma_{k \neq j, k \neq i} |U(j-k)| + |\mu|$
Hence (2.3.13.1) $\leq sup_s|\frac{1}{Z^s} - \frac{1}{Z^t}| + sup_s|(\frac{1}{Z^s} - \frac{exp(-\beta U(i-j))}{Z^t}| exp(\beta[\Sigma_{k \neq j, k \neq i} |U(j-k)| + |\mu|)$

$\rho_{i,j} = \frac{1}{2} sup_{s,t}$ s=t except at i$|\mu_j^s - \mu_j^t| \leq \frac{1}{2} sup_{s,t}$ s=t except at i$|exp(-\beta\sigma U(i-j))|_\infty \leq$

$\frac{\beta}{2}|U(i-j)|$

Hence $\Sigma_{i \neq j} \rho_{i,j} = \Sigma_{i \neq j} \frac{\beta}{2} |U(i-j)| = \Sigma_{i \neq 0} U(i) \frac{\beta}{2}$

which is less than ϵ if $\beta < \frac{2}{\Sigma_i U(i)}$. Hence by Lemma V.I.4 in Simon[13] we conclude that we have a unique Gibbs state. This solves the anti-ferromagnetic Ising model.

CHAPTER 3

Hard Core Stochastic Ising Model

We will begin by defining a flip rate $c(x,\eta)$ for now we will use

$$c(x,\eta) = \begin{cases} 0 & \text{if } \eta(x) = 0 \text{ and } \eta(y) = 1 \text{ where } |y-x| = 1 \\ exp(\Sigma_R J_R \chi_R(\eta)) & \text{otherwise} \end{cases}$$

Notice that $c(x,\eta)$ is defined on all of $X = \{0,1\}^{Z^d}$, let $c_T(x,\eta)$ denote the flip rate on T$\subset Z^d$. From the flip rate we create a Markov pregenerator as we did in Chapter 1. Let A= $\{(\eta(i) : i \in Z^d : \eta(i)\eta(j) = 0 \text{ if } |i-j| = 1)\}$. Let's show A is compact, since A is a subset of χ we need only show A is closed. Assume A is not closed, say $\exists\eta$ such that $\eta_n \in A$ and $\eta_n \to \eta$ and $\eta \notin A$. This implies that there are two points x and y with $|y-x| = 1$ with $\eta(x) = 1$ and $\eta(y) = 1$. Since $\eta_n \to \eta$ then $\eta_n(x)$ and $\eta_n(y)$ must equal 1 for all N $\geq N_0$, but then $\eta_n \notin A$ which is a contradiction. Hence A is closed and compact.

For the hard core stochastic Ising model we have the potential below

$$J_R = \begin{cases} \beta H & \text{if } |R| = 1 \\ \beta J(y-x) & \text{if } |R| = 2 \; x,y \in R \\ 0 & \text{if } |R| > 2 \end{cases}$$

$J(y-x) < 0$ if $|y-x|$ is an odd boxcar distance from the origin, and $J(y-x) > 0$ if $|y-x|$ is an even boxcar distance from the origin. We will simply investigate the nearest neighbor model where J(y-x)=0 for $|y-x| \geq 2$, J(y-x)=-∞ if $\eta(x) = \eta(y) = 1$, and J(y-x)=-1 otherwise.

Proposition 3.1.1: Assume that $sup_{x \in Z^d} \Sigma_{x \in T} c_T(x, \eta) < \infty$

1) For $f \in D(A)$, the series $\Omega f(\eta) = \Sigma_T \int_{\{0,1\}^T} c_T(x,\eta)[f(\eta^\zeta) - f(\eta)]$ converges uniformly and defines a function in C(A), and

$\|\Omega f\| \leq (sup_{x \in Z^d} \Sigma_{x \in T} c_T(x,\eta))\ |||f|||$, where $|||f||| = \Sigma_{x \in Z^d} \|\Delta_f(x)\|$

2) Ω is a Markov pregenerator.

Proof: $\int_{\{0,1\}^T} c_T(x,\eta)[f(\eta^\zeta) - f(\eta)]$

is in C(A) for each T and each f$\in C(A)$. By regarding η^ζ as the result of changing the coordinates of η corresponding to sites in T one at a time, it is clear that

$|f(\eta^\zeta) - f(\eta)| \leq \Sigma_{x \in T} \Delta_f(x)$, $\Delta_f(x) = sup_\eta |f(\eta) - f(\eta_x)|$

therefore

$$\| \int c_T(x,\eta)[f(\eta^\zeta) - f(\eta)] \| \leq c_T \Sigma_{x \in T} \|\Delta_f(x)\|$$

$$\Sigma_T \| \int c_T(x,\eta)[f(\eta^\zeta) - f(\eta)]\| \leq (sup_x \Sigma_{x \in T} c_T)\ |||f|||$$

for any $f \in D(A)$. Hence the series defining Ωf converges uniformly. Since the summands are continuous, it follows that $\Omega f \in C(A)$.

To prove 2) we must simply show property 3) of Definition 1.1.6. Suppose $f \in D(A)$ and $f(\eta) = min(f(\zeta) : \zeta \in A)$. Then $f(\zeta) \geq f(\eta)$ for all $\zeta \in A$, so $\Omega f(\eta) \geq 0$.

Thus simply giving a transition function, $c(x,\eta)$ defines the Markov pregenerator it is necessary to show that R(I-$\lambda\Omega$) is dense in C(A) for all sufficiently small $\lambda > 0$. To prove this we approximate Ω by a sequence of bounded pregenerators Ω^n, since bounded pregenerators are generators we conclude

$$R(I - \lambda\Omega^n) = C(A)$$

for each n and each $\lambda \geq 0$. Therefore given a g$\in D(A)$, there are $f_n \in C(A)$ so that $f_n - \lambda\Omega^n f_n = g$. Thus if $g_n = f_n - \lambda\Omega f_n$ and it will follow that

$$||g_n - g|| = \lambda||(\Omega - \Omega^n)f_n|| \to 0$$

$R(I-\lambda\Omega)$ is dense is a consequence of the fact that $g_n \in R(I-\lambda\Omega)$ for each n, and that D(A) is dense. Therefore, Ω is a Markov generator, which is uniquely associated with a Markov semigroup, which is then associated with a Markov process.

By Theorem 4.3 a modified Hille-Yosida Theorem in Chapter 1 from Ethier and Kurtz our generator relates to a semigroup on $\overline{D(A)}$

Definition 3.2: Given a potential J_R, and $\eta, \eta_x \in A$ a Markov process with non-negative rates, $c(x,\eta)$, is called a stochastic Ising model relative to the potential if c(x,η) exp[$\Sigma_{x\in R} J_R \chi_R(\eta)$] does not depend on the coordinate η(x), where $\chi_R(\eta) = \prod_{x\in R}[2\eta(x) - 1]$

Definition 3.2.1: A Gibbs state with respect to a given potential on $T \subset Z^d$ $|T| < \infty$ is given by

$$(3.2.1)\ \nu(\eta) = \begin{cases} 0 & \text{if } \eta(x)=1 \text{ and } \eta(y)=1 \text{ where } |y-x|=1 \\ \frac{1}{Z}exp(\Sigma_{R\subset T} J_R \chi_R(\eta)) & \text{otherwise} \end{cases}$$

Definition 3.2.2: For general T we define a Gibbs state as a measure, ν with $\nu(A^c)=0$ where

(3.2.2) $\nu(\eta : \eta(x) = \zeta(x) | \eta(u) = \zeta(u)\ \forall u \neq x)$

$$= \begin{cases} 1 & \text{if } \zeta \in A, \zeta_x \notin A \\ 0 & \text{if } \zeta \notin A, \zeta_x \in A \\ \frac{1}{1+exp(-2\Sigma_{x\in R} J_R \chi_R(\zeta))} & \text{if } \zeta \in A, \zeta_x \in A \end{cases}$$

Theorem 3.1: Suppose $T \in Z^d$ is finite then 3.2.1 is equivalent to 3.2.2.

Proof: Suppose 3.2.1, then for $\zeta \notin A$ both 3.2.1, and 3.2.2 equal 0, thus we simply look to prove the theorem for $\zeta \in A$ $\nu(\eta : \eta(x) = \zeta(x) | \eta(u) = \zeta(u) \forall u \neq x) =$

$\frac{\nu(\zeta)}{\nu(\zeta)+\nu(\zeta_x)} =$

$\frac{exp[\Sigma_R J_R \chi_R(\zeta)]}{exp[\Sigma_R J_R \chi_R(\zeta)]+exp[\Sigma_R J_R \chi_R(\zeta_x)]} =$

$\frac{1}{1+exp[-2\Sigma_{x\in R} J_R \chi_R(\zeta)]}$

since $\chi_R(\eta_x) = -\chi_R(\eta)$ if $x \in R$ and $\chi_R(\eta)$ if $x \notin R$

For the converse suppose ν is a Gibbs state as in 3.2.2. Then ν satisfies 3.2.1 for possibly some other potential, say J_R^*, since it is non-zero in A and T is finite, but then we must have $\Sigma_{x\in R} J_R \chi_R(\eta) = \Sigma_{x\in R} J_R^* \chi_R(\eta)$ for all x. Since χ_R are linearly independant we conclude $J_R = J_R^*\ \forall R \neq \emptyset$. Changing $J_\emptyset$ is just changing the normilization constant.

Before we prove the next Theorem we must prove the FKG inequality for our Gibbs measure in this Chapter. Consider

$$\nu(\eta) = \frac{1}{Z}exp(\beta\Sigma_{x,y\ |x-y|=1} J_R^* \eta(x)\eta(y) + H\beta\Sigma_i \eta(i) + \beta\Sigma_{x,y\ |x-y|=1} J_R \chi_R(\eta))$$

where J_R^* is any negative value for $|y-x| = 1$ and 0 if $|R| > 2$. J_R is a fixed potential of an anti-ferromagnetic Ising model from Chapter 2. Writing $\chi_R(\eta)$ as $(2\eta(x)-1)(2\eta(y)-1)$ we conclude

$$\nu(\eta) = \frac{1}{Z}exp(\beta\Sigma_{x,y\ |x-y|=1}J_R^*\eta(x)\eta(y) + H\beta\Sigma_i\eta(i) + \beta\Sigma_{x,y\ |x-y|=1}J_R(2\eta(x)-1)(2\eta(y)-1))$$

$$\nu(\eta) = \frac{1}{Z}exp(\beta\Sigma_{x,y\ |x-y|=1}J_R^*\eta(x)\eta(y) + (H\beta - 4\beta J)\Sigma_i\eta(i) + 4\beta\Sigma_{x,y\ |x-y|=1}J_R\eta(x)\eta(y))$$

This ν satisfies the FKG inequality for any fixed $J_R^* \leq 0$, we will let $J_R^* \to -\infty$ this limiting measure is the Gibbs state introduced in this Chapter, thus this measure satisfies the FKG inequality. According to Chapter 2 we have phase transition if $\beta > (1 - \frac{\tilde{H}}{d})^{-1}c_d$ and $d > |\tilde{H}|$. With $J_R = -1$ we can rewrite the Hamiltonian as

$$\nu(\eta) = \frac{1}{Z}exp(-4\beta(J_R^* - 1)\Sigma_{x,y\ |x-y|=1}(-1)\eta(x)\eta(y) + (H\beta + 4\beta)\Sigma_i\eta(i)$$

For the model to have phase transition we need to solve:

1) $d > |\frac{H+12}{4} - d| \Leftrightarrow -12 < H < 8d - 12$
2) $\beta > (1 - \frac{\frac{H+12}{4} - d}{d})^{-1}c_d \Leftrightarrow \beta > (\frac{4d}{8d - H - 12})c_d$

Thus we have exact conditions for phase transition of the hard-core Ising model once we prove a few preliminary results.

We will define $\nu_{T,\zeta}$ analogous to the previous Chapters, we set $\nu_{T,\zeta} = 0$ if $\zeta \notin A$. For f$\in C(A)$, define $\nu_{T,\zeta}(f) = \Sigma_\eta f(\eta)\nu_{T,\zeta}(\eta)$ for T fixed this is a function of ζ, with $\nu_{T,\zeta}(f) = 0$ if $\zeta \notin A$. We will say that ν is a Gibbs measure if

1) $\nu(f|\Sigma_{T^c})(\zeta) = \nu_{T,\zeta}(f)$, where Σ_{T^c} is the sigma algebra generated by all configurations outside of T

2) $\nu(A^c) = 0$

Theorem 3.2: Suppose that the potential above is given with $\beta \geq 0$ and $J(x) \leq 0$. Then $\zeta_1 \preceq \zeta_2$ implies that $\nu_{T,\zeta_1} \preceq \nu_{T,\zeta_2}$ for any finite $T \subset Z^d$ $\zeta_1, \zeta_2 \in A$.

Proof: $\nu_{T,\zeta}$ satisfies the FKG inequality by the argument after Theorem 3.1.

$$\nu_{T,\zeta}(\eta) = \frac{1}{Z}exp(\beta\Sigma_{x,y\ |x-y|=1}J_R^*\eta^\zeta(x)\eta^\zeta(y)+H\beta\Sigma_i\eta^\zeta(i)+\beta\Sigma_{x,y\ |x-y|=1}J_R\chi_R(\eta^\zeta))$$

where J_R^* is any negative value for $|y-x|=1$ and 0 if $|R|>2$. J_R is a fixed potential of an anti-ferromagnetic Ising model from Chapter 2. Writing $\chi_R(\eta^\zeta)$ as $(2\eta^\zeta(x)-1)(2\eta^\zeta(y)-1)$ we conclude

$$\nu_{T,\zeta}(\eta) = \frac{1}{Z}exp(\beta\Sigma_{x,y\ |x-y|=1}J_R^*\eta^\zeta(x)\eta^\zeta(y)+ H\beta\Sigma_i\eta^\zeta(i)+\beta\Sigma_{x,y\ |x-y|=1}J_R(2\eta^\zeta(x)-1)(2\eta^\zeta(y)-1))$$

$$\nu_{T,\zeta}(\eta) = \frac{1}{Z}exp(\beta\Sigma_{x,y\ |x-y|=1}J_R^*\eta^\zeta(x)\eta^\zeta(y)+ (H\beta-4\beta J)\Sigma_i\eta^\zeta(i)+4\beta\Sigma_{x,y\ |x-y|=1}J_R\eta^\zeta(x)\eta^\zeta(y)$$

This ν satisfies the FKG inequality for any fixed $J_R^* \leq 0$, we will let $J_R^* \to -\infty$ this limiting measure is the Gibbs state introduced above, thus this measure satisfies the FKG inequality.

Note for configurations $\zeta \notin A$ $\nu_{T,\zeta} = 0$. Let A_T represent configurations in A restricted to T. By the FKG inequality, it suffices to check that for $\eta_1, \eta_2 \in A_T$

$\Sigma_{R\cap T\neq\emptyset}J_R[\chi_R(\eta_1^{\zeta_1}\wedge\eta_2^{\zeta_1})+\chi_R(\eta_1^{\zeta_2}\wedge\eta_2^{\zeta_2})] \geq \Sigma_{R\cap T\neq\emptyset}J_R[\chi_R(\eta_1^{\zeta_1})+\chi_R(\eta_2^{\zeta_2})]$ whenever $\zeta_1 \preceq \zeta_2$

Using the special form for J_R which we have assumed, this can be rewritten as the statement that the expression

$$2\beta H\Sigma_{x\in T}[(\eta_1\wedge\eta_2)(x)+(\eta_1\vee\eta_2)(x)-\eta_1(x)-\eta_2(x)]$$
$$+2\beta\Sigma_{x,y\in T;x\neq y}J(y-x)[(\eta_1\wedge\eta_2)(x)(\eta_1\wedge\eta_2)(y)+(\eta_1\vee\eta_2)(x)(\eta_1\vee\eta_2)(y)$$
$$-\eta_1(x)\eta_1(y)-\eta_2(x)\eta_2(y)$$
$$+4\beta\Sigma_{x\in T,y\notin T}J(y-x)[(\eta_1\wedge\eta_2)(x)\zeta_1(y)+(\eta_1\vee\eta_2)(x)\zeta_2(y)-\eta_1(x)\zeta_1(y)-\eta_2(x)\zeta_2(y)$$

is nonnegative. The terms in the first sum are all zero so the sign of the H is irrelevant in verifying the non-negativity of this expression. The terms in brackets in the second sum is zero unless $\eta_1(x)=\eta_2(y)=0$ and $\eta_2(x)=\eta_1(y)=1$ or $\eta_1(x)=\eta_2(y)=1$ and $\eta_2(x)=\eta_1(y)=0$ in which case it is equal to 1. The term in brackets in the third sum is zero unless $\eta_1(x)=1$ and $\eta_2(x)=0$, in which case it is equal to $\zeta_2(y)-\zeta_1(y)$. So, since $\beta\geq 0$ and $J(y-x)\geq 0$ the required sums are nonnegative whenever $\zeta_1\leq\zeta_2$.

Let $\nu_{\underline{T}}$ denote the Gibbs state on T taking the value $\zeta\equiv B(x)$ outside T, and $\nu_{\bar{T}}$ is a Gibbs state on T taking the value $\zeta\equiv A(x)$ outside T

Corollary 3.2.1: Under the assumptions of Theorem 3.2, $T_1\subset T_2$ implies that $\nu_{\underline{T}_1}\preceq\nu_{\underline{T}_2}$ and $\nu_{\bar{T}_1}\succeq\nu_{\bar{T}_2}$

Proof: By Theorem 3.2 $\nu_{T_1,\zeta}\preceq\nu_{\underline{T}_1}\ \forall\zeta$. Therefore $\nu_{\bar{T}_1}=\Sigma_{\gamma:\gamma\equiv A(x)\ on T_2/T_1}\nu^{\text{-}}\nu_{T_2}$ which clearly implies that $\nu_{\bar{T}_2}\leq\nu_{\bar{T}_1}$ The opposite statement holds true analogously.

Corollary 3.2.2: Under the assumptions of Theorem 3.2 we have

1)$\nu_{\_}=lim_{T\uparrow S}\nu_{\underline{T}}$ exists

2)$\nu^{\text{-}}=lim_{T\uparrow S}\nu_{\bar{T}}$ exists

3) $\nu_{\_}\preceq\nu\preceq\nu^{\text{-}}\ \forall\nu\in G$

4) phase transition occurs if and only if $\nu_{\_}\neq\nu^{\text{-}}$

5) phase transition occurs if and only if $\nu_{-}(\eta : \eta(x) = A(x)) \neq \nu^{-}(\eta : \eta(x) = A(x))$

Proof: 1) and 2) exist since Corollary 3.2.1 implies monotonicity.
3) By Theorem 3.2 we have $\nu_{\mathcal{I}} \preceq \nu_{T,\zeta} \preceq \nu_{\bar{T}}$
for any Gibbs state ν_T is a convex combination of $\nu_{T,\zeta}$ which are less thann or equal to $\nu_{\bar{T}}$. Therefore $\nu \preceq \nu_{\bar{T}}$ Likewise we conclude $\nu \succeq \nu_{\mathcal{I}}$

4) This follows from 3) and the definition of phase transition.

5) follows from 4) since $\nu_{-} \neq \nu^{-}$

Theorem 3.3: Suppose that ν is a probability measure on X with $\eta, \eta_x \in A$ and that $c(x,\eta)$ are the rates for a Markov process with state space A. Then ν is reversible for the spin system if and only if
(3.2.1) $\int c(x,\eta)[f(\eta_x) - f(\eta)]d\nu = 0$
$\forall x \in Z^d$ and $f \in C(A)$.
(3.2.2) $\nu(\eta : \eta(x) = \zeta(x) | \eta(u) = \zeta(u)\ \forall u \neq x) = \frac{c(x,\zeta_x)}{c(x,\zeta)+c(x,\zeta_x)}$

Proof: If (3.2.1) holds for all f∈C(A), then it can be applied to the function $f(\eta_x)g(\eta)$ for f,g∈D(A) to obtain

$$\int c(x,\eta)f(\eta)g(\eta_x)d\nu = \int c(x,\eta)f(\eta_x)g(\eta)d\nu$$

or equivalently

$$\int c(x,\eta)f(\eta)[g(\eta_x) - g(\eta)]d\nu = \int c(x,\eta)g(\eta)[f(\eta_x) - f(\eta)]d\nu$$

summing on x we get that ν is reversible for the spin system.

To prove the converse, assume that ν is reversible.

For a finite subset T of Z^d and an x∈T, let $f(\eta) = \Pi_{y \in T}\eta(y)$ and $g(\eta) = f(\eta_x)$. Then

$$g(\eta)\Omega f(\eta) = f(\eta_x)\Sigma_{y \in T}c(y,\eta)[f(\eta_x) - f(\eta)] = c(x,\eta)f(\eta_x) \text{ and}$$

$$f(\eta)\Omega g(\eta) = f(\eta)\Sigma_{y \in T}c(y,\eta)[g(\eta_x) - g(\eta)] = c(x,\eta)f(\eta)$$

so that (3.2.1) holds for that f. By linearity, it holds for all f∈D(A) since D(A) is dense in C(A) (3.2.1) holds for all f∈C(A).

Now assume that $c(x,\eta) > 0$ $\forall x \in Z^d$ and $\eta, \eta_x \in A$. Fix an $x \in Z^d$ and let $c_D(\eta)$ and $c_B(\eta)$ be the unique functions on X which do not depend on $\eta(x)$ such that then (3.2.2) can be rewritten as the statement that

$\int \eta(x)f(\eta)d\nu = \int \frac{c_D(\eta)}{c_D(\eta)+c_B(\eta)}f(\eta)d\nu$ for all $f \in C(A)$ which do not depend on $\eta(x)$.

Since $c_D(\eta) + c_B(\eta)$ does not depend on $\eta(x)$ and is strictly positive, this is equivalent to the statement that

$\int \eta(x)g(\eta)[c_D(\eta) + c_B(\eta)]d\nu = \int c_D(\eta)g(\eta)d\nu$ for all g∈C(A) which do not depend on $\eta(x)$.

But this can be rewitten as

$\int g(\eta)(\eta(x)c_D(\eta) - [1 - \eta(x)]c_D(\eta))d\nu = 0$ or

(3.2.3) $\int c(x,\eta)g(\eta)[2\eta(x) - 1]d\nu = 0$

On the other hand, if f∈C(A) is written as

$f(\eta) = f_D(\eta)[1-\eta(x)] + f_B(\eta)\eta(x)$ where f_D and f_B do not depend on $\eta(x)$ then

$f(\eta_x) - f(\eta) = [f_D(\eta) - f_B(\eta)][2\eta(x) - 1]$ so that (3.2.1) can be rewritten as

$\int c(x,\eta)[f_D(\eta) - f_B(\eta)][2\eta(x) - 1]d\nu = 0$

Theorem 3.4: Suppose that c(x,η) is non-negative, given η c(x,η) depends on x. If the Markov process is reversible with respect to some probability measure ν, then it is a stochastic Ising model relative to some potential $\{J_R\}$.

Proof: By Theorem 3.3 ν has conditional probabilities given by (3.3.2), ν is a Gibbs state relative to some potential (note any finite state measure that never equals zero on A can be written in the form defined as a Gibbs state). Using (3.3.2)

$$\frac{c(x,\zeta_x)}{c(x,\zeta)+c(x,\zeta_x)} = \begin{cases} \frac{1}{1+exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)]} & \text{if } \zeta \in A\ \zeta_x \in A \\ 0 & \text{if } \zeta \in A\ \zeta_x \notin A \end{cases}$$

which implies

$\frac{c(x,\zeta)}{c(x,\zeta_x)} = exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)]$ for $\zeta, \zeta_x \in A$

using the multiplicative property of χ we conclude

$c(x,\zeta)exp[\Sigma_{x\in R}J_R\chi_R(\zeta)] = c(x,\zeta_x)exp[\Sigma_{x\in R}J_R\chi_R(\zeta_x)]$

which implies that our Markov process is a stochastic Ising model (independent of the coordinate ζ_x)

Theorem 3.5: Suppose that $c(x,\eta)$ are the rates for a stochastic Ising model relative to the potential $\{J_R\}$. Then $G = R^*$ where G denotes the set of all Gibbs states relative to the same potential, and R^* is an extension of reversible measures on A, taking the value of 0 outside A.

Proof: By the Theorem 3.3 it suffices to show that for a stochastic Ising model,

$$\frac{c(x,\zeta_x)}{c(x,\zeta)+c(x,\zeta_x)} = \frac{1}{1+exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)]} \text{ for } \zeta,\zeta_x \in A$$

since both are equal on configurations outside of A, but this is shown just as in the proof of Theorem 3.3. Since we have a stochastic Ising model we have

$$c(x,\zeta)exp[\Sigma_{x\in R}J_R\chi_R(\zeta)] = c(x,\zeta_x)exp[\Sigma_{x\in R}J_R\chi_R(\zeta_x)] \text{ for } \zeta,\zeta_x \in A$$

therefore

$$\frac{c(x,\zeta)}{c(x,\zeta_x)} = exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)] \text{ for } \zeta,\zeta_x \in A$$

and we conclude $\frac{c(x,\zeta_x)}{c(x,\zeta)+c(x,\zeta_x)} = \frac{1}{1+exp[-2\Sigma_{x\in R}J_R\chi_R(\zeta)]}$ for $\zeta,\zeta_x \in A$

Theorem 3.6: Consider a stochastic Ising model relative to the potential $\{J_R\}$, and let G be the corresponding Gibbs states. Then $G \subset \Im$. In particular, if the stochastic Ising model is ergodic, then there is no phase transition for that potential.

Proof: $R \subset \Im$ follows from the definition and by the previous Theorem R=G, therefore $G \subset \Im$. If the process is ergodic, then $\Im$ is a singleton, therefore G is a singleton as well, so $\{J_R\}$ shows no phase transition.

Theorem 3.7: Given a stochastic Ising model with J_R as before we have the following:

1. The model is ergodic if and only if there is no phase transition.
2. If $-12 < H < 8d - 12$, and $\beta > (\frac{4d}{8d-H-12})c_d$ then there is phase transition, which implies the model is not ergodic

3. If $0 < \beta <<$ then there is no phase transition hence the model is ergodic.

Proof: 1) One direction is proven in Theorem 3.6. Since there is no phase transition $\nu_{\_} = \nu^{-}$ if $\mu \in \wp$, then the family of probability measures $\{\mu S(t), t \geq 0\}$ is relatively compact. Hence all subsequential limits of this family are equal as t $\to \infty$ to $\nu_{\_}$ and ν^{-} which is equal to $lim_{t\to\infty}\mu S(t)$ hence the process is ergodic

3) Suppose we have a family of probability measures on X, say $\{\mu_j^s : j \in Z^d, s \in X\}$ with μ_j^s independant of s_j. Let $||\mu_j^s - \mu_j^t||$ be the total variation between the two configurations s and t.

$$\rho_{i,j} := \frac{1}{2} sup_{s,t \ s=t \ except \ at \ i} ||\mu_j^s - \mu_j^t||$$

Dobrushkins Uniqueness Theorem states suppose $sup_j(\Sigma_{i\neq j}\rho_{i,j}) < 1$, then there exists at most one Gibbs state. Since we already know at least one Gibbs state exists we conclude that it is unique.

$\rho_{i,j} = \frac{1}{2} sup_{s,t \ s=t \ except \ at \ i} ||\mu_j^s - \mu_j^t||$

$\leq \frac{1}{2} sup_{s,t \ s=t \ except \ at \ i} |exp(-\beta\sigma U(i-j))|_\infty$

By Lemma V.1.4 in Simon[13] since our measure takes the form $\mu_h = \frac{exp(-h)}{\int exp(-h)d\mu_0}$ we conclude $||\mu_h - \mu_g|| \leq ||h-g||_\infty$

$\leq \frac{\beta}{2}|U(i-j)|$

Hence $\Sigma_{i\neq j}\rho_{i,j} = \Sigma_{i\neq j}\frac{\beta}{2}|U(i-j)| = \Sigma_{i\neq 0}U(i)\frac{\beta}{2}$

which is less than ϵ if $\beta < \frac{2}{\Sigma_i U(i)}$. Hence we have a unique Gibbs state for the ferromagnetic and anti-ferromagnetic Ising model. If i and j are not nearest neighbors then t and s have the same set of allowable configurations. Additionally, if both are finite we use the procedure above. If a configuration is not allowable for s then it is not allowable for t, hence the difference is 0-0=0.

Hence for $|j-i| \geq 2$ we conclude as before that $\rho_{i,j} \leq \frac{\beta|U(i-j)|}{2}$.

As before we rewrite our Hamiltonian as
$-H\Sigma_i x_i + \frac{1}{2}\Sigma_{i\in V, i\neq j}\Sigma_{j\in V}U(i-j) + \Sigma_{t\notin V}\Sigma_{i\in V}U(i-t)$
using this form we can express
$\mu_j^s(\sigma) = \frac{1}{Z^s}exp(-\beta(\Sigma_{k\neq j}\sigma s_k U(j-k) - \mu\sigma) =$
$\frac{1}{Z^s} - \beta\Sigma_{k\neq j,k\neq i}\sigma s_k U(j-k) - \beta\mu\sigma$
where $\sigma = 0$ or 1.
Analogously $\mu_j^t = \frac{1}{Z^t}exp(-\beta(\Sigma_{k\neq j}\sigma t_k U(j-k) - \mu\sigma) =$
$\frac{1}{Z^t} - \beta\Sigma_{k\neq j,k\neq i}\sigma s_k U(j-k) - \beta\mu\sigma - \beta\sigma U(i-j)$

Now we can assume $s_i = 0$ and $t_i = 1$
(3.7.1) $sup_{s,t \ s=t \text{ except at } i}\{||\mu_j^t - \mu_j^s||\} =$
$sup_{s,t \ s=t^i}\{|\frac{1}{Z^s} - \frac{1}{Z^t}| + |(\frac{1}{Z^s} - \frac{exp(-\beta U(i-j))}{Z^t}exp(-\beta\Sigma_{j\neq k,k\neq i}s_k U(j-k) - \beta\mu)|\}$

Finding the sup of the above is equivalent to finding the
$inf_s\beta[\Sigma_{k\neq j,k\neq i}s_k U(j-k) + \mu]$
$\geq \beta inf_s[-\Sigma_{k\neq j,k\neq i}s_k|U(j-k)| - |\mu|]$
$\geq -\beta\Sigma_{k\neq j,k\neq i}|U(j-k)| + |\mu|$
Hence (3.7.1)
$\leq sup_{s,t \ s=t^i}|\frac{1}{Z^s} - \frac{1}{Z^t}| + sup_{s,t \ s=t^i}|(\frac{1}{Z^s} - \frac{exp(-\beta U(i-j))}{Z^t}|exp(\beta[\Sigma_{k\neq j,k\neq i}|U(j-k)| +$
$|\mu|)$

Looking at configurations where s_j is surrounded by zero's except for it's neighbor at t_{j+1} which is 1. Evaluating $||\mu_j^s - \mu_j^t||$ and substituting in $s_j = 0$ and $t_{j+1} = 1$ we get $||\mu_j^s - \mu_j^t|| = |\frac{1}{Z^s}exp(-\beta\Sigma_{j\neq k,k\neq i}s_k U(j-k) - \beta\mu - 0| + |\frac{1}{Z^s} - 1|$ since $Z^s \geq 1 + exp(-\beta(sup_{s,t \ s=t \text{ except at } i}\Sigma_{k\neq j,k\neq i}s_k U(j-k) - \beta\mu \geq 1 +$ e:We conclude that $||\mu_j^s - \mu_j^t|| \leq \frac{1}{2}|\frac{1-Z^s}{Z^s}| \leq \frac{1}{2}\frac{exp(-\beta\Sigma U(j-k)-\beta\mu)+1}{1+exp(-\beta\Sigma U(j-k))} < \frac{1}{2}$ hence we
We conclude that $||\mu_j^s - \mu_j^t|| \leq \frac{1}{2}|\frac{1-Z^s}{Z^s}| \leq \frac{1}{2}\frac{exp(-\beta\Sigma U(j-k)-\beta\mu)+1}{1+exp(-\beta\Sigma U(j-k))} < \frac{1}{2}$ hence we have a unique Gibbs state.

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