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THE SURVIVAL OF MODULARITY UNDER CONGRUENCE RESTRICTIONS

A Dissertation
Submitted to
the Temple University Graduate Board

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in Partial Fulfillment
of the Requirements for the Degree
DOCTOR OF PHILOSOPHY

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by
Kurt E. Ludwick
August, 2001

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ABSTRACT

THE SURVIVAL OF MODULARITY UNDER
CONGRUENCE RESTRICTIONS

Kurt E. Ludwick
DOCTOR OF PHILOSOPHY

Temple University, August, 2001

Professor Marvin I. Knopp, Chair

Given $f(z)$, a modular form on a congruence subgroup (of the full modular group), we construct the function $f(z;r,t)$ by summing over the terms of the Fourier expansion of $f(z)$ with index congruent to r modulo t. Our object is to study the properties and applications of such congruence restricted sums. In the first chapter, we determine a condition on the multiplier system of $f(z)$ which guarantees that $f(z;r,t)$ is itself a modular form on a (smaller) congruence subgroup.

In the second chapter, we investigate the effects of congruence restrictions on modular forms on $\Gamma(1)$. When $f(z)$ is such a modular form of ***half-integral weight***, we show that $f(z;r,t)$ is a modular form on a certain congruence subgroup, and we investigate the growth of $f(z;r,t)$ at each rational point. We then determine explicitly the Fourier expansion of $f(z;r,t)$ at each rational point when t is prime. Finally, we use congruence restrictions of $\eta(z)$ (the Dedekind eta function) and of $\frac{1}{\eta(z)}$ to construct modular ***functions***.

In the third chapter, we study two different types of congruence restrictions of the theta function. We show that each type of congruence restriction yields an entire modular form. We then use these findings to generalize a result of Bateman, Datskovsky and Knopp on the ratio $\frac{r_s(n)}{r_s^*(n)}$, where $r_s(n)$ ($r_s^*(n)$, respectively) denotes the number of representations of n as an ordered sum of s integer squares (s *odd* integer squares).

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DEDICATION

To my parents, Ann and Ralph Ludwick.

With all my love and gratitude.

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CHAPTER 1

INTRODUCTION

1.1 Definitions and Notation

We will use the following standard notation:

$\mathbb{C}$:= the set of complex numbers $\mathbb{R}$:= the set of real numbers

$\mathbb{Q}$:= the set of rational numbers $\mathbf{Z}$:= the set of integers

$\mathcal{H} := \{z \in \mathbb{C} : \Im z > 0\}$ $S := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $T := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\begin{aligned}
\Gamma(1) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d, \in \mathbf{Z}, ad - bc = 1 \right\} \\
\Gamma_0(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : N | c \right\} \\
\Gamma_1(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \ (N) \right\} \\
\Gamma(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) : N | b \right\} \\
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (N) \right\}
\end{aligned}$$

Remark 1.1 *Here and throughout, we write* $n \equiv r\ (t)$ *to indicate the equivalence of* n *and* r *modulo* t. *With respect to matrices, we write* $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \equiv \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\ (t)$ *to indicate that* $a_j \equiv b_j\ (t)$ *for each* j.

Remark 1.2 $\Gamma(1)$ *is called the* **modular group**. *It is generated by* S *and* T.

Remark 1.3 *Any group* Γ *such that* $\Gamma(N) \subset \Gamma \subset \Gamma(1)$ *is called a* **congruence subgroup of level** N. *In particular,* $\Gamma_0(N)$, $\Gamma_1(N)$ *and* $\Gamma(N)$ *are all congruence subgroups of level* N. $\Gamma(N)$ *itself is called the* **principal** *congruence subgroup of level* N.

Remark 1.4 *We view* $\Gamma(1)$ *as a group of linear functional transformations:*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) := \frac{az+b}{cz+d}.$$

Definition 1.1 *Let all of the following be given:* Γ, *a subgroup of finite index in* $\Gamma(1)$; $k \in \mathbb{R}$; *and* v, *a function from* Γ *to the set* $\{z \in \mathbb{C} : |z| = 1\}$. *Let* f *be a function, meromorphic on* $\mathcal{H}$, *with at most finitely many poles in a fundamental region for* Γ *in* $\mathcal{H}$ *and which satisfies the transformation law*

$$f(Mz) = v(M)(cz+d)^k f(z) \tag{1.1}$$

for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ *and for all* $z \in \mathcal{H}$. *Then,* f *has a Fourier expansion at* $i\infty$ *and at each rational point, of the form:*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i(n+\kappa)z/\lambda}, \quad \text{as } z \to i\infty, \tag{1.2}$$

$$f(z) = (z-q)^{-k} \sum_{n\in\mathbb{Z}} a_n(q) e^{2\pi i(n+\kappa_q)A_q^{-1}(z)/\lambda_q}, \quad \text{as } z \to q = \tfrac{a}{c} \in \mathbb{Q}, \tag{1.3}$$

where $A_q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ *(so that* $A_q(i\infty) = q$*). If the expansions given by Equations (1.2) and (1.3) are all left-finite, then we say that* $f(z)$ **is a** *modular form on Γ of weight k with multiplier system v. Further, if $f(z)$ is holomorphic on $\mathcal{H}$, $n_0 + \kappa \geq 0$, and $n_0(q) + \kappa_q \geq 0$ $\forall q \in \mathbb{Q}$, we say $f(z)$ is an* ***entire modular form****. If $f(z)$ is an entire modular form such that $n_0 + \kappa > 0$ and $n_0(q) + \kappa_q > 0$ $\forall q \in \mathbb{Q}$, then we call $f(z)$ a* ***cusp form****.*

Remark 1.5 *In (1.2), we define λ to be the smallest positive integer such that $S^\lambda \in \Gamma$, and we define κ to be the unique real number such that $0 \leq \kappa < 1$ and $e^{2\pi i\kappa} = v(S^\lambda)$. In (1.3), we define λ_q to be the smallest positive integer such that $A_q S^{\lambda_q} A_q^{-1} \in \Gamma$, and we define κ_q to be the unique real number such that $0 \leq \kappa_q < 1$ and $e^{2\pi i\kappa_q} = v(A_q S^{\lambda_q} A_q^{-1})$.*

Remark 1.6 *By "$z \to i\infty$" above, we mean more precisely that $\Im(z) \to \infty$ while $\Re(z) \in [-\frac{1}{2}, \lambda - \frac{1}{2})$. Similarly, we mean by "$z \to q$" that $\Im(A^{-1}(z)) \to \infty$ while $\Re(A^{-1}(z)) \in [-\frac{1}{2}, \lambda - \frac{1}{2})$.*

Remark 1.7 *The transformation law for $f(z)$ dictates that the multiplier system v must satisfy the following* ***consistency condition****: for all $z \in \mathcal{H}$ and for all $M_1, M_2, M_3 \in \Gamma$ with $M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ and $M_3 = M_1 M_2$,*

$$v(M_3)(c_3 z + d_3)^k = v(M_1)(c_1 M_2 z + d_1)^k v(M_2)(c_2 z + d_2)^k. \tag{1.4}$$

Remark 1.8 *The requirements on the Fourier expansions in Definition 1.1 may be restated as follows: for each $\frac{a}{c} \in \mathbb{Q}$ with $\gcd(a, c) = 1$, choose $A_{\frac{a}{c}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, and put $A_{i\infty} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then, for $q \in \mathbb{Q} \cup \{i\infty\}$, $f(z)$ is a modular form if there exists a finite constant N_q such that, as $A_q^{-1}(z) \to i\infty$ (in the manner described by Remark 1.6),*

$$|z - q|^k |f(z)| = \mathcal{O}(e^{N_q \Im(A_q^{-1}(z))}), \text{ if } q \in \mathbb{Q}; \tag{1.5}$$

$$|f(z)| = \mathcal{O}(e^{N_q \Im(A_q^{-1}(z))}), \text{ if } q = i\infty. \tag{1.6}$$

(Note that $\Im(A_q^{-1}(z)) = \frac{1}{|cz-a|^2}$*; therefore,* $\Im(A_q^{-1}(z))$ *does not depend on the choice of* A_q*.)*

Further, $f(z)$ *is an entire modular form if* $N_q \leq 0 \ \forall \ q \in \mathbb{Q} \cup \{i\infty\}$*, and* $f(z)$ *is a cusp form if* $N_q < 0$ *for all such* q*.*

Note: Throughout, when v is a multiplier system and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, we will write $v\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in place of $v\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ in order to simplify notation.

1.2 Congruence Restrictions

Let f be a function defined on $\mathcal{H}$ with the Fourier expansion

$$f(z) = \sum_{n=n_0}^{\infty} a_n e^{2\pi i(n+\kappa)z}. \tag{1.7}$$

Let $r, t \in Z$, with $t > 1$, be given. Then,

Definition 1.2

$$f(z; r, t) := \sum_{\substack{n=n_0 \\ n \equiv r(t)}}^{\infty} a_n e^{2\pi i(n+\kappa)z}. \tag{1.8}$$

Within the past few years, several people who work with modular forms have made and used the following observation: often, when $f(z)$ is a modular form on a congruence subgroup Γ of level N, $f(z; r, t)$ turns out to also be a modular form on a congruence subgroup of level N', where $N|N'$. Furthermore,the modular form $f(z; r, t)$ inherits the weight and the multiplier system of $f(z)$.

Ken Ono, in [7], [8] and [9], has applied congruence restrictions to the modular forms related to the Dedekind eta function to obtain results on the arithmetic function $p(n)$ (the partition function). His results include new modular identities satisfied by $p(n)$ in certain arithmetic progressions on n (of the type originally discovered by Ramanujan), as well as proofs of conjectures on the existence and/or frequency of values of n satisfying certain modular restrictions. For example: in [9] he proves a conjecture of Erdös:

Conjecture (Erdös) *If m is prime, then there exists an integer $n_m \geq 0$ such that $p(n_m) \equiv 0 \ (m)$.*

Earlier, in [7], Ono had proved the following results on the parity of $p(n)$ in arithmetic progressions on n:

Theorem 1 (Ono) *For any arithmetic progression $r \pmod t$, there are infinitely many integers $N \equiv r \ (t)$ for which $p(N)$ is even.*

Theorem 2 (Ono) *For any arithmetic progression $r \pmod t$, there are infinitely many integers $M \equiv r \ (t)$ for which $p(M)$ is odd,* ***provided there is one such*** *M.*

Ono also goes on to state an upper bound for the the smallest such $M \equiv r \ (t)$, provided it exists. Of particular interest, though, is the "all or nothing" nature of the second theorem: in the arithmetic progression $M \equiv r \ (t)$, $p(M)$ is odd either *infinitely many times*, or *never*.

In another recent paper [3], Bateman, Datskovsky and Knopp apply congruence restrictions to the theta function to study the behavior of the quotient of the arithmetic functions $r_s(n)$ and $r_s^*(n)$ (the number of ways to write n as the sum of s squares and of s odd squares, respectively). They prove that, when $s \geq 8$, the ratio $\frac{r_s^*(n)}{r_s(n)}$ is not constant in n on the arithmetic progression $n \equiv s \ (8)$. (Notice that $r_s^*(n) = 0$ whenever $n \not\equiv s \ (8)$.)

The contents of this thesis are partially motivated by these recent applications. In fact, in Chapter 3 we generalize Bateman, Datskovsky and Knopp's result. However, in each of the cases cited above, congruence restrictions are applied only to a certain type of modular form, in order to achieve a very specific result. Ono's treatment is of use only for modular forms with Nebentypus character (a certain type of multiplier system), while the arguments of Bateman, Datskovsky and Knopp are valid only for congruence restrictions of the theta function (as defined in Chapter 3) whose modulus divides 8.

Our object here is to study this "survival of modularity under congruence restrictions." We wish to determine more generally, and as precisely as possible, conditions on a modular form which will guarantee that the function which results from restricting its Fourier expansion to terms in an arithmetic progression will again be a modular form. We then seek to apply our results to discover new modular forms, to study the coefficients of existing modular forms, and to generalize previous results obtained through the use of modular forms.

We wish to show that $f(z;r,t)$ satisfies the same transformation law as $f(z)$ – that is, that we could replace $f(z)$ with $f(z;r,t)$ in Equation (1.1) – on some congruence subgroup of $\Gamma(1)$. We may rewrite $f(z;r,t)$ as the double-sum:

$$f(z;r,t) = \frac{1}{t}\sum_{\nu(t)} e^{-2\pi i(r+\kappa)\nu/t} \sum_{n=n_0}^{\infty} a_n e^{2\pi i(n+\kappa)(z+\nu/t)} \qquad (1.9)$$

$$= \frac{1}{t}\sum_{\nu(t)} e^{-2\pi i(r+\kappa)\nu/t} f(z+\frac{\nu}{t}). \qquad (1.10)$$

To simplify notation, we define

$$\gamma_{\nu,t} := \begin{pmatrix} 1 & \nu/t \\ 0 & 1 \end{pmatrix}; \qquad (1.11)$$

that is, $z + \frac{\nu}{t} = \gamma_{\nu,t}(z)$. (Note that $\gamma_{\nu,t}$ is *not* an element of $\Gamma(1)$ when $t \not| \nu$.) Thus, we may rewrite (1.10) in the form

$$f(z; r, t) = \frac{1}{t} \sum_{\nu\,(t)} e^{-2\pi i(r+\kappa)\nu/t} f(\gamma_{\nu,t}(z)). \tag{1.12}$$

To derive the strongest possible result from our argument, we consider the following congruence subgroup (and introduce our own notation, since none seems to have existed previously for this particular group):

Definition 1.3

$$\Gamma_{0,n}(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \mid a \equiv d\ (n) \right\} \tag{1.13}$$

Before we continue with congruence restrictions, we note a few interesting properties of the group $\Gamma_{0,n}(N)$:

Remark 1.9 *If $n|N$, then $\Gamma_{0,n}(N)$ is a congruence subgroup, and $\Gamma_1(N) \subseteq \Gamma_{0,n}(N) \subseteq \Gamma_0(N)$.*

Remark 1.10 *For all $N \in \mathbf{Z}^+, \Gamma_{0,1}(N) = \Gamma_0(N)$.*

Proposition 1.1 *If $n|N$, then*

$$\Gamma_{0,n}(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : N|c, a^2 \equiv 1\ (n) \right\}.$$

Proof Given $N|c$ and $n|N$, we must show that $a \equiv d\ (n)$ if and only if $a^2 \equiv 1\ (n)$:

$$\begin{aligned} & d \equiv a\ (n) \\ \Longleftrightarrow\ & ad \equiv a^2\ (n) \text{ (since } \gcd(a, n) = 1) \\ \Longleftrightarrow\ & ad - bc \equiv a^2 - bc\ (n) \\ \Longleftrightarrow\ & 1 \equiv a^2\ (n) \text{ (since } n|c). \end{aligned}$$

This proves the proposition.

It is sometimes the case that $\Gamma_{0,N}(n) = \Gamma_0(N)$. In particular, Proposition 1.1 implies $\Gamma_{0,n}(N) = \Gamma_0(N)$ if and only if, for all $a \in \mathbf{Z}$ such that $\gcd(a, N) = 1$, $a^2 \equiv 1\ (n)$. This is possible only under certain conditions on N and n.

Proposition 1.2

$$a^2 \equiv 1\ (8) \iff \gcd(a, 2) = 1,$$

and

$$a^2 \equiv 1\ (3) \iff \gcd(a, 3) = 1.$$

However, for any prime integer $p > 3$, there exists no such nontrivial modulus m such that $a^2 \equiv 1\ (m) \iff \gcd(a, p) = 1$.

Proof It is well known (and easy to prove) that $a^2 \equiv 1\ (8)$ if and only if a is odd, and that $a^2 \equiv 1\ (3)$ if and only if $\gcd(a, 3) = 1$. Simply note that, for all $a, j \in \mathbf{Z}$, $(a+2j)^2 = 4j(j+a)+a^2 \equiv a^2\ (8)$, and $(a+3j)^2 = 3j(3j+2a)+a^2 \equiv a^2\ (3)$.

On the other hand, suppose that $p > 3$ is prime. Then, the integers 1,2 and 3 are all relatively prime to p. The squares of these integers are 1,4 and 9, respectively. Suppose $\gcd(a,p) = 1 \implies a^2 \equiv 1\ (m)$, for $m \in \mathbf{Z}$. Then, we must have $1 \equiv 4 \equiv 9\ (m)$. This is possible only when $m = 1$. This proves the proposition.

Lemma 1.1 *Suppose $n > 1$ and $n|N$. If $2|N$, then $\Gamma_{0,n}(N) = \Gamma_0(N)$ if and only if $n|8$. If $3|N$, then $\Gamma_{0,n}(N) = \Gamma_0(N)$ if and only if $n = 3$. If $\gcd(6, N) = 1$, then $\Gamma_{0,n}(N) \neq \Gamma_0(N)$.*

Proof $\Gamma_{0,n}(N) = \Gamma_0(N)$ if and only if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \iff d \equiv a\ (n)$, which in turn (by Proposition 1.1 is true if and only if, for all $a \in \mathbf{Z}$ such that $\gcd(a,N)=1$, $a^2\equiv 1\ (n)$. The condition $\gcd(a,N)=1$ is equivalent to the condition $\gcd(a,p)=1, \forall$ prime $p|N$. The hypothesis then follows directly from Proposition 1.2. This proves the lemma.

Definition 1.4 *For $N,t\in\mathbf{Z}^+$, we define*

$$G_{N,t} := \{M\in\Gamma(1) : \gamma_{\nu,t}M\gamma_{-\nu,t}\in\Gamma_0\,(N)\,,\forall\nu\in\mathbf{Z}\}.$$

Proposition 1.3 *For all $\nu,\lambda,t\in\mathbf{Z}$,*

$$\gamma_{\nu,t}M\gamma_{-\nu,t}\in\Gamma_0\,(N)\Longleftrightarrow\gamma_{\nu+\lambda t,t}M\gamma_{-(\nu+\lambda t),t}\in\Gamma_0\,(N)\,.$$

Proof Let $\lambda\in\mathbf{Z}$, Then,

$$\begin{aligned}\gamma_{\nu+\lambda t,t} &= \begin{pmatrix}1 & \frac{\nu+\lambda t}{t}\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}1 & \frac{\nu}{t}+\lambda\\ 0 & 1\end{pmatrix}\\ &= S^\lambda\gamma_{\nu,t}=\gamma_{\nu,t}S^\lambda\end{aligned}$$

Therefore,

$$\gamma_{\nu+\lambda t,t}M\gamma_{-(\nu+\lambda),t}=S^\lambda\gamma_{\nu,t}M\gamma_{-\nu,t}S^{-\lambda},$$

which is an element of $\Gamma_0\,(N)$ if and only if $\gamma_{\nu,t}M\gamma_{-\nu,t}\in\Gamma_0\,(N)$. This proves the proposition.

Remark 1.11 *It follows from Proposition 1.3 that the statement "$\gamma_{\nu,t}M\gamma_{-\nu,t}\in\Gamma_0\,(N)\,,\forall\nu\in\mathbf{Z}$" is equivalent to the statement "$\gamma_{\nu,t}M\gamma_{-\nu,t}\in\Gamma_0\,(N)\,,\forall\nu\in\mathbf{Z}, 0\le\nu\le t-1.$"*

Definition 1.5

$$\Sigma_{0,n}(N) := \left\{M=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma(1) : c\equiv\frac{N}{2}\ (N), d-a\equiv\frac{n}{2}\ (n)\right\}.$$

Remark 1.12 *If either N or n is odd, then $\Sigma_{0,n}(N)$ is empty.*

Lemma 1.2 $\Gamma_{0,t}(t^2) \cup \Sigma_{0,t}(t^2)$ *is a group.*

Proof First, we note that $A \in \Gamma_{0,t}(t^2) \Rightarrow A^{-1} \in \Gamma_{0,t}(t^2)$ (since $\Gamma_{0,t}(t^2)$ is itself a group) and $B \in \Sigma_{0,t}(t^2) \Rightarrow B^{-1} \in \Sigma_{0,t}(t^2)$ (this is clear from the definition of $\Sigma_{0,t}(t^2)$). Therefore, it remains only to show that $\Gamma_{0,t}(t^2) \cup \Sigma_{0,t}(t^2)$ is closed under matrix multiplication.

Let $A_1 \in \Gamma_{0,t}(t^2)$ and $A_2, A_3 \in \Sigma_{0,t}(t^2)$. We will show: $A_1A_2 \in \Sigma_{0,t}(t^2)$, $A_2A_1 \in \Sigma_{0,t}(t^2)$, and $A_2A_3 \in \Gamma_{0,t}(t^2)$.

We may write

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ t^2c_1 & a_1 + td_1 \end{pmatrix},$$

with $a_1, b_1, c_1, d_1 \in \mathbf{Z}$, and

$$A_i = \begin{pmatrix} a_i & b_i \\ \frac{t^2c_i}{2} & a_i + \frac{td_i}{2} \end{pmatrix},$$

with $a_i, b_i, c_i, d_i \in \mathbf{Z}$ and c_i, d_i odd, for $i = 2, 3$. Note that, for each i, the lower left entry is even (since t is even, by Remark 1.12); therefore, for each i, a_i is *odd.*

With this notation, we have

$$A_1A_2 = \begin{pmatrix} a_1a_2 + \frac{t^2b_1c_1}{2} & a_1b_2 + b_1\left(a_2 + \frac{td_2}{2}\right) \\ t^2c_1a_2 + (a_1 + td_1)\frac{t^2c_2}{2} & t^2c_1b_2 + (a_1 + td_1)\left(a_2 + \frac{td_2}{2}\right) \end{pmatrix}.$$

The lower-left entry of A_1A_2 is an integer multiple of t^2, plus $\frac{t^2a_1c_2}{2}$. Since a_1c_2 is odd, the lower-left entry of A_1A_2 is congruent to $\frac{t^2}{2}$ modulo t^2.

The difference between the lower-right and upper-left entries of A_1A_2 is an integer multiple of t, plus $\frac{ta_1d_2}{2}$. Since a_1d_2 is odd, this difference is congruent to $\frac{t}{2}$ modulo t. Thus, A_1A_2 is an element of $\Sigma_{0,t}(t^2)$. A similar calculation shows that $A_2A_1 \in \Sigma_{0,t}(t^2)$.

We also have

$$A_2A_3 = \begin{pmatrix} a_2a_3 + \frac{t^2b_2c_3}{2} & a_2b_3 + b_2\left(a_3 + \frac{td_3}{2}\right) \\ \frac{t^2c_2a_3}{2} + \left(a_2 + \frac{td_2}{2}\right)\frac{t^2c_3}{2} & \frac{t^2b_2c_3}{2} + \left(a_2 + \frac{td_2}{2}\right)\left(a_3 + \frac{td_3}{2}\right) \end{pmatrix}.$$

The lower-left entry of A_2A_3 may be rewritten as $t^2\left(\frac{c_2a_3}{2} + \frac{c_3}{2}\left(a_2 + \frac{td_2}{2}\right)\right)$. Since c_2a_3 and $a_2 + \frac{td_2}{2}$ are both odd, this sum is an integer multiple of t^2.

The difference between the lower-right and upper-left entries of A_2A_3 is a half-integer multiple of t^2 (which is thus an integer multiple of t), plus $t\left(\frac{a_3d_2}{2} + \frac{d_3a_2}{2}\right)$. Since a_i and d_i are even for $i = 2, 3$, this quantity is an integer multiple of t. Thus, A_2A_3 is an element of $\Gamma_{0,t}(t^2)$. This proves the lemma.

Lemma 1.3 *If* $16 \nmid t$, *then* $\Sigma_{0,t}(t^2)$ *is empty.*

Proof First, we note that $\Sigma_{0,t}(t^2)$ is clearly empty when t is odd, since the condition $c \equiv \frac{t^2}{2}\ (t^2)$ would not be satisfied for any $c \in \mathbf{Z}$.

Assume that t is even, but $4 \nmid t$. (In this case, $\frac{t}{2}$ is odd.) Suppose there exists $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $M \in \Sigma_{0,t}(t^2)$. Then, the condition $d-a \equiv \frac{t}{2}\ (t)$ implies that d and a are of opposite parity; this requires either d or a to be even, and so the product ad is even. On the other hand, the condition $c \equiv \frac{t^2}{2}\ (t^2)$ implies that c is even, and so bc is even. But $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \Rightarrow ad - bc = 1$, which is odd. Therefore, no such $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ exists, and so $\Sigma_{0,t}(t^2)$ is empty.

Finally, assume $4|t$, and suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_{0,t}(t^2)$. Since $d - a \equiv \frac{t}{2}\ (t)$, we may write $d = a + \frac{ht}{2}$, where $h \in \mathbf{Z}$ is odd. Also, it follows from the condition $c \equiv \frac{t^2}{2}\ (t^2)$ that $t^2|2c$; since $4|t$, this implies $8|c$, and thus $2 \nmid a$.

Therefore,

$$\begin{aligned} ad - bc = 1 &\implies ad \equiv 1\ (c) \\ &\implies a\left(a + \frac{ht}{2}\right) \equiv 1\ (8) \\ &\implies a^2 + \frac{aht}{2} \equiv 1\ (8) \\ &\implies \frac{aht}{2} \equiv 0\ (8) \\ &\implies aht \equiv 0\ (16). \end{aligned}$$

(Recall from Proposition 1.2 that a odd $\Rightarrow a^2 \equiv 1\ (8)$.) Therefore,

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_{0,t}(t^2) &\Rightarrow 16 | aht \\ &\Rightarrow 16 | t \text{ (since } ah \text{ is odd).} \end{aligned}$$

This proves the lemma.

Proposition 1.4 *If $N, t \in \mathbf{Z}^+$, then*

$$G_{N,t} = \Gamma_0(N) \cap \left(\Gamma_{0,t}(t^2) \cup \Sigma_{0,t}(t^2)\right).$$

Proof Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then,

$$\begin{aligned} \gamma_{\nu,t} M \gamma_{-\nu,t} &= \begin{pmatrix} 1 & \nu/t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -\nu/t \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a + \frac{c\nu}{t} & b + \frac{d\nu - a\nu}{t} - \frac{c\nu^2}{t^2} \\ c & d - \frac{c\nu}{t} \end{pmatrix} \end{aligned}$$

We will first show that $M \in \Gamma_0(N) \cap (\Gamma_{0,t}(t^2) \cup \Sigma_{0,t}(t^2)) \implies M \in G_{N,t}$. If $M \in \Gamma_{0,t}(t^2) \cap \Gamma_0(N)$, then $\gamma_{\nu,t} M \gamma_{-\nu,t}$ is clearly an element of $\Gamma_0(N)$. On the other hand, if $M \in \Sigma_{0,t}(t^2) \cap \Gamma_0(N)$, then $\frac{d-a}{t} = \frac{h_1}{2}$ and $\frac{c}{t^2} = \frac{h_2}{2}$, where $h_1, h_2 \in \mathbf{Z}$ are both *odd*; therefore, in this case we have $\frac{(d-a)\nu}{t} - c\frac{\nu^2}{t^2} \in \mathbf{Z}$ for all $\nu \in \mathbf{Z}$. Thus, $\Gamma_0(N) \cap (\Gamma_{0,t}(t^2) \cup \Sigma_{0,t}(t^2)) \subseteq G_{N,t}$.

Next, we will show that $M \in G_{N,t} \Longrightarrow M \in \Gamma_0(N) \cap (\Gamma_{0,t}(t^2) \cup \Sigma_{0,t}(t^2))$. Suppose $\gamma_{\nu,t} M \gamma_{-\nu,t} \in \Gamma_0(N), \forall \nu \in \mathbf{Z}$. This immediately implies $N|c$; thus, we have $M \in \Gamma_0(N)$. We also have $\frac{(d-a)\nu}{t} - \frac{c\nu^2}{t^2} \in \mathbf{Z}, \forall \nu \in \mathbf{Z}$ - in particular, $\frac{d-a}{t} - \frac{c}{t^2} \in \mathbf{Z}$ and $\frac{2(d-a)}{t} - \frac{4c}{t^2} \in \mathbf{Z}$. We consider the following cases:

- Suppose $t^2|c$. In this case, $\frac{c}{t^2} \in \mathbf{Z}$, which implies $\frac{d-a}{t} \in \mathbf{Z}$ as well. Thus, we have $t^2|c$ and $t|(d-a)$, and so $M \in \Gamma_{0,t}(t^2)$.

- Suppose $t^2 \not| c$. In this case, $\frac{c}{t^2} \notin \mathbf{Z}$, which implies $\frac{d-a}{t} \notin \mathbf{Z}$. However, as previously noted, we must have $\frac{2(d-a)}{t} - \frac{4c}{t^2} \in \mathbf{Z}$. Therefore,

$$\begin{aligned} 2\left(\frac{d-a}{t} - \frac{c}{t^2}\right) - \left(\frac{2(d-a)}{t} - \frac{4c}{t^2}\right) &\in \mathbf{Z} \\ \Longrightarrow -\frac{2c}{t^2} + \frac{4c}{t^2} &\in \mathbf{Z} \\ \Longrightarrow \frac{2c}{t^2} &\in \mathbf{Z}, \end{aligned}$$

  and

$$\begin{aligned} 4\left(\frac{d-a}{t} - \frac{c}{t^2}\right) - \left(\frac{2(d-a)}{t} - \frac{4c}{t^2}\right) &\in \mathbf{Z} \\ \Longrightarrow \frac{4(d-a)}{t} - \frac{2(d-a)}{t} &\in \mathbf{Z} \\ \Longrightarrow \frac{2(d-a)}{t} &\in \mathbf{Z}. \end{aligned}$$

  Thus, $t^2 \not| c \Longrightarrow t^2|2c, t|2(d-a)$, and $t \not| (d-a)$. It follows that $c \equiv \frac{t^2}{2}\ (t^2)$ and $d - a \equiv \frac{t}{2}\ (t)$; therefore, $M \in \Sigma_{0,t}(t^2)$.

Thus, $G_{N,t} \subseteq \Gamma_0(N) \cap (\Gamma_{0,t}(t^2) \cup \Sigma_{0,t}(t^2))$. This proves the proposition.

Corollary 1.1 *$G_{N,t}$ is a group.*

Proof This follows from Proposition 1.4, in which it is shown that $G_{N,t}$ is the intersection of two groups.

Corollary 1.2 *If* $16 \not| t$, *then* $G_{N,t} = \Gamma_{0,t}(t^2)$.

Proof This follows from Proposition 1.4 and Lemma 1.3.

Remark 1.13 *It also follows from Proposition 1.4 that*

$$\Gamma_0(N) \cap \Gamma_{0,t}(t^2) \subseteq G_{N,t}, \forall\, t \in \mathbf{Z}^+.$$

Definition 1.6 *Given* v*, a multiplier system on* $\Gamma_0(N)$,

$$\mathcal{S}_{v,t} := \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{N,t} : v(\gamma_{\nu,t} M \gamma_{-\nu,t}) = v(M), \forall \nu \in \mathbf{Z} \right\}.$$

Remark 1.14 *If* v *is the trivial multiplier system on* $\Gamma_0(N)$, *then* $\mathcal{S}_{v,t} = G_{N,t}$.

Proposition 1.5 *If* v *is a multiplier system of weight* k*, where* $k \in \mathbf{Z}$*, then* $\mathcal{S}_{v,t}$ *is a group.*

Proof Choose $M_1, M_2 \in \mathcal{S}_{v,t}$, with $M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$. Then, for $\nu \in \mathbf{Z}$,

$$\begin{aligned} v(\gamma_{\nu,t} M_1 M_2 \gamma_{-\nu,t}) &= v(\gamma_{\nu,t} M_1 \gamma_{-\nu,t} \gamma_{\nu,t} M_2 \gamma_{-\nu,t}) \\ &= v(\gamma_{\nu,t} M_1 \gamma_{-\nu,t}) v(\gamma_{\nu,t} M_2 \gamma_{-\nu,t}) \\ &= v(M_1) v(M_2). \end{aligned}$$

Thus, $\mathcal{S}_{v,t}$ is closed under matrix multiplication. Also, $v(I) = 1 \Longrightarrow v(M_i M_i^{-1}) = 1 \Longrightarrow v(M_i) = \overline{v(M_i^{-1})}$. Therefore,

$$\begin{aligned} v(\gamma_{\nu,t} M_i^{-1} \gamma_{-\nu,t}) &= v((\gamma_{\nu,t} M_1 \gamma_{-\nu,t})^{-1}) \\ &= \overline{v(\gamma_{\nu,t} M_1 \gamma_{-\nu,t})} \\ &= \overline{v(M_i)} = v(M_i^{-1}). \end{aligned}$$

Thus, $M \in \mathcal{S}_{v,t} \Longrightarrow M^{-1} \in \mathcal{S}_{v,t}$. This proves the proposition.

Theorem 1.1 *Let f be a modular form on $\Gamma_0(N)$ of weight k. Then, $f(z;r,t)$ satisfies the transformation law*

$$f(Mz;r,t) = v(M)(cz+d)^k f(z;r,t) \tag{1.14}$$

for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S_{v,t}$.

Proof We begin with equation (1.10):

$$\begin{aligned} f(Mz;r,t) &= \sum_{\nu=0}^{t-1} e^{-2\pi i(r+\kappa)\nu/t} f(\gamma_{\nu,t}Mz) \\ &= \sum_{\nu=0}^{t-1} e^{-2\pi i(r+\kappa)\nu/t} f(\gamma_{\nu,t}M\gamma_{-\nu,t}\gamma_{\nu,t}z) \end{aligned}$$

Now, since $\gamma_{\nu,t}M\gamma_{-\nu,t} \in \Gamma_0(N)$, we may apply the transformation law for $f(z)$:

$$\begin{aligned} f(Mz;r,t) &= \sum_{\nu=0}^{t-1} e^{-2\pi i(r+\kappa)\nu/t} v(\gamma_{\nu,t}M\gamma_{-\nu,t})\left(c\gamma_{\nu,t}z + d - \frac{c\nu}{t}\right)^k f(\gamma_{\nu,t}z) \\ &= \left(c\left(z+\frac{\nu}{t}\right) + d - \frac{c\nu}{t}\right)^k \sum_{\nu=0}^{t-1} e^{-2\pi i(r+\kappa)\nu/t} v(M) f(\gamma_{\nu,t}z) \\ &= v(M)\left(cz + \frac{c\nu}{t} + d - \frac{c\nu}{t}\right)^k \sum_{\nu=0}^{t-1} e^{-2\pi i(r+\kappa)\nu/t} f(\gamma_{\nu,t}z) \\ &= v(M)(cz+d)^k \sum_{\nu=0}^{t-1} e^{-2\pi i(r+\kappa)\nu/t} f(\gamma_{\nu,t}z) \\ &= v(M)(cz+d)^k f(z;r,t) \end{aligned}$$

This proves the theorem.

We wish to show that $f(z;r,t)$ is, in fact, a *modular form*. To do so, we will need the following lemma:

Lemma 1.4 *Let $q = \frac{a}{c} \in \mathbf{Q}$, with* $\gcd(a,c) = 1$ *and $c \in \mathbf{Z}^+$. For each $\nu \in \mathbf{Z}, 0 \le \nu \le t-1$, put* $\delta_\nu := \gcd(at+\nu c, ct)$, *and choose*

$$A_{q+\frac{\nu}{t}} := \begin{pmatrix} a_\nu & b_\nu \\ c_\nu & d_\nu \end{pmatrix} \in \Gamma(1),$$

with $a_\nu := \frac{at+\nu c}{\delta_\nu}$ *and* $c_\nu := \frac{ct}{\delta_\nu}$. *(Thus,* $A_{q+\frac{\nu}{t}}(i\infty) = \frac{a}{c} + \frac{\nu}{t}$.*) Then,*

$$A^{-1}_{q+\frac{\nu}{t}}\left(z+\frac{\nu}{t}\right) - \frac{\delta_\nu^2}{t^2}A^{-1}z = \frac{\delta_\nu}{ct^2}(d\delta_\nu - td_\nu). \tag{1.15}$$

Proof Write $A_{q+\frac{\nu}{t}}$ in the form

$$A_{q+\frac{\nu}{t}} = \begin{pmatrix} c_\nu(a/c+\nu/t) & b_\nu \\ c_\nu & d_\nu \end{pmatrix}.$$

This gives us

$$\begin{aligned} A^{-1}_{q+\frac{\nu}{t}}\left(z+\frac{\nu}{t}\right) &= -\frac{d_\nu}{c_\nu} + \frac{1}{c_\nu(-c_\nu(z+\nu/t)+c_\nu(a/c+\nu/t)} \\ &= -\frac{d_\nu}{c_\nu} - \frac{1}{c_\nu^2(z-a/c)} \\ &= -\frac{d_\nu}{c}\frac{\delta_\nu}{t} - \frac{\delta_\nu^2}{t^2}\frac{1}{c^2(z-a/c)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\delta_\nu^2}{t^2}A_q^{-1}z &= -\frac{\delta_\nu^2}{t^2}\left(\frac{d}{c}+\frac{1}{c^2(z-a/c)}\right) \\ &= -\frac{\delta_\nu^2}{t^2}\frac{d}{c} - \frac{\delta_\nu^2}{t^2}\frac{1}{c^2(z-a/c)} \\ &= -\frac{\delta_\nu^2}{t^2}\frac{d}{c} + A^{-1}_{q+\frac{\nu}{t}}\left(z+\frac{\nu}{t}\right) + \frac{d_\nu}{c}\frac{\delta_\nu}{t} \\ &= -\frac{\delta_\nu^2}{t^2}\frac{d}{c} + A^{-1}_{q+\frac{\nu}{t}}\left(z+\frac{\nu}{t}\right) + \frac{d_\nu}{c}\frac{\delta_\nu}{t} \\ &= A^{-1}_{q+\frac{\nu}{t}}\left(z+\frac{\nu}{t}\right) - \frac{\delta_\nu}{ct^2}(d\delta_\nu - td_\nu). \end{aligned}$$

This completes the proof of Lemma 1.4.

Remark 1.15 *The preceding lemma implies that, for* $q \in \mathbb{Q}$,

$$\exp\left(\Im\left(A^{-1}_{q+\frac{\nu}{t}}\left(z+\frac{\nu}{t}\right)\right)\right) = C_{q,\nu,t}\exp\left(\frac{\delta_\nu^2}{t^2}\Im(A_q^{-1}(z))\right),$$

where $C_{q,\nu,t}$ *is a constant with absolute value* $\exp\left(\frac{\delta_\nu}{ct^2}(d\delta_\nu - td_\nu)\right)$.

Remark 1.16 *Note that*

$$\begin{aligned}\delta_\nu &= \gcd(c,t)\gcd(a\frac{t}{\gcd(c,t)}+\nu\frac{c}{\gcd(c,t)},\frac{c}{\gcd(c,t)}t)\\ &= \gcd(c,t)\gcd(a\frac{t}{\gcd(c,t)}+\nu\frac{c}{\gcd(c,t)},t)\\ &\leq t\gcd(c,t)\leq t^2.\end{aligned}$$

Therefore,

$$t^{-2}\leq\frac{\delta_\nu^2}{t^2}\leq t^2, \forall\ \nu\in\mathbf{Z}.$$

Theorem 1.2 *Suppose* $\exists\ \Gamma_{v,t}\subset S_{v,t}$ *such that* $\Gamma_{v,t}$ *is a subgroup of finite index in* $\Gamma(1)$. *Then,* $f(z;r,t)$ *is a* modular form *on* $\Gamma_{v,t}$ *of weight* k *with multiplier system* v. *Furthermore, if* $f(z)$ *is an entire modular form (or a cusp form, respectively) on* $\Gamma_0(N)$, *then* $f(z;r,t)$ *is an entire modular form (or cusp form) on* $\Gamma_{v,t}$.

Proof We must determine the behavior of $f(z;r,t)$ at each $q\in\mathbf{Q}$ and at $i\infty$. Throughout this proof, we will use the notation introduced in Remark 1.8.

It is clear from the definition of $f(z;r,t)$ that, as $z\to i\infty, |f(z;r,t)| = \mathcal{O}(\exp(N_{i\infty,t}\Im(z)))$, for some $N_{i\infty,t}\geq N_{i\infty}=n_0+\kappa$. Therefore, the growth of $f(z;r,t)$ as $z\to i\infty$ is sufficiently bounded.

Let $q=\frac{a}{c}\in\mathbf{Q}$, with $\gcd(a,c)=1$ and $c\in\mathbf{Z}^+$. Then, $\exists N_q\in\mathbf{R}$ such that

$$|z-q|^{-k}|f(z)| = \mathcal{O}\left(e^{N_q\Im(A_q^{-1}(z))}\right),$$

where $A_q=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma(1)$. We may restate this condition: there exists $M_q>0$ such that, as $z\to q$,

$$|f(z)|\leq M_q|z-q|^{-k}e^{N_q\Im(A_q^{-1}(z))}.$$

Note that, as $z \to q$, $z + \frac{\nu}{t} \to q + \frac{\nu}{t}$. Therefore,

$$\begin{aligned}
|f(z;r,t)| &= \frac{1}{t}\left|\sum_{\nu=0}^{t-1} e^{-2\pi i(r+\kappa)\nu/t} f\left(z+\frac{\nu}{t}\right)\right| \\
&\leq \frac{1}{t}\sum_{\nu=0}^{t-1}\left|\left(z+\frac{\nu}{t}\right)-\left(q+\frac{\nu}{t}\right)\right|^{-k} \times \\
&\qquad M_{q+\frac{\nu}{t}} \exp\left(N_{q+\frac{\nu}{t}}\Im\left(A^{-1}_{q+\frac{\nu}{t}}\left(z+\frac{\nu}{t}\right)\right)\right) \\
&= \frac{1}{t}\sum_{\nu=0}^{t-1}|z-q|^{-k} M_{q+\frac{\nu}{t}} \exp\left(N_{q+\frac{\nu}{t}}\Im\left(A^{-1}_{q+\frac{\nu}{t}}\left(z+\frac{\nu}{t}\right)\right)\right) \\
&\leq \frac{1}{t}\sum_{\nu=0}^{t-1}|z-q|^{-k} M'_{q+\frac{\nu}{t}} \exp\left(N_{q+\frac{\nu}{t}}\frac{\delta_\nu^2}{t^2}\Im\left(A_q^{-1}(z)\right)\right) \\
&\quad \text{(where } M'_q := M_q C_{q,\nu,t}\text{)} \\
&\leq \frac{1}{t}\sum_{\nu=0}^{t-1}|z-q|^{-k} M'_{q+\frac{\nu}{t}} \exp\left(\tilde{N}_{q+\frac{\nu}{t}}\Im\left(A_q^{-1}(z)\right)\right),
\end{aligned}$$

where

$$\tilde{N}_{q+\frac{\nu}{t}} := \begin{cases} t^2 N_{q+\frac{\nu}{t}}, & \text{if } N_{q+\frac{\nu}{t}} \geq 0, \\ t^{-2} N_{q+\frac{\nu}{t}}, & \text{if } N_{q+\frac{\nu}{t}} < 0. \end{cases}$$

We put

$$M_{q,t} := \max\{M_{q+\frac{\nu}{t}} : 0 \leq \nu \leq t-1\},$$

and

$$N_{q,t} := \max\{\tilde{N}_{q+\frac{\nu}{t}} : 0 \leq \nu \leq t-1\}.$$

Then, as $z \to q$,

$$|f(z;r,t)| \leq t|z-q|^{-k} M_{q,t} \exp\left(N_{q,t}\Im(A_q^{-1}(z))\right).$$

Thus, $|z-q|^k|f(z;r,t)| = \mathcal{O}(\exp\left(N_{q,t}\Im(A_q^{-1}(z))\right))$ as $z \to q$, for all $q \in \mathbb{Q}$, and so $f(z;r,t)$ is a modular form. Furthermore, if $N_q \leq 0$ (or $N_q < 0$, respectively) for all $q \in \mathbb{Q}$, then $N_{q,t} \leq 0$ ($N_{q,t} < 0$) for all $q \in \mathbb{Q}$ as well.

Therefore, if $f(z)$ is an entire modular form (or cusp form, respectively), then $f(z;r,t)$ is also an entire modular form (cusp form). This completes the proof of Theorem 1.2.

1.3 Objectives

In Chapter 2, we apply congruence restrictions to modular forms on the full modular group. We pay special attention to the Dedekind eta function,

$$\eta(z) := e^{\pi iz/12}\prod_{n>0}(1-e^{2\pi inz}),$$

and to its multiplier system, since all multiplier systems on the full modular group may be expressed in terms of the Dedekind eta multiplier system. We then combine this result with an analysis of the growth of $f(z;r,t)$ at each rational point to determine that $f(z;r,t)$ is in fact a modular form on the group $\Gamma_{0,24t}(24t^2)$. Next, we study the Fourier expansions of $f(z;r,t)$ at rational points. (In particular, we may express these Fourier expansions explicitly when t is prime.) We then apply these results by showing that the functions

$$F_{r_1,r_2,t}(t) = \eta(z;r_1,t)\tfrac{1}{\eta}(z;r_2,t),\ r_1,r_2 \in \mathbf{Z},\ t \in \mathbf{Z}^+,$$

are modular forms of weight zero with the trivial multiplier system on the group $\Gamma_{0,24t}(24t^2)$. We also determine the orders of the poles of these particular modular forms at each rational point, and we provide a pair of criterion which, if met, would guarantee that the function $F_{r_1,r_2,t}(t)$ is constant on $\mathcal{H}$.

In Chapter 3, we apply congruence restrictions to the Fourier expansion of the theta function.

$$\vartheta(z) := \sum_{n\in\mathbf{Z}} e^{\pi i n^2 z} \tag{1.16}$$

$$= \sum_{m=0}^{\infty} a_m e^{\pi i m z}, \text{ where } a_m := \begin{cases} 1, & m=0 \\ 2, & \sqrt{m}\in\mathbf{Z}^+ \\ 0, & \text{otherwise,} \end{cases} \tag{1.17}$$

is a modular form on the congruence subgroup $\Gamma_0(4)$. We use Therorem 1.2 to prove that the result of any congruence restriction $m \equiv r\ (t)$ of the standard Fourier expansion of $\vartheta(2z)$ (given by (1.17) above) is an entire modular form. This "congruence-restricted theta function" inherits the weight $\frac{1}{2}$ and the multiplier sytem v_ϑ from $\vartheta(2z)$. We also consider another type of congruence restriction of $\vartheta(2z)$, by restricting the sum given by (1.16) above to terms in an arithmetic progression on n. We apply a result from a recent paper of Sinai Robins [11] to show that these functions (called congruence restrictions of the *second kind*) also are entire modular forms. Finally, we apply these results to extend Bateman, Datskovsky and Knopp's result on $\frac{r_s^*(n)}{r_s(n)}$ to arithmetic progressions on n. We prove the following:

Theorem 3.3 Let $s,t,u\in\mathbf{Z}$ be given such that $0\le t<u$, $8|u$, $s\ge u$ and $s\equiv t\ (8)$. Put $\mathcal{M} := \{m\in\mathbf{Z} : m\ge 0, m\equiv t\ (u)\}$. Then, $\frac{r_s(m)}{r_s^*(m)}$ is not constant on $\mathcal{M}$.

Theorem 3.3 is a generalization of the main result of [3]. The cited result is the special case $u=8$ of this theorem. We also prove the following weaker result in the case $s<u$:

Theorem 3.4 Let $s,t,u\in\mathbf{Z}$ be given such that $0\le t<u$, $8|u$, $0<s<u$ and $s\equiv t\ (8)$. Put $\mathcal{M} := \{m\in\mathbf{Z} : m\ge 0, m\equiv t\ (u)\}$. Then, if $\exists m\in\mathcal{M}$

for which $\frac{r_s^*(8m+s)}{r_s(8m+s)} \neq C$, then there exists **no** positive integer N such that $\frac{r_s^*(8m+s)}{r_s(8m+s)} = C$ for all $m \in \mathcal{M}, m \geq N$.

This theorem provides an experimental method for possibly establishing the result of Theorem 3.3 even when $s < u$.

CHAPTER 2

MODULAR FORMS ON THE FULL MODULAR GROUP

2.1 The Dedekind Eta Function

Definition 2.1 *The Dedekind eta function,*

$$\eta(z) = e^{\pi i z/12} \prod_{n \geq 1} (1 - e^{2\pi i n z}), \tag{2.1}$$

is a modular form on $\Gamma(1)$ *of weight* $\frac{1}{2}$ *with multiplier system ([5], Chapter 3, Theorem 2)*

$$v_\eta \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{cases} \left(\frac{d}{c}\right)^* e^{\frac{\pi i}{12}\left[(a+d)c - bd(c^2-1) - 3c\right]}, & \text{if } c \text{ is odd}, \\ \left(\frac{c}{d}\right)_* e^{\frac{\pi i}{12}\left[(a+d)c - bd(c^2-1) + 3d - 3 - 3cd\right]} & \text{if } c \text{ is even}, \end{cases} \tag{2.2}$$

where

$$\left(\frac{d}{c}\right)^* := \begin{cases} \left(\frac{d}{|c|}\right), & d \neq 0 \\ 1, & d = 0 \end{cases}$$

and

$$\left(\frac{c}{d}\right)_* := \begin{cases} \left(\frac{c}{|d|}\right)(-1)^{\frac{sign(c)-1}{2}\frac{sign(d)-1}{2}}, & c \neq 0 \\ d, & c = 0 \end{cases}$$

(Note: In the above definition of v_η, the expression $\left(\frac{c}{d}\right)$ denotes the Jacobi symbol.)

In this section, we will show that, for $h \in \mathbf{Z}$, $\eta(z; r, t)^h$ is a cusp form of weight $\frac{h}{2}$ with multiplier system v_η^h on the group $\Gamma_{0,24t}(24t^2)$.

Theorem 2.1 *Let $r, t \in \mathbf{Z}^+$ be given, such that $0 \leq r < t$, and let $h \in \mathbf{Z}$. Then, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{0,24t}(24t^2)$ and $\nu \in \mathbf{Z}$,*

$$v_\eta^h \begin{pmatrix} a + \frac{c\nu}{t} & b + \frac{(d-a)\nu}{t} - \frac{c\nu^2}{t^2} \\ c & d - \frac{c\nu}{t} \end{pmatrix} = v_\eta^h \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Remark 2.1 *If the theorem holds for $h = 1$, then it holds for all $h \in \mathbf{Z}$. Therefore, we may assume in the following proof that $h = 1$.*

Remark 2.2 *It will suffice to show that $v_\eta \begin{pmatrix} a + \frac{c\nu}{t} & b + \frac{(d-a)\nu}{t} - \frac{c\nu^2}{t^2} \\ c & d - \frac{c\nu}{t} \end{pmatrix}$ is independent of ν; once this is shown, the theorem is proved by substuting $\nu = 0$.*

Proof We have the following formula ([5], Chapter 4, Theorem 2) for $v_\eta \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ when c is even:

$$v_\eta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\frac{c}{|d|}\right)(-1)^{\frac{\mathrm{sign}\, c-1}{2}\frac{\mathrm{sign}\, d-1}{2}} e^{\frac{\pi i}{12}[(a+d)c - bd(c^2-1) + 3d - 3 - 3cd]}, & \text{if } c \neq 0 \\ e^{\pi i b/12}, & \text{if } c = 0. \end{cases}$$

We first dispense with the case $c = 0$. To prove the theorem in this case, simply note that $M = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ (for some $b \in \mathbf{Z}$), and so

$$\begin{pmatrix} a + \frac{c\nu}{t} & b + \frac{(d-a)\nu}{t} - \frac{c\nu^2}{t^2} \\ c & d - \frac{c\nu}{t} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = M.$$

Now, suppose $c \neq 0$. Since $\eta(z)$ is periodic with period 24, the consistency condition for multiplier systems guarantees that $v_\eta(AS^\lambda) = v_\eta(A), \forall \lambda \in 24\mathbf{Z}$. Therefore,

$$v_\eta\begin{pmatrix} a+\frac{c\nu}{t} & b+\frac{(d-a)\nu}{t}-\frac{c\nu^2}{t^2} \\ c & d-\frac{c\nu}{t} \end{pmatrix} = v_\eta\begin{pmatrix} a+\frac{c\nu}{t} & \lambda(a+\frac{c\nu}{t})+b+\frac{(d-a)\nu}{t}-\frac{c\nu^2}{t^2} \\ c & \lambda c+d-\frac{c\nu}{t} \end{pmatrix}.$$

It is to our advantage to choose λ in such a way that the lower-right entry of the matrix on the right-hand side of this equation is positive, since this simplifies the calculation of v_η. In particular, choose λ such that $\lambda c + d - \frac{c\nu}{t} > 0$ (which is possible since $c \neq 0$) *and* such that $24t|\lambda$.

Thus, we have

$$v_\eta\begin{pmatrix} a+\frac{c\nu}{t} & b+\frac{(d-a)\nu}{t}-\frac{c\nu^2}{t^2} \\ c & d-\frac{c\nu}{t} \end{pmatrix} = \left(\frac{c}{\lambda c+d-\frac{c\nu}{t}}\right)\exp(\pi i\Psi(a,b,c,d)/12),$$

where we define

$$\Psi(a,b,c,d) :=$$
$$\left\{(c\lambda+a+d)c-3-3c\left(\lambda c+d-\frac{c\nu}{t}\right)+3\left(\lambda c+d-\frac{c\nu}{t}\right)-\left(\lambda\left(a+\frac{c\nu}{t}\right)+b+\frac{(d-a)\nu}{t}+\frac{c\nu^2}{t^2}\right)\left(\lambda c+d-\frac{c\nu}{t}\right)(c^2-1)\right\}.$$

This complicated exponential factor may be simplified considerably by eliminating all terms which are integer multiples of 24 from the quantity inside the brackets. Using our assumptions that $24t^2|c$, $24t|(d-a)$ and $24t|\lambda$, we achieve the following simplification:

$$v_\eta\begin{pmatrix} a+\frac{c\nu}{t} & b+\frac{(d-a)\nu}{t}-\frac{c\nu^2}{t^2} \\ c & d-\frac{c\nu}{t} \end{pmatrix} = \left(\frac{c}{\lambda c+d-\frac{c\nu}{t}}\right)e^{\pi i(bd+3d-3)/12}.$$

Recall that our objective is to show that the above quantity is *independent of* ν. Since the exponential factor is clearly independent of ν, it remains only to show that $\left(\frac{c}{\lambda c+d-\frac{c\nu}{t}}\right)$ is also independent of ν.

We put $c_1 := \frac{c}{t^2}$ and $c_2 := \frac{c_1}{2^\alpha} = \frac{c}{2^\alpha t^2}$, where $\alpha \in \mathbf{Z}$ and c_2 is odd. Since $24t^2|c$, we know that $24|c_1$, $3|c_2$ and $\alpha \geq 3$.

Now, substitute:

$$\begin{aligned}
\left(\frac{c}{\lambda c + d - \frac{c\nu}{t}}\right) &= \left(\frac{t^2}{\lambda c + d - \frac{c\nu}{t}}\right)\left(\frac{c_1}{\lambda c + d - \frac{c\nu}{t}}\right) \\
&= \left(\frac{2^\alpha}{\lambda c + d - \frac{c\nu}{t}}\right)\left(\frac{c_2}{\lambda c + d - \frac{c\nu}{t}}\right) \\
&= \left(\frac{2}{\lambda c + d - \frac{c\nu}{t}}\right)^\alpha \left(\frac{c_2}{\lambda c + d - \frac{c\nu}{t}}\right). \qquad (2.3) \\
&= \left(\frac{2}{d}\right)^\alpha \left(\frac{\lambda c + d - \frac{c\nu}{t}}{c_2}\right)(-1)^{\frac{c_2-1}{2}\frac{\lambda c+d-\frac{c\nu}{t}-1}{2}} \qquad (2.4) \\
&= \left(\frac{2}{d}\right)^\alpha \left(\frac{d}{c_2}\right)(-1)^{\frac{c_2-1}{2}\frac{d-1}{2}},
\end{aligned}$$

which is independent of ν. This proves the theorem.

Remark 2.3 *For odd $n \in \mathbf{Z}^+$,*

$$\left(\frac{2}{n}\right) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \ (8) \\ -1 & \text{else.} \end{cases}$$

That is, $\left(\frac{2}{n}\right)$ depends only on the value of n modulo 8. Therefore, $\left(\frac{2}{\lambda c+d-\frac{c\nu}{t}}\right) = \left(\frac{2}{d}\right)$. This observation, together with the Law of Quadratic Reciprocity, allow us to proceed from (2.3) to (2.4) in the above argument.

Corollary 2.1 *For all $h \in \mathbf{Z}$, $\eta^h(Mz; r, t) = v_\eta^h(M)(cz+d)^{h/2}\eta^h(z; r, t)$ is a cusp form on $\Gamma_{0,24t}(24t^2)$ of weight $\frac{h}{2}$ with multiplier system v_η^h.*

Proof This follows from Theorems 2.2 and 1.1.

2.2 Modular Forms of Half-Integer Weight

Suppose that $f(z)$ is a modular form on $\Gamma(1)$ of half-integer weight – that is, $2k \in \mathbf{Z}$ – with nontrivial multiplier system v. In this section, we will show that $f(z;r,t)$ is a modular form on $\Gamma_{0,24t}(24t^2)$ of weight k with multiplier system v. Furthermore, if $f(z)$ is an entire modular form (or cusp form, respectively) on $\Gamma(1)$, then $f(z;r,t)$ is an entire modular form (cusp form) on $\Gamma_{0,24t}(24t^2)$.

Theorem 2.2 *If v is a multiplier system of weight k on $\Gamma(1)$, with $2k \in \mathbf{Z}$, then*

$$v\begin{pmatrix} a+\frac{c\nu}{t} & b+\frac{(d-a)\nu}{t}-\frac{c\nu^2}{t^2} \\ c & d-\frac{c\nu}{t} \end{pmatrix} = v\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{0,24t}(24t^2)$ and $\nu \in \mathbf{Z}$.

Proof If v is a multiplier system of weight k, with $2k \in \mathbf{Z}$, then v_η^{2k} is the multiplier system for $\eta^{2k}(z)$, and therefore $v\ \overline{v_\eta}^{2k}$ is a multiplier system of weight *zero* on the full modular group. That is, we can write:

$$v = v_\eta^{2k} w_{\beta,\gamma},$$

where $w_{\beta,\gamma}$ is one of the six multiplier systems of weight zero on the full modular group as given by [10]. In particular,

$$w_{\beta,\gamma} := w_1^\beta w_2^\gamma,$$

where

$$0 \le \beta \le 1, \quad 0 \le \gamma \le 2,$$

$$w_1\begin{pmatrix} a & b \\ c & d \end{pmatrix} := e^{\pi i(bd-ac+bc)}, \quad \text{and}$$

$$w_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} := e^{\frac{2\pi i}{3}(a+d)(b-c)(ad+bc)}.$$

As a consequence of the previous theorem, it remains only to show that

$$w_1\begin{pmatrix} a+\frac{c\nu}{t} & b+\frac{(d-a)\nu}{t}-\frac{c\nu^2}{t^2} \\ c & d-\frac{c\nu}{t} \end{pmatrix}$$

and

$$w_2\begin{pmatrix} a+\frac{c\nu}{t} & b+\frac{(d-a)\nu}{t}-\frac{c\nu^2}{t^2} \\ c & d-\frac{c\nu}{t} \end{pmatrix}$$

are independent of ν when $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{0,24t}(24t^2)$.

Since $24t^2|c$, we may simplify w_1 and w_2 as follows:

$$\begin{aligned} w_1\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= e^{\pi i b}, \text{ and} \\ w_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= e^{\frac{2\pi i}{3}abd(a+d)}. \end{aligned}$$

Therefore,

$$\begin{aligned} w_1\begin{pmatrix} a+\frac{c\nu}{t} & b+\frac{(d-a)\nu}{t}-\frac{c\nu^2}{t^2} \\ c & d-\frac{c\nu}{t} \end{pmatrix} &= e^{\pi i(b+\frac{(d-a)\nu}{t}-\frac{c\nu^2}{t^2})} \\ &= e^{\pi i(b+24n)}\text{, for some integer } n \\ &= e^{\pi i b}, \end{aligned}$$

and

$$\begin{aligned} w_2\begin{pmatrix} a+\frac{c\nu}{t} & b+\frac{(d-a)\nu}{t}-\frac{c\nu^2}{t^2} \\ c & d-\frac{c\nu}{t} \end{pmatrix} &= e^{\frac{2\pi i}{3}(a+\frac{c\nu}{t})(b+\frac{(d-a)\nu}{t}-\frac{c\nu^2}{t^2})(d-\frac{c\nu}{t})(a+d)} \\ &= e^{\frac{2\pi i}{3}(a+24n_1)(b+24n_2)(c+24n_3)(a+d)} \\ &\qquad \text{(where each } n_i \text{ is an integer)} \\ &= e^{\frac{2\pi i}{3}abd(a+d)}, \end{aligned}$$

as desired. This proves the theorem.

Corollary 2.2 *Let $f(z)$ be a modular form on $\Gamma(1)$ of half-integer weight k with nontrivial multiplier system v. Then, $f(z;r,t)$ is a modular form on*

$\Gamma_{0,24t}(24t^2)$ *of weight* k *with multiplier system* v*. Furthermore, if* $f(z)$ *is an entire modular form (or cusp form, respectively), then* $f(z;r,t)$ *is an entire modular form (cusp form).*

Proof This follows from Theorems 2.2 and 1.2.

If the multiplier system of $f(z)$ is trivial, we have the following stronger result.

Corollary 2.3 *If* $f(z)$ *is a modular form on* $\Gamma(1)$ *of half-integer weight* k *with* trivial *multiplier system* v*, then* $f(z;r,t)$ *is a modular form (or entire modular form, or cusp form, as determined by* $f(z)$*) on the group* $\Gamma_{0,t}(t^2)$*.*

Proof This follows from Theorem 1.2, Proposition 1.4 and Remark 1.14.

2.3 Growth at Rational Points

Throughout the remainder of this chapter, let $a, c, r, t \in \mathbf{Z}$ be given such that $c > 0$, $\gcd(a, c) = 1$ and $0 \leq r < t \in \mathbf{Z}$. Put $q := \frac{a}{c}$.

Suppose $f(z)$ is a modular form on $\Gamma(1)$ of half-integer weight. In this section, we will study the growth of $f(z;r,t)$ at q. That is, we will investigate the order of the zero or pole of $f(z;r,t)$ at q for each $q \in \mathbf{Q}$.

We proceed from Equation (1.10) by determing the expansion of $f(z)$ at $\frac{a}{c} + \frac{\nu}{t}$ for each $\nu \in \mathbf{Z}, 0 \leq \nu \leq t-1$. Given ν, we choose $A_\nu = \begin{pmatrix} a_\nu & b\nu \\ c_\nu & d_\nu \end{pmatrix}$, as described by Equation 1.15. (Note that, to simplify notation, we will use A_ν rather than $A_{q+\frac{\nu}{t}}$ throughout this chapter.)

Now we determine (as far as possible at this point) the expansion of f at $\frac{a}{c}+\frac{\nu}{t}$:

$$\begin{aligned} f(z) &= f(A_\nu(A_\nu^{-1}z)) \\ &= (c_\nu A_\nu^{-1}z + d_\nu)^k v(A_\nu) f(A_\nu^{-1}z) \\ &= (-c_\nu z + a_\nu)^{-k} v(A_\nu) f(A_\nu^{-1}z) \\ &= e^{\pi i k}(c_\nu)^{-k}\left(z - \left(\frac{a}{c}+\frac{\nu}{t}\right)\right)^{-k} v(A_\nu)\sum_n a_n e^{2\pi i(n+\kappa)A_\nu^{-1}z} \\ &= e^{\pi i k}\left(z - \left(\frac{a}{c}+\frac{\nu}{t}\right)\right)^{-k}\left(\frac{\delta_\nu}{ct}\right)^k v(A_\nu)\sum_n a_n e^{2\pi i(n+\kappa)A_\nu^{-1}z}, \end{aligned}$$

Therefore, for z near $\frac{a}{c}$, equation (1.10) becomes

$$f(z;r,t) = \tag{2.5}$$
$$e^{\pi i k}c^{-k}t^{-k-1}\left(z-\frac{a}{c}\right)^{-k}\sum_{\nu=0}^{t-1} e^{-2\pi i(r+\kappa)\nu/t}\delta_\nu^k v(A_\nu)\sum_n a_n e^{2\pi i(n+\kappa)A_\nu^{-1}(z+\frac{\nu}{t})}.$$

We wish to manipulate this expansion into the form required by Equation (1.3); that is, the exponential terms should include the expression $A^{-1}z$, rather than $A_\nu^{-1}\left(z+\frac{\nu}{t}\right)$. This is the object of the following lemma.

Definition 2.2 *Given t,*

$$\Delta := \{\delta_\nu : \nu \in \mathbf{Z}, 0 \le \nu \le t-1\},$$

and

$$\Upsilon_\delta := \{\nu \in \mathbf{Z}, 0 \le \nu \le t-1 : \delta_\nu = \delta\}.$$

Lemma 2.1

$$f(z;r,t) = \tag{2.6}$$
$$e^{\pi ik}c^{-k}t^{-k-1}(z-q)^{-k}\sum_{\delta\in\Delta}\sum_{n\in\mathbf{Z}}\exp\left(\frac{2\pi i(n+\kappa)\delta^2}{t^2}A^{-1}(z)\right)\times$$
$$a_n\delta^{-k}\sum_{\nu\in\Upsilon_\delta}v(A_\nu)\exp\left(\frac{-2\pi i(r+\kappa)\nu}{t}\right)\exp\left(\frac{2\pi i(n+\kappa)\delta(d\delta-td_\nu)}{ct^2}\right).$$

Proof We apply Lemma 1.4 to (2.5):

$$f(z;r,t) =$$
$$e^{\pi ik}c^{-k}t^{-k-1}\left(z-\frac{a}{c}\right)^{-k}\sum_{\nu=0}^{t-1}e^{-2\pi i(r+\kappa)\nu/t}\delta_\nu^k v(A_\nu)\times$$
$$\sum_n a_n\exp\left(2\pi i(n+\kappa)\left(\frac{\delta_\nu^2}{t^2}A^{-1}(z)+\frac{\delta_\nu}{ct^2}(d\delta_\nu-td_\nu)\right)\right)$$
$$= e^{\pi ik}c^{-k}t^{-k-1}\left(z-\frac{a}{c}\right)^{-k}\sum_n a_n\times$$
$$\sum_{\nu=0}^{t-1}v(A_\nu)e^{-2\pi i(r+\kappa)\nu/t}\delta_\nu^k\exp\left(2\pi i(n+\kappa)\left(\frac{\delta_\nu^2}{t^2}A^{-1}(z)+\frac{\delta_\nu(d\delta_\nu-td_\nu)}{ct^2}\right)\right)$$
$$= e^{\pi ik}c^{-k}t^{-k-1}\left(z-\frac{a}{c}\right)^{-k}\sum_{\delta\in\Delta}\sum_n\exp\left(\frac{2\pi i(n+\kappa)\delta^2}{t^2}A^{-1}(z)\right)\times$$
$$a_n\delta^k\sum_{\nu\in\Upsilon_\delta}v(A_\nu)\exp\left(\frac{-2\pi i(r+\kappa)\nu}{t}\right)\exp\left(\frac{2\pi i(n+\kappa)\delta(d\delta-td_\nu)}{ct^2}\right).$$

This completes the proof of Lemma 2.1.

Theorem 2.3 *Let δ_{min} and δ_{max} denote the minimum and maximum values of $\delta\in\Delta$, respectively. If $n_0+\kappa\ge 0$, then $f(z;r,t)$ has a zero of order at least $(n_0+\kappa)\frac{\delta_{min}^2}{t^2}$ at each $q\in\mathbf{Q}$. If $n_0+\kappa<0$, then $f(z;r,t)$ has either a zero (of nonnegative order) or a pole of order at most $(n_0+\kappa)\frac{\delta_{max}^2}{t^2}$ at each $q\in\mathbf{Q}$.*

Proof The order of the zero or pole at q is $|N_q|$, where N_q is the minimal real number such that $|z-q|^k|f(z;r,t)|=\mathcal{O}(\exp(N_q\Im(A^{-1}z))$, as $\Im(A^{-1}z)\to\infty$.

If $N_q \geq 0$, then we say $f(z; r, t)$ has a zero of order N_q at q. If $N_q < 0$, then we say $f(z; r, t)$ has a pole of order $-N_q$ at q.

The terms which appear in (2.6) are of the form $\exp\left(2\pi i \frac{(n+\kappa)\delta^2}{t^2} A^{-1}(z)\right)$ times some constant factor independent of z. Therefore, we determine N_q by minimizing $\frac{(n+\kappa)\delta^2}{t^2}$.

If $f(z)$ is an entire modular form or a cusp form, then $n_0 + \kappa \geq 0$, so we minimize $\frac{(n+\kappa)\delta^2}{t^2}$ by minimizing $\frac{\delta^2}{t^2}$. Therefore, $|z - q|^k |f(z; r, t)| = \mathcal{O}(\exp\left(\frac{(n_0+\kappa)\delta_{min}^2}{t^2} \Im(A^{-1}z)\right)$. Now, we must allow for the possibility that the sum over $\nu \in \Upsilon_{\delta_{min}}$ *may* be zero when $n = n_0$. Therefore, in this case $N_q \geq \frac{(n_0+\kappa)\delta_{min}^2}{t^2}$, with equality if and only if

$$\sum_{\nu \in \Upsilon_{\delta_{min}}} v(A_\nu) \exp\left(\frac{-2\pi i(r+\kappa)\nu}{t}\right) \exp\left(\frac{2\pi i(n_0+\kappa)\delta(d\delta_{min} - td_\nu)}{ct^2}\right) \neq 0.$$

If $f(z)$ is a nonentire modular form on $\Gamma(1)$, then $n_0 + \kappa < 0$, so we minimize $\frac{(n+\kappa)\delta^2}{t^2}$ by maximizing $\frac{\delta^2}{t^2}$. Therefore,

$$|z - q|^k |f(z; r, t)| = \mathcal{O}(\exp\left(\frac{(n_0+\kappa)\delta_{max}^2}{t^2} \Im(A^{-1}z)\right).$$

As in the previous paragraph, we must allow for the possibility that the sum over $\nu \in \Upsilon_{\delta_{max}}$ may vanish when $n = n_0$; therefore, $N_q \geq \frac{(n_0+\kappa)\delta_{max}^2}{t^2}$, with equality if and only if

$$\sum_{\nu \in \Upsilon_{\delta_{max}}} v(A_\nu) \exp\left(\frac{-2\pi i(r+\kappa)\nu}{t}\right) \exp\left(\frac{2\pi i(n_0+\kappa)\delta(d\delta_{max} - td_\nu)}{ct^2}\right) \neq 0.$$

This completes the proof of Theorem 2.3.

2.4 Fourier Expansions

In this section, we will derive an expression for the Fourier expansion of $f(z;r,t)$ at $q = \frac{a}{c}$. We will also give a more explicit expression in the case where t is prime.

2.4.1 Expansions at Rational Points

If $f(z)$ is a modular form on $\Gamma(1)$ of half-integer weight k with multiplier system v, then $f(z)$ satisfies the transformation law given in Corollary 2.2 on $\Gamma_{0,24t}(24t^2)$, or on $\Gamma_{0,t}(t^2)$ if v is trivial. Therefore ([5], Chapter 2, Theorem 4), at q, $f(z;r,t)$ has a Fourier expansion of the form

$$f(z;r,t) = (z-q)^{-k} \sum_n a_n(q) e^{2\pi i(n+\kappa_q)(A^{-1}z)/\lambda_q}, \tag{2.7}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, λ_q is the smallest positive integer such that

$$AS^{\lambda_q}A^{-1} \in \begin{cases} \Gamma_{0,t}(t^2), & \text{if } v \equiv 1 \\ \Gamma_{0,24t}(24t^2), & \text{otherwise,} \end{cases}$$

and κ_q is the unique real number such that $0 \leq \kappa_q < 1$ and $e^{2\pi i\kappa_q} = v(AS^{\lambda_q}A^{-1})$.

Lemma 2.2 *If v is a nontrivial multiplier system on $\Gamma(1)$, then we may write $f(z;r,t)$ in the form*

$$f(z;r,t) = (z-q)^{-k} \sum_n a_n(q) e^{2\pi i(n+\kappa_q)(A^{-1}z)/\lambda_q},$$

where

$$\lambda_q = \frac{24t^2}{\gcd(24t^2, c\gcd(2t,c))}. \tag{2.8}$$

Remark 2.4 *In this lemma we determine λ_q when v is nontrivial; in this case, λ_q is the width of $\Gamma_{0,24t}(24t^2)$ at $\frac{a}{c}$.*

Proof We know that λ_q is defined to be the minimal positive integer for which $AS^{\lambda_q}A^{-1} \in \Gamma_{0,24t}(24t^2)$. Since $AS^{\lambda_q}A^{-1} = \begin{pmatrix} 1-\lambda ac & a^2\lambda \\ -c^2\lambda & 1+\lambda ac \end{pmatrix}$, this is equivalent to finding $\lambda_q \in \mathbf{Z}^+$ minimal such that $24t^2|c^2\lambda$ and $24t|2ac\lambda$. We may combine these two conditions on λ_q into one by rewriting them in the form $48at^2|2ac^2\lambda_q$ and $24ct|2ac^2\lambda_q$, respectively. From these two conditions, we see that we must choose $\lambda_q \in \mathbf{Z}^+$ minimal such that $\mathrm{lcm}(48at^2, 24ct)|2ac^2\lambda_q$:

$$\begin{aligned}
\mathrm{lcm}(48at^2, 24ct)|2ac^2\lambda_q &\iff \frac{(48at^2)(24ct)}{\gcd(48at^2, 24ct)} \;\Big|\; 2ac^2\lambda_q \\
&\iff \frac{2(24^2)act^3}{24t\gcd(2at, c)} \;\Big|\; 2ac^2\lambda_q \\
&\iff \frac{48act^2}{\gcd(2t, c)} \;\Big|\; 2ac^2\lambda_q \\
&\iff \frac{24t^2}{\gcd(2t, c)} \;\Big|\; c\lambda_q \\
&\iff 24t^2 \mid \gcd(2t, c)c\lambda_q. \qquad (2.9)
\end{aligned}$$

The smallest positive integer λ_q which satisfies (2.9) is $\frac{24t^2}{\gcd(24t^2, c\gcd(2t,c))}$. This proves the lemma.

Lemma 2.3 *If v is the trivial multiplier system on $\Gamma(1)$, then we may write $f(z;r,t)$ in the form*

$$f(z;r,t) = (z-q)^{-k}\sum_n a_n(q)e^{2\pi i(n+\kappa_q)(A^{-1}z)/\lambda_q},$$

where

$$\lambda_q = \frac{t^2}{\gcd(t^2, c\gcd(2t,c))}. \qquad (2.10)$$

Remark 2.5 *In this lemma we calculate λ_q when v is trivial; in this case, λ_q is the width of $\Gamma_{0,t}(t^2)$ at $\frac{a}{c}$.*

Proof Proceeding as in the previous lemma, we note that, in this case, λ_q is defined to be the minimal positive integer for which $AS^{\lambda_q}A^{-1} \in \Gamma_{0,t}(t^2)$. This is equivalent to finding $\lambda_q \in \mathbf{Z}^+$ minimal such that $t^2|c^2\lambda$ and $t|2ac\lambda$. We may combine these two conditions on λ_q into the single condition: $\lambda_q \in \mathbf{Z}^+$ minimal such that $\text{lcm}(2at^2, ct)|2ac^2\lambda_q$:

$$\begin{aligned}
\text{lcm}(2at^2, ct)|2ac^2\lambda_q &\iff \left.\frac{(2at^2)(ct)}{\gcd(2at^2, ct)}\,\right|\, 2ac^2\lambda_q \\
&\iff \left.\frac{act^2}{\gcd(2t, c)}\,\right|\, 2ac^2\lambda_q \\
&\iff \left.\frac{t^2}{\gcd(2t, c)}\,\right|\, c\lambda_q \\
&\iff t^2|\gcd(2t, c)c\lambda_q. \qquad (2.11)
\end{aligned}$$

The smallest positive integer λ_q which satisfies (2.11) is $\frac{t^2}{\gcd(t^2, c\gcd(2t,c))}$. This proves the lemma.

Corollary 2.4 *Put $c' = \frac{c}{(c,t)}$ and $t' = \frac{t}{(c,t)}$. Then,*

$$\lambda_q = \begin{cases} \frac{24t'^2}{\gcd(3,c')}, & \text{if } v \not\equiv 1, c' \text{ odd} \\ \frac{12t'^2}{\gcd(12,c')}, & \text{if } v \not\equiv 1, c' \text{ even} \\ t'^2, & \text{if } v \equiv 1, c' \text{ odd or } t' \text{ odd} \\ \frac{t'^2}{2}, & \text{if } v \equiv 1, c' \text{ and } t' \text{ both even.} \end{cases} \qquad (2.12)$$

Proof We first consider the case $v \not\equiv 1$:

$$\begin{aligned}
\lambda_q &= \frac{24t^2}{\gcd(24t^2, c\gcd(2t,c))} \\
&= \frac{24t^2}{\gcd(c,t)^2 \gcd(24t'^2, c'\gcd(2t',c'))} \\
&= \frac{24t'^2}{\gcd(24t'^2, c'\gcd(2,c'))} \\
&= \begin{cases} \frac{24t'^2}{\gcd(24,c')}, & \text{if } c' \text{ is odd,} \\ \frac{24t'^2}{\gcd(24,2c')}, & \text{if } c' \text{ is even.} \end{cases}
\end{aligned}$$

Next, we consider the case $v \equiv 1$:

$$\begin{aligned}
\lambda_q &= \frac{t^2}{\gcd(t^2, c\gcd(2t,c))} \\
&= \frac{t'^2}{\gcd(t'^2, c'\gcd(2,c'))} \\
&= \frac{t'^2}{\gcd(t'^2, \gcd(2,c'))} \\
&= \begin{cases} t'^2, & \text{if } c' \text{ is odd or if } t' \text{ is odd,} \\ \frac{t'^2}{2}, & \text{if } c' \text{ and } t' \text{ are both even.} \end{cases}
\end{aligned}$$

Since v is a multiplier system on $\Gamma(1)$, we may establish the following relationship between λ_q and κ_q:

Lemma 2.4 *If v is a multiplier system on $\Gamma(1)$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ and $\lambda \in \mathbf{Z}$, then $v(AS^\lambda A^{-1}) = v(S)^\lambda$.*

Corollary 2.5 *If v is a multiplier system on $\Gamma(1)$, $\lambda \in \mathbf{Z}$, and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ with $c > 0$, then*

$$e^{2\pi i\kappa_q} = e^{2\pi i\lambda_q\kappa}. \tag{2.13}$$

Proof of Lemma 2.4 This follows directly from Chapter 2 of [5]. In particular, Theorem 6 of Chapter 2 shows that $v(ASA^{-1}) = v(S) = \kappa$, since (under $\Gamma(1)$) q is equivalent to $i\infty$. From Theorem 4, it follows that for *any* $\lambda \in \mathbf{Z}$,

$$f(z) = (z-q)^{-k} \sum_n a_n(q,\lambda) e^{2\pi i(n+\kappa(\lambda))A^{-1}(z)/\lambda},$$

where we define $\kappa(\lambda)$ as the unique real number such that $0 \leq \kappa(\lambda) < 1$ and $e^{2\pi i\kappa(\lambda)} = v(AS^{\lambda}A^{-1})$, and $a_n(q,\lambda)$ as the n^{th} coefficient of the Fourier expansion corresponding to λ. Note that this Fourier expansion must be independent of the choice of λ; thus, for *any* $\lambda \in \mathbf{Z}$, we must have

$$\sum_n a_n(q,1) e^{2\pi i(n+\kappa)A^{-1}(z)} = \sum_n a_n(q,\lambda) e^{2\pi i(n+\kappa(\lambda))A^{-1}(z)/\lambda}.$$

Therefore, for any $n \in \mathbf{Z}$ for which $a_n(q,1) \neq 0$, there must exist $n' \in \mathbf{Z}$ such that $\frac{n'+\kappa(\lambda)}{\lambda} = n + \kappa$. Thus, $\kappa(\lambda) - \lambda\kappa \in \mathbf{Z}$, and so $e^{2\pi i\kappa(\lambda)} = e^{2\pi i\lambda\kappa}$. This proves the lemma. Corollary 2.5 is proved by substituting λ_q for λ in Lemma 2.4.

Corollary 2.5 implies that $\lambda_q\kappa - \kappa_q = h_q$ for some $h_q \in \mathbf{Z}$,; therefore, we may rewrite equation (2.7) as follows:

$$\begin{aligned} f(z;r,t) &= (z-q)^{-k} \sum_n a_n(q) e^{2\pi i(n+\lambda_q\kappa-h_q)(A^{-1}z)/\lambda_q} \\ &= (z-q)^{-k} \sum_n a_{n+h_q}(q) e^{2\pi i(n+\lambda_q\kappa)(A^{-1}z)/\lambda_q}. \end{aligned}$$

To simplify notation, define $b_n(q) := a_{n+h_q}(q)$. Then, the above may be written:

$$f(z;r,t) = (z-q)^{-k} \sum_n b_n(q) e^{2\pi i(\frac{n}{\lambda_q}+\kappa)(A^{-1}z)}. \tag{2.14}$$

We now have two expressions for $f(z;r,t)$, given by Equations (2.6) and (2.14). Combining these two expressions into one equation gives us

$$
\begin{aligned}
& e^{\pi ik}t^{-k-1}c^{-k}(z-q)^{-k}\sum_{n\in\mathbf{Z}}\sum_{\delta\in\Delta}\exp\left(\frac{2\pi i(n+\kappa)\delta^2}{t^2}A^{-1}(z)\right)\times \qquad (2.15)\\
& \quad a_n\delta^{-k}\sum_{\nu\in\Upsilon_\delta}v(A_\nu)\exp\left(\frac{-2\pi i(r+\kappa)\nu}{t}\right)\exp\left(\frac{2\pi i(n+\kappa)\delta(d\delta-td_\nu)}{ct^2}\right)\\
= \; & (z-q)^{-k}\sum_m b_m(q)e^{2\pi i(\frac{m}{\lambda_q}+\kappa)(A^{-1}z)}.
\end{aligned}
$$

Therefore,

$$
b_m(q) = e^{\pi ik}t^{-k-1}c^{-k}\times
$$

$$
\sum_{\substack{n\in\mathbf{Z},\delta\in\Delta \\ \frac{m}{\lambda_q}+\kappa=(n+\kappa)\frac{\delta^2}{t^2}}} a_n\delta^{-k}\sum_{\nu\in\Upsilon_\delta}v(A_\nu)\exp\left(\frac{-2\pi i(r+\kappa)\nu}{t}\right)\exp\left(\frac{2\pi i(n+\kappa)\delta(d\delta-td_\nu)}{ct^2}\right).
$$

2.4.2 Congruence Restrictions of Prime Modulus

Suppose t is prime. Given $a,c\in\mathbf{Z}$ such that $\gcd(a,c)=1$ and $c>0$, we select $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma(1)$ and determine (as precisely as possible) the entries of $A_\nu=\begin{pmatrix} a_\nu & b_\nu\\ c_\nu & d_\nu\end{pmatrix}$ for $0\le\nu\le t-1$. In particular, given a_ν, we must choose $d_\nu\in\mathbf{Z}$ such that $a_\nu d_\nu\equiv 1\ (c_\nu)$.

Lemma 2.5 *Suppose t is prime and $1\le\nu\le t-1$. If $c|t$, we may choose A_ν in such a way that*

$$
A_\nu(z+\frac{\nu}{t}) = A^{-1}(z)+h_\nu,
$$

where

$$\begin{aligned} h_\nu &= 0, \quad \text{if } a \equiv \tfrac{-\nu c}{t} \ (t); \\ h_\nu &\equiv (a + \frac{\nu c}{t})^{-1} \frac{\nu c d}{t} \ (c), \ \text{otherwise.} \end{aligned}$$

If $c \not| t$, then we may choose A_ν such that

$$A_\nu(z + \frac{\nu}{t}) = A^{-1}(z) + \frac{d}{ct^2} - \left(\frac{h_\nu}{c} + \frac{j_\nu}{t}\right),$$

where $h_\nu, j_\nu \in \mathbf{Z}$ are chosen such that

$$h_\nu a t^2 + j_\nu \nu c^2 \equiv 1 \ (ct).$$

Remark 2.6 *Note that $A_\nu = A$ when $\nu = 0$.*

Proof First we determine δ_ν when t is prime:

$$\begin{aligned} \delta_\nu &= \gcd(at + \nu c, ct) \\ &= \gcd(c,t)\gcd(at' + \nu c', ct') \\ &= \gcd(c,t)\gcd(a + \nu c', \gcd(c,t)c') \\ &= \gcd(c,t)\gcd(a + \nu c', \gcd(c,t)). \end{aligned}$$

Therefore,

$$\delta_\nu = \begin{cases} 1, & \text{if } t \not| c \\ t, & \text{if } t|c \text{ but } t \not| (a + \frac{\nu c}{t}) \\ t^2, & \text{if } t|c \text{ and } t|(a + \frac{\nu c}{t}). \end{cases} \qquad (2.16)$$

We consider separately each of the three cases listed above.

Case 1 Suppose $a \equiv \frac{-\nu c}{t}\ (t)$. (This implies that $t|c$, but $t^2 \not| c$.) In this case, $\delta_\nu = t^2$, and so we may select $A_\nu = \begin{pmatrix} \frac{at+\nu c}{t^2} & bt+\nu d \\ \frac{c}{t} & dt \end{pmatrix}$. This choice of A_ν gives us

$$\frac{\delta_\nu(d\delta_\nu - td_\nu)}{ct^2} = \frac{t^2(dt^2 - dt^2)}{ct^2} = 0.$$

Therefore, by Lemma 1.4, we have $A_\nu^{-1}(z + \frac{\nu}{t}) = A^{-1}(z)$.

Case 2 Suppose $t|c$ but $a \not\equiv \frac{-\nu c}{t}\ (t)$. In this case, $\delta_\nu = t$, and so $a_\nu = a + \frac{\nu c}{t}$, $c_\nu = c$, and $\gcd(c, a + \frac{\nu c}{t}) = 1$.

Choose $h_\nu \in \mathbf{Z}$ such that $h_\nu \equiv (a + \frac{\nu c}{t})^{-1}\frac{\nu cd}{t}\ (c)$. (Here the superscript -1 indicates the multiplicative inverse, modulo c.) Then,

$$\begin{aligned}(a + \frac{\nu c}{t})(d - ch_\nu) &= ad + \frac{\nu cd}{t} - h_\nu(a + \frac{\nu c}{t}) \\ &\equiv 1 + \frac{\nu cd}{t} - h_\nu(a + \frac{\nu c}{t})\ (c) \\ &\equiv 1 + \frac{\nu cd}{t} - \frac{\nu cd}{t}\ (c) \\ &\equiv 1\ (c), \quad \text{as required.}\end{aligned}$$

Therefore, we may put $d_\nu = d - ch_\nu$, and so by Lemma 1.4,

$$\begin{aligned}A_\nu^{-1}(z + \frac{\nu}{t}) &= A^{-1}(z) + \frac{\delta_\nu(d\delta_\nu - td_\nu)}{ct^2} \\ &= A^{-1}(z) + \frac{t(dt - td + tch_\nu)}{ct^2} \\ &= A^{-1}(z) + \frac{t(tch_\nu)}{ct^2} \\ &= A^{-1}(z) + h_\nu.\end{aligned}$$

Remark 2.7 *Note that if $t^2|c$, then $\delta_\nu = t$ for all values of ν. In this case, we may choose $h_\nu = 0$.*

Case 3 Suppose $t \not| c$. In this case, $\delta - \nu = 1$, and so $a_\nu = at + \nu c$ and $c_\nu = ct$.

Claim There exist $h_\nu, j_\nu \in \mathbf{Z}$ such that $h_\nu at^2 + j_\nu \nu c^2 \equiv 1\ (ct)$.

Proof of Claim Since $\gcd(at^2, \nu c^2) = \gcd(at^2, \nu) = \gcd(a, \nu)$, there exist $h'_\nu, j'_\nu \in \mathbf{Z}$ such that $h'_\nu at^2 + j'_\nu \nu c^2 = \gcd(a_\nu)$. Now, since $\gcd(a, c) = 1$ and $\gcd(\nu, t) = 1$, any common divisor of a and ν must be relatively prime to both c and t; therefore, $\gcd(a, \nu)$ is relatively prime to ct, and so there exists $k_\nu \in \mathbf{Z}$ such that $k_\nu \gcd(a, \nu) \equiv 1\ (ct)$. Put $h_\nu := k_\nu h'_\nu, j_\nu := k_\nu j'_\nu$; then, $h_\nu at^2 + j_\nu \nu c^2 \equiv 1\ (ct)$, as desired. This proves the claim.

(Note that, once we find such a pair h_ν, j_ν, we also have $H_\nu at^2 + J_\nu \nu c^2 \equiv 1\ (ct)$ for all H_ν, J_ν such that $H_\nu \equiv h_\nu\ (c), J_\nu \equiv j_\nu\ (t)$. That is, h_ν and j_ν are determined only up to their residue classes modulo c and modulo t, respectively.)

Put $d_\nu = h_\nu t + j_\nu c$. Then,

$$\begin{aligned} a_\nu d_\nu &= (at + \nu c)(h_\nu t + j_\nu c) \\ &= ah_\nu t^2 + \nu j_\nu c^2 + (aj_\nu + \nu h_\nu)ct \\ &\equiv ah_\nu t^2 + \nu j_\nu c^2\ (ct) \\ &\equiv 1\ (ct), \end{aligned}$$

as required. Therefore, by Lemma 1.4,

$$\begin{aligned} A_\nu^{-1}(z + \frac{\nu}{t}) &= A^{-1}(z) + \frac{\delta_\nu(d\delta_\nu - td_\nu)}{ct^2} \\ &= A^{-1}(z) + \frac{d - t(h_\nu t + j_\nu c)}{ct^2} \\ &= A^{-1}(z) + \frac{d}{ct^2} - \left(\frac{h_\nu}{c} + \frac{j_\nu}{t}\right) \end{aligned}$$

This proves the lemma.

We may now combine the results of Lemmas 2.1 and 2.5 to arrive at the following expansions for $f(z;r,t)$ at q. (Note: we put $a_n := 0$ for all $n \notin \mathbf{Z}$ and for all $n \in \mathbf{Z}$ such that $n < n_0$.)

Case 1 $t^2|c$.

In this case, Lemma 2.1 simplifies as follows:

$$\begin{aligned} & f(z;r,t) \\ = \quad & e^{\pi i k} t^{-2k-1} c^{-k} (z-q)^{-k} \sum_{\nu=0}^{t-1} v(A_\nu) e^{-2\pi i \frac{(r+\kappa)\nu}{t}} \sum_{n\in\mathbf{Z}} a_n e^{2\pi i (n+\kappa) A^{-1}(z)} \qquad (2.17) \\ = \quad & e^{\pi i k} t^{-2k-1} c^{-k} (z-q)^{-k} \sum_{\nu=0}^{t-1} v(A_\nu) e^{-2\pi i \frac{(r+\kappa)\nu}{t}} \sum_{m} a_{\frac{m}{\lambda_q}} e^{2\pi i (\frac{m}{\lambda_q}+\kappa) A^{-1}(z)}. \end{aligned}$$

Remark 2.8 *From (2.17), it follows that the expansion of $f(z;r,t)$ at q is essentially that of $f(z)$ at $i\infty$. In particular, if $f(z;r,t)$ is nontrivial, then $f(z;r,t)$ has a zero (or pole) at q of the same order as the zero (or pole) of $f(z)$ at $i\infty$. (The order of this zero or pole is, of course, $|n_0+\kappa|$.)*

Case 2 $t|c$, $t^2 \nmid c$

Let μ stand for the unique integer such that $0 \le \mu \le t-1$ and $a \equiv -\frac{\mu c}{t}\ (t)$. Then,

$$
\begin{aligned}
&f(z;r,t)\\
&= e^{\pi i k}t^{-k-1}c^{-k}(z-q)^{-k}\times \qquad (2.18)\\
&\left\{t^{-k}\sum_{n\in\mathbf{Z}} e^{2\pi i(n+\kappa)A^{-1}(z)}a_n \sum_{\substack{\nu=0\\ \nu\neq\mu}}^{t-1} v(A_\nu)e^{-2\pi i\frac{(r+\kappa)\nu}{t}}e^{2\pi i\frac{(n+\kappa)h_\nu}{t}}+\right.\\
&\left. t^{-2k}\sum_{n\in\mathbf{Z}} e^{2\pi i(n+\kappa)t^2A^{-1}(z)}a_n v(A_\mu)e^{-2\pi i\frac{(r+\kappa)\mu}{t}}\right\}\\
&= e^{\pi i k}t^{-k-1}c^{-k}(z-q)^{-k}\times\\
&\left\{t^{-k}\sum_{m\in\mathbf{Z}} e^{2\pi i(\frac{m}{\lambda_q}+\kappa)A^{-1}(z)}a_{\frac{m}{\lambda_q}} \sum_{\substack{\nu=0\\ \nu\neq\mu}}^{t-1} v(A_\nu)e^{-2\pi i\frac{(r+\kappa)\nu}{t}}e^{2\pi i\frac{(\frac{m}{\lambda_q}+\kappa)h_\nu}{t}}+\right.\\
&\left. t^{-2k}\sum_{m\in\mathbf{Z}} e^{2\pi i(\frac{m}{\lambda_q}+\kappa)A^{-1}(z)}a_{m'} v(A_\mu)e^{-2\pi i\frac{(r+\kappa)\mu}{t}}\right\}\\
&= e^{\pi i k}t^{-2k-1}c^{-k}(z-q)^{-k}\sum_{m\in\mathbf{Z}} e^{2\pi i(\frac{m}{\lambda_q}+\kappa)A^{-1}(z)}\times \qquad (2.19)\\
&\left\{a_{\frac{m}{\lambda_q}} \sum_{\substack{\nu=0\\ \nu\neq\mu}}^{t-1} v(A_\nu)e^{-2\pi i\frac{(r+\kappa)\nu}{t}}e^{2\pi i\frac{(\frac{m}{\lambda_q}+\kappa)h_\nu}{t}} + t^{-k}a_{m'}v(A_\mu)e^{-2\pi i\frac{(r+\kappa)\mu}{t}}\right\},
\end{aligned}
$$

where $m' := \frac{m+\lambda_q\kappa(1-t^2)}{\lambda_q t^2}$ (so that $t^2(m'+\kappa) = \frac{m}{\lambda_q}+\kappa$).

Remark 2.9 *Since v is a multiplier system of half-integer weight, we must have $24\kappa \in \mathbf{Z}$ ([10]). If t is a prime greater than 3, then $24|(1-t^2)$. If $t=3$, then it follows from Corollary 2.4 that $3|\lambda_q$; if $t=2$, Corollary 2.4 implies $8|\lambda_q$. Thus, for all prime $t\in\mathbf{Z}, 24|\lambda_q(1-t^2)$, and it follows that $\lambda_q\kappa(1-t^2)\in\mathbf{Z}$. Therefore, for all prime $t\in\mathbf{Z}$, there exist values of $m\in\mathbf{Z}$ for which $m'\in\mathbf{Z}$.*

Remark 2.10 *If $f(z)$ has a pole at $i\infty$ (that is, if $n_0+\kappa<0$), then the term with exponent $t^2(n_0+\kappa)$ cannot be cancelled by any of the other terms which* *appear in (2.18). In particular, if $f(z)$ has a pole of order $-(n_0+\kappa)>0$ at $i\infty$, then $f(z;r,t)$ has a pole of order $-t^2(n_0+\kappa)$ at q.*

Remark 2.11 *Define*

$$\Lambda_1(n) := \sum_{\substack{\nu=0\\ \nu\neq\mu}}^{t-1} v(A_\nu)e^{-2\pi i\frac{(r+\kappa)\nu}{t}}e^{2\pi i\frac{(n+\kappa)h_\nu}{t}}.$$

(This is the sum on ν which appears in (2.18).) We make the following observation:

$$\begin{aligned}\Lambda_1(n+t) &= \sum_{\substack{\nu=0\\ \nu\neq\mu}}^{t-1} v(A_\nu)e^{-2\pi i\frac{(r+\kappa)\nu}{t}}e^{2\pi i\frac{(n+t+\kappa)h_\nu}{t}}\\ &= e^{2\pi i h_\nu}\Lambda_1(n) = \Lambda_1(n).\end{aligned}$$

That is, $\Lambda_1(n)$ is periodic with period t. Therefore, either $\Lambda_1(n)=0$ for all n, or there exists a smallest integer m such that $0\le m\le t-1$ and $\Lambda_1(n_0+m)\neq 0$.

Case 3 $t \nmid c$.

In this case, $\delta=1$ when $\nu\neq 0$; therefore,

$$\begin{aligned} f(z;r,t) &= e^{\pi i k}t^{-k-1}c^{-k}(z-q)^{-k}\Bigg\{\sum_{n\in\mathbf{Z}} a_n t^{-k}v(A)e^{2\pi i(n+\kappa)A^{-1}(z)}+\\ &\quad \sum_{n\in\mathbf{Z}} a_n e^{2\pi i\frac{(n+\kappa)}{t^2}A^{-1}(z)}\sum_{\nu=1}^{t-1} v(A_\nu)e^{-2\pi i\frac{(r+\kappa)\nu}{t}}e^{2\pi i\frac{(n+\kappa)h_\nu}{t^2}}\Bigg\} \qquad (2.20)\\ &= e^{\pi i k}t^{-k-1}c^{-k}(z-q)^{-k}\sum_{m\in\mathbf{Z}} e^{2\pi i(\frac{m}{\lambda_q}+\kappa)A^{-1}(z)}\times\\ &\quad \left\{a_{\frac{m}{\lambda_q}}t^{-k}v(A)+\sum_{\nu=1}^{t-1} v(A_\nu)e^{-2\pi i\frac{(r+\kappa)\nu}{t}}e^{2\pi i\frac{(\tilde{m}+\kappa)h_\nu}{t^2}}\right\}, \qquad (2.21)\end{aligned}$$

where $\tilde{m} := \frac{t^2m+\lambda_q\kappa(t^2-1)}{\lambda_q}$ (so that $\frac{\tilde{m}+\kappa}{t^2}=\frac{m}{\lambda_q}+\kappa$).

Remark 2.12 *If $f(z)$ has a pole at $i\infty$ (that is, if $n_0+\kappa<0$), then the term with exponent $n_0+\kappa$ cannot be cancelled by any of the other terms which appear in (2.20). In particular, if $f(z)$ has a pole of order $-(n_0+\kappa)>0$ at $i\infty$, then $f(z;r,t)$ has a pole of order $-(n_0+\kappa)$ at q.*

Remark 2.13 *Define*

$$\Lambda_2(n) := \sum_{\nu=1}^{t-1} v(A_\nu) e^{-2\pi i \frac{(r+\kappa)\nu}{t}} e^{2\pi i \frac{(n+\kappa)h_\nu}{t^2}}.$$

(This is the sum on ν which appears in (2.20).) We observe that Λ_2 is periodic with period t^2:

$$\begin{aligned} \Lambda_2(n+t^2) &= \sum_{\nu=1}^{t-1} v(A_\nu) e^{-2\pi i \frac{(r+\kappa)\nu}{t}} e^{2\pi i \frac{(n+t^2+\kappa)h_\nu}{t^2}} \\ &= e^{2\pi i h_\nu} \Lambda_2(n). \end{aligned}$$

Therefore, either $\Lambda_2(n)=0$ for all n, or there exists a smallest integer m such that $0\le m\le t^2-1$ and $\Lambda_2(n_0+m)\neq 0$.

We summarize our results on the growth of $f(z;r,t)$ at $q=\frac{a}{c}$ with the following two theorems.

Theorem 2.4 *Suppose $f(z)$ has a zero at $i\infty$ of order $N=n_0+\kappa\ge 0$.*

Case 1 *If $t^2|c$, then $f(z;r,t)$ has a zero of order N at $\frac{a}{c}$.*

Case 2 *Suppose $t|c$ but $t^2\not|c$. If $\Lambda_1(n)=0$, $\forall n\in\mathbf{Z}$, then $f(z;r,t)$ has a zero of order t^2N at $\frac{a}{c}$. Otherwise, there exists a least nonnegative integer m_1 such that $m_1<t$ and $\Lambda_1(n_0+m_1)\neq 0$. In this case, $f(z;r,t)$ has a zero of order*

$$\min(t^2N, N+m_1) = \begin{cases} t^2N, & \text{if } N\le \frac{m_1}{t^2-1} \\ N+m_1, & \text{otherwise.} \end{cases}$$

Case 3 *Suppose* $t \not| c$. *If* $\Lambda_2(n) = 0,\ \forall n \in \mathbf{Z}$, *then* $f(z;r,t)$ *has a zero of order* N *at* $\frac{a}{c}$. *Otherwise, there exists a least nonnegative integer* m_2 *such that* $m_2 < t^2$ *and* $\Lambda_2(n_0 + m_2) \neq 0$. *In this case,* $f(z;r,t)$ *has a zero of order*

$$\min\left(N, \frac{N+m_2}{t^2}\right) = \begin{cases} N, & \text{if } N \leq \frac{m_2}{t^2-1} \\ \frac{N+m_2}{t^2}, & \text{otherwise.} \end{cases}$$

Remark 2.14 *Since* $m_1 \leq t-1$, $\frac{m_1}{t^2-1} \leq \frac{1}{t+1}$.

Remark 2.15 *If* $\Lambda_1(n) \neq 0$ *for all* $n \in \mathbf{Z}$, *then* $m_1 = 0$, *and thus* $f(z;r,t)$ *has a zero of order* N *at* $\frac{a}{c}$.

Theorem 2.5 *If* $f(z)$ *has a pole of order* $P = -(n_0 + \kappa) > 0$ *at* $i\infty$, *then* $f(z;r,t)$ *has a pole at* $\frac{a}{c}$ *of order*

$$\begin{array}{rl} P, & \text{if } t^2|c, \\ t^2P, & \text{if } t|c \text{ but } t^2 \not| c, \\ P, & \text{if } t \not| c. \end{array} \tag{2.22}$$

2.5 Construction of Invariant Functions

In this section, we use congruence restrictions of integer powers of the Dedekind eta function to construct modular functions (that is, modular forms of weight zero with trivial multiplier systems) on congruence subgroups of $\Gamma(1)$. We also state conditions which, if met, would allow us to construct *entire* modular functions, which must be constant ([5], Chapter 2, Theorem 7).

For $\ell \in \mathbf{Z}^+$ and $r_1, r_2 \in \mathbf{Z}$,

$$\begin{aligned} f_\ell(z) &:= \eta(z)^\ell \\ g_\ell(z) &:= \eta(z)^{-\ell} \\ F^{(\ell)}_{r_1,r_2,t}(z) &:= f(z;r_1,t)g(z;r_2,t) \end{aligned}$$

We have shown that $f_\ell(z;r_1,t)$ is a modular form on the group $\Gamma_{0,24t}(24t^2)$ of weight $\frac{\ell}{2}$ with multiplier system v_η^ℓ, and $g_\ell(z;r_2,t)$ is a modular form on the same group of weight $-\frac{\ell}{2}$ with multiplier system $\overline{v_\eta}^\ell$. Also, since η has no zeros or poles in $\mathcal{H}$, it is clear (from (1.10), for example) that $f_\ell(z;r_1,t)$ and $g_\ell(z;r_2,t)$ are holomorphic on $\mathcal{H}$. Therefore, we may conclude that $F^{(\ell)}_{r_1,r_2,t}(z)$ is a modular *function* on $\Gamma_{0,24t}(24t^2)$ which is holomorphic on $\mathcal{H}$.

To simplify notation, we shall replace f_ℓ, g_ℓ and $F^{(\ell)}_{r_1,r_2,t}$, with f, g and F, respectively, to simplify notation throughout the remainder of this section.

Let $a, c \in \mathbf{Z}$, with $c > 0$ and $\gcd(a,c) = 1$, and put $q = \frac{a}{c}$. From Theorems 2.4 and 2.5, we see that $f(z;r_1,t)$ has a zero at q, $g(z;r_2,t)$ has a pole at q, and that the order of the pole of $g(z;r_2,t)$ at q is greater than or equal to the order of the zero of $f(z;r_1,t)$ at q. Therefore, $F(z)$ has a pole at q of order $P - N$, where P is the order of the pole of $g(z;r_2,t)$ at q and N is the order of the zero of $f(z;r_1,t)$ at q.

In particular, if Λ_1 and Λ_2 (as defined in Theorem 2.4) are not identically zero, then $F(z)$ has a pole at q of order

$$\begin{cases} \frac{\ell}{24} - \frac{\ell}{24} = 0, & \text{if } t^2 | c \\ \frac{t^2\ell}{24} - \min\left(\frac{t^2\ell}{24}, \frac{\ell}{24} + m_1\right), & \text{if } t | c \text{ but } t^2 \not| c \\ \frac{\ell}{24} - \min\left(\frac{\ell}{24}, \frac{1}{t^2}\left\{\frac{\ell}{24} + m_2\right\}\right), & \text{if } t \not| c. \end{cases}$$

(Here m_1 and m_2 are as defined in Theorem 2.4.) If $m_1 = m_2 = 0$ (that is, if $\Lambda_1(n_0) \neq 0$ and $\Lambda_2(n_0) \neq 0$), then $F(z)$ has a pole at q of order

$$\begin{array}{rl} 0, & \text{if } t^2 | c \\ \frac{(t^2-1)\ell}{24}, & \text{if } t | c \text{ but } t^2 \not| c \\ \frac{(t^2-1)\ell}{24t^2}, & \text{if } t \not| c. \end{array}$$

On the other hand, if there exist r_1, t such that Λ_1 and Λ_2 are both identically zero, then it follows from Theorem 2.4 that $F(z)$ the order of the zero of $f(z; r_1, t)$ is *equal to* the order of the pole of $g(z; r_2, t)$ at q. (Notice that the order of the pole of $g(z; r_2, t)$ at q does not depend on r_2.) This allows us to state the following theorem:

Theorem 2.6 *Suppose there exist $r, t \in \mathbf{Z}$, with $t > 0$, such that Λ_1 and Λ_2 (as defined in Remarks 2.11 and 2.13) are both identically zero. Then, for any $r' \in \mathbf{Z}$, $F^{(\ell)}_{r,r',t}(z)$ is constant on $\mathcal{H}$.*

Proof The preceding discussion implies that $F^{(\ell)}_{r,r',t}(z)$ is holomorphic on $\mathcal{H}$ and regular at each rational point $q \in \mathbf{Q}$. Therefore, $F^{(\ell)}_{r,r',t}$ is an *entire modular function* and thus constant on $\mathcal{H}$. This proves the theorem.

CHAPTER 3

THE THETA FUNCTION

3.1 Introduction

Definition 3.1 *The "theta function," ϑ, is defined on $\mathcal{H}$ as follows:*

$$\vartheta(z) := \sum_{n\in\mathbf{Z}} e^{\pi i n^2 z} \tag{3.1}$$

$$= \sum_{m=0}^{\infty} a_m e^{\pi i m z}, \text{ where } a_m := \begin{cases} 1, & m=0 \\ 2, & \sqrt{m}\in\mathbf{Z}^+ \\ 0, & \text{else.} \end{cases} \tag{3.2}$$

It is known (see, for example, Chapters 3 and 4 of [5]) that $\vartheta(2z)$ is a modular form on $\Gamma_0(4)$ of weight $\frac{1}{2}$ with multiplier system

$$v_\vartheta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\frac{c/2}{|d|}\right) e^{\pi i(d-1)/4}(-1)^{(sign(c)+sign(d)-2)/2}, & \text{if } c\neq 0 \\ de^{\pi i(d-1)/4}, & \text{if } c=0. \end{cases} \tag{3.3}$$

Remark 3.1 *The functions $\vartheta(2z)$ and $\eta(z)$ are very closely related, because we may use Jacobi's Triple-Product Identity [1] to rewrite $\vartheta(2z)$ as follows:*

$$\vartheta(2z) = \prod_{n\geq 1}(1-e^{4\pi inz})(1+e^{(4n-2)\pi iz})^2. \qquad (3.4)$$

This allows us to write $\vartheta(2z)$ in terms of $\eta(z)$:

$$\begin{aligned}\vartheta(2z) &= \prod_{n\geq 1}(1-e^{2\pi i(2n)z})^2(1+e^{2\pi i(2n-1)z})^2(1-e^{2\pi in(2z)})^{-1} \\ &= e^{\pi i(2z+1)/12}\eta(2z+1)^{-1}\prod_{n\geq 1}(1-e^{\pi in}e^{2\pi inz})^2 \\ &= e^{\pi i(2z+1)/12}\eta(2z+1)^{-1}\prod_{n\geq 1}(1-e^{2\pi in(z+1/2)})^2 \\ &= e^{\pi i(2z+1)/12}\eta(2z+1)^{-1}e^{-2\pi i(z+1/2)/12}\eta\left(z+\frac{1}{2}\right)^2 \\ &= \frac{\eta\left(z+\frac{1}{2}\right)^2}{\eta(2z+1)}. \qquad (3.5)\end{aligned}$$

We will apply congruence restrictions to the theta function in two different ways:

Definition 3.2 *We shall refer to the following as a "congruence restriction of the first kind:"*

$$\begin{aligned}\vartheta(2z;r,t) &:= \sum_{m\equiv r\,(t)} a_m e^{2\pi imz} \qquad (3.6) \\ &= \sum_{n^2\equiv r\,(t)} e^{2\pi in^2z}.\end{aligned}$$

Definition 3.3 *We shall refer to the following as a "congruence restriction of the second kind:"*

$$\vartheta_{r,t}(2z) := \sum_{n\equiv r\,(t)} e^{2\pi in^2z}. \qquad (3.7)$$

Remark 3.2 *Since the Fourier expansion of $\vartheta(2z)$ can be expressed in two different ways, both of which are of interest, it is useful to study congruence restrictions of both of these expressions. Congruence restrictions of the second kind are of interest only with respect to theta functions; with all other modular forms, we consider only congruence restrictions of the first kind.*

We now introduce another type of Dedekind eta product which, as we shall see, is closely related to congruence restricted theta functions of the second kind.

Definition 3.4 *Let $g, \delta \in \mathbf{Z}$, with $\delta > 0$. In [11], Robins defines the generalized Dedekind η-product, $\eta_{\delta,g}(z)$, as follows:*

$$\eta_{\delta,g}(z) := e^{\pi i P_2(g/\delta)\delta z} \prod_{\substack{n>0 \\ n\equiv g(\delta)}} (1-e^{2\pi i n z}) \prod_{\substack{n>0 \\ n\equiv -g(\delta)}} (1-e^{2\pi i n z}), \tag{3.8}$$

where $P_2(x) := \{x\}^2 - \{x\} + \frac{1}{6}$ is the second Bernoulli function and $\{x\} := x - [x]$ is the fractional part of x. (For $0 \le x < 1$, $\{x\} = x$.)

Robins goes on to show that, when $\delta \not| g$, $\eta_{\delta,g}(z)$ is a modular form of weight 0 on $\Gamma_1(\delta)$ which satisfies the transformation law

$$\eta_{\delta,g}(Mz) = e^{\pi i \mu_{\delta,g}}\, \eta_{\delta,g}(z),\ \forall M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(\delta), \tag{3.9}$$

where

$$\mu_{\delta,g} = \frac{\delta a}{c} P_2\left(\frac{g}{\delta}\right) + \frac{\delta d}{c} P_2\left(\frac{ag}{\delta}\right) - 2s\left(a, \frac{c}{\delta}; 0, \frac{g}{\delta}\right) \tag{3.10}$$

Furthermore, Robins proves that, when $\gcd(a, 6) = 1$,

$$\mu_{\delta,g} = ab\delta P_2\left(\frac{g}{\delta}\right) - \frac{ac}{6\delta} + (a-1)\frac{g}{\delta} - \frac{1}{2}(a-1). \tag{3.11}$$

Therefore, (3.9) may be written:

$$\frac{\eta_{\delta,g}(Mz)}{\eta_{\delta,g}(z)} = \exp\left\{\pi i\left(ab\delta P_2\left(\frac{g}{\delta}\right) - \frac{ac}{6\delta} + (a-1)\left(\frac{g}{\delta} - \frac{1}{2}\right)\right)\right\}, \tag{3.12}$$

for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(\delta)$ such that $\gcd(a, 6) = 1$.

We now establish a connection between the two types of congruence restrictions of $\vartheta(2z)$:

Lemma 3.1

$$\vartheta(2z; r, t) = \sum_{\substack{R=0 \\ R^2 \equiv r(t)}}^{t-1} \vartheta_{R,t}(2z). \tag{3.13}$$

Proof We may rearrange the right-hand side:

$$\begin{aligned} \sum_{\substack{R=t \\ R^2 \equiv r(t)}} \vartheta_{R,t}(2z) &= \sum_{\substack{R=0 \\ R^2 \equiv r(t)}}^{t-1} \sum_{\substack{m \in \mathbf{Z} \\ m \equiv R(t)}} e^{2\pi i m^2 z} \\ &= \sum_{\substack{m \in \mathbf{Z} \\ m \equiv R(t) \\ 0 \leq R \leq t-1, R^2 \equiv r(t)}} e^{2\pi i m^2 z} \end{aligned}$$

To prove the lemma, we must show:

$$\left\{n \in \mathbf{Z} : n^2 \equiv r\ (t)\right\} = \bigcup_{\substack{0 \leq R \leq t-1 \\ R^2 \equiv r\ (t)}} \left\{m \in \mathbf{Z} : m \equiv R\ (t)\right\}.$$

We first show that

$$\left\{n \in \mathbf{Z} : n^2 \equiv r\ (t)\right\} \subseteq \bigcup_{\substack{0 \leq R \leq t-1 \\ R^2 \equiv r\ (t)}} \left\{m \in \mathbf{Z} : m \equiv R\ (t)\right\}.$$

Choose $n \in \mathbf{Z}$ such that $n^2 \equiv r\ (t)$. Then, $n = tQ_n + R_n$ for some $Q_n, R_n \in \mathbf{Z}$ with $0 \leq R_n \leq t-1$. It then follows:

$$n^2 = t(tQ_n^2 + 2Q_nR_n) + R_n^2,$$

which implies

$$n^2 \equiv R_n^2\ (t),$$

and therefore

$$r \equiv R_n^2\ (t).$$

Thus, $n^2 \equiv r(t) \Longrightarrow n \equiv R\ (t)$ for some $R \in \mathbf{Z}$ such that $0 \leq R \leq t-1$ and $R^2 \equiv r\ (t)$.

It remains to show that

$$\bigcup_{\substack{0 \leq R \leq t-1 \\ R^2 \equiv r\ (t)}} \{m \in \mathbf{Z} : m \equiv R\ (t)\} \subseteq \left\{n \in \mathbf{Z} : n^2 \equiv r\ (t)\right\}.$$

Choose $m \in \mathbf{Z}$ such that $m \equiv R_m\ (t)$ for some $R_m \in \mathbf{Z}$, where $0 \leq R_m \leq t-1$ and $R_m^2 \equiv r\ (t)$. Clearly, $m \equiv R_m \pmod t \Longrightarrow m^2 \equiv R_m^2 \equiv r\ (t)$. Therefore, if $m \equiv R$ and $R^2 \equiv r$, then $m^2 \equiv r\ (t)$.

The two sets in question are equal; therefore, the corresponding sums are equal, and so the lemma is proved.

Example 3.1

$$\vartheta(2z; 9, 16) = \vartheta_{3,16}(2z) + \vartheta_{5,16}(2z) + \vartheta_{11,16}(2z) + \vartheta_{13,16}(2z),$$

since $r^2 \equiv 9\ (16) \Longleftrightarrow r \equiv 3, 5, 11$ *or* $13\ (16)$.

$$\vartheta(2z; 9, 18) = \vartheta_{3,18}(2z) + \vartheta_{15,18}(2z) + \vartheta_{9,18}(2z),$$

since $r^2 \equiv 9\ (18) \Longleftrightarrow r \equiv 3, 9$ *or* $15\ (18)$.

Another useful property of $\vartheta_{r,t}(2z)$ is the following:

$$\begin{aligned}\vartheta_{r,t}(2z) &= \sum_{\substack{n\in\mathbf{Z}\\ n\equiv r(t)}} e^{2\pi i n^2 z}\\ &= \sum_{\substack{n\in\mathbf{Z}\\ -n\equiv r(t)}} e^{2\pi i(-n)^2 z}\\ &= \sum_{\substack{n\in\mathbf{Z}\\ n\equiv -r(t)}} e^{2\pi i n^2 z}.\end{aligned}$$

Therefore,

$$\vartheta_{r,t}(2z) = \vartheta_{-r,t}(2z).$$

(Note that, for $r = 0$ or $r = \frac{t}{2}$, the above is trivial, since in each case $r \equiv -r(t)$.) This allows us to rewrite the result proved in the preceding lemma as follows:

$$\vartheta(2z;r,t) = 2\sum_{\substack{R=1\\ R^2\equiv r(t)}}^{\lfloor\frac{t-1}{2}\rfloor} \vartheta_{R,t}(2z) + \chi_{r,t}(0)\vartheta_{0,t}(2z) + \chi_{r,t}(t/2)\vartheta_{t/2,t}(2z),$$

where we define $\chi_{r,t}(R) := \begin{cases} 1 & \text{if } r \equiv R\ (t)\\ 0 & \text{else.}\end{cases}$

Example 3.2 *The previous examples may be rewritten as follows:*

$$\begin{aligned}\vartheta(2z;9,16) &= 2\vartheta_{3,16}(2z) + 2\vartheta_{5,16}(2z),\\ \vartheta(2z;9,18) &= 2\vartheta_{3,18}(2z) + \vartheta_{9,18}(2z).\end{aligned}$$

3.2 Congruence Restrictions of the First Kind

It follows from Theorem 1.2 that $\vartheta(2z;r,t)$ is a modular form of weight $\frac{1}{2}$ with multiplier system v_ϑ on the group $\Gamma_{v_\vartheta,t} \subset S_{v_\vartheta,t}$, provided such a Γ_{v_ϑ} exists. The object of our next theorem is to determine a subgroup of finite index in $\Gamma(1)$ which is also a subset of S_{v_ϑ}.

Theorem 3.1 *Let* $h_t := \frac{8}{\gcd(4,t)}$. *Then, for all* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{0,t}\left(h_t t^2\right)$, $v_\vartheta\left(\gamma_{\nu,t} A \gamma_{-\nu,t}\right) = v_\vartheta$. *Therefore, we may take*

$$\Gamma_{v_\vartheta,t} = \Gamma_{0,t}\left(h_t t^2\right). \tag{3.14}$$

Remark 3.3 *This is not necessarily the largest group which may serve as* $\Gamma_{v_\vartheta,t}$. *This is simply the best result obtainable with our argument. There may exist a larger (and, therefore, better) choice for* $\Gamma_{v_\vartheta,t}$ *which* contains $\Gamma_{0,t}\left(h_t t^2\right)$. Finding a larger $\Gamma_{v_\vartheta,t}$ would strengthen the result of this theorem.

Remark 3.4 *Note that our choice of* $\Gamma_{v_\vartheta,t}$ *depends only on* t*, and not on* r.

Proof To prove Theorem 3.1, it will suffice to show that

$$v_\vartheta\begin{pmatrix} a + \frac{c\nu}{t} & b + \frac{(d-a)\nu}{t} + \frac{c\nu^2}{t^2} \\ c & d - \frac{c\nu}{t} \end{pmatrix}$$

is independent of ν; once this is shown, the theorem is proved by putting $\nu = 0$.

The assumptions $t^2|c$, $8t|c$ and $a \equiv d(t)$ guarantee that

$$\begin{pmatrix} a + \frac{c\nu}{t} & b + \frac{(d-a)\nu}{t} + \frac{c\nu^2}{t^2} \\ c & d - \frac{c\nu}{t} \end{pmatrix}$$

is an element of $\Gamma_0(4)$. Therefore,

$$v_\vartheta\begin{pmatrix} a+\frac{c\nu}{t} & b+\frac{(d-a)\nu}{t}+\frac{c\nu^2}{t^2} \\ c & d-\frac{c\nu}{t} \end{pmatrix}$$

is defined.

We first dispense with the case $c = 0$. In this case, the value of v_ϑ is simply the lower-right entry of the matrix, which (when $c = 0$) does not depend on ν.

Now, suppose $c \neq 0$. If follows from $v_\vartheta(S) = 1$ that $v_\vartheta(A) = v_\vartheta(AS^\lambda)$, $\forall\lambda \in \mathbf{Z}$ and $A \in \Gamma_0(4)$. We choose $\lambda \in \mathbf{Z}$ so that $d - c\nu/t + c\lambda > 0$:

$$\begin{aligned} & v_\vartheta\begin{pmatrix} a+\frac{c\nu}{t} & b+\frac{d-a}{\nu}+\frac{c\nu^2}{t^2} \\ c & d-\frac{c\nu}{t} \end{pmatrix} \\ = \; & v_\vartheta\left(\begin{pmatrix} * & * \\ c & d-\frac{c\nu}{t} \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\right) \\ = \; & v_\vartheta\begin{pmatrix} * & * \\ c & d-c\nu/t+c\lambda \end{pmatrix} \\ = \; & \left(\frac{c/2}{d-c\nu/t+c\lambda}\right) e^{\pi i(d-c\nu/t+c\lambda-1)/4}(-1)^{(sign(c)-1)/2} \end{aligned}$$

(Note that the value of v_ϑ depends only on the *lower row* of its argument.) Now, we define $c_1 := \frac{c}{2^\alpha t^2}$, where $\alpha \in \mathbf{Z}$ is chosen so that c_1 is an odd integer. In other words: c_1 is the largest *odd* factor of c, and α is the largest power of 2 which divides c.

Note that $8t|c$ implies $-c\nu/t + c\lambda \equiv 0$ (8), while $2t^2|c$ implies $\alpha \geq 1$. With these two observations in mind, we proceed:

$$\begin{aligned}
&v_\vartheta\begin{pmatrix} * & * \\ c & d-\frac{c\nu}{t}\end{pmatrix} \\
&= \left(\frac{2}{d-c\nu/t+c\lambda}\right)^{\alpha-1}\left(\frac{t}{d-c\nu/t+c\lambda}\right)^2\left(\frac{c_1}{d-c\nu/t+c\lambda}\right)\times \\
&\qquad e^{\pi i(d-1)/4}(-1)^{(sign(c)-1)/2} \\
&= \left(\frac{c_1}{d-c\nu/t+c\lambda}\right)\left((-1)^{((d-c\nu/t+c\lambda)^2-1)/8}\right)^{\alpha-1}e^{\pi i(d-1)/4}(-1)^{(sign(c)-1)/2} \\
&= \left(\frac{c_1}{d-c\nu/t+c\lambda}\right)(-1)^{(d^2-1)(\alpha-1)/8}e^{\pi i(d-1)/4}(-1)^{(sign(c)-1)/2} \\
&= \left(\frac{d-c\nu/t+c\lambda}{c_1}\right)(-1)^{(c_1-1)/2}(-1)^{(d-c\nu/t+c\lambda-1)/2}\times \\
&\qquad (-1)^{(d^2-1)(\alpha-1)/8}e^{\pi i(d-1)/4}(-1)^{(sign(c)-1)/2} \\
&= \left(\frac{d}{c_1}\right)(-1)^{(c_1-1)/2}(-1)^{(d-1)/2}(-1)^{(d^2-1)(\alpha-1)/8}e^{\pi i(d-1)/4}(-1)^{(sign(c)-1)/2},
\end{aligned}$$

which is independent of ν.

We have shown that $v_\vartheta\begin{pmatrix} * & * \\ c & d-\frac{c\nu}{t}\end{pmatrix} = v_\vartheta\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \forall\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(8t) \cap \Gamma_{0,t}(2t^2)$. We note that

$$\operatorname{lcm}(8t, 2t^2) = \frac{16t^3}{\gcd(8t, 2t^2)} = \frac{8t^2}{\gcd(4,t)} = h_t t^2.$$

Therefore,

$$\Gamma_0(8t) \cap \Gamma_{0,t}(2t^2) = \Gamma_{0,t}(h_t t^2).$$

This proves the theorem.

Corollary 3.1 *$\vartheta(2z; r, t)$ is an entire modular form of weight $\frac{1}{2}$ with multiplier system v_ϑ on the group $\Gamma_{0,t}(h_t t^2)$, with $h_t := \frac{8}{\gcd(4,t)}$.*

Proof This follows from Theorems 1.1 and 3.1.

3.3 Congruence Restrictions of the Second Kind

Theorem 3.2 *$\vartheta_{r,t}(2z)$ is an entire modular form on $\Gamma_1(4t^2) \cap \Gamma_0(3)$, of weight $\frac{1}{2}$, with multiplier system v_ϑ.*

Proof First, we make the assumption $-\frac{t}{2} < r \le \frac{t}{2}$. There is no loss of generality in doing so, since $r' \equiv r\ (t) \Longrightarrow \vartheta_{r',t}(2z) = \vartheta_{r,t}(2z)$. We then carry out the proof of the theorem by considering three separate cases.

Case 1 Suppose $r = 0$. Then,

$$\begin{aligned}\vartheta_{0,t}(2z) &= \sum_{n\equiv 0(t)} e^{2\pi i n^2 z} \\ &= \sum_{n} e^{2\pi i (nt)^2 z} \\ &= \vartheta(2t^2 z).\end{aligned}$$

Since $\vartheta(2z)$ is a modular form on $\Gamma_0(4)$, it follows that $\vartheta(2t^2z)$ is a modular form on $\Gamma_0(4t^2)$. We now proceed to show that $\vartheta(2t^2z)$ has multiplier system v_ϑ.

Suppose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4t^2)$. Then,

$$\begin{aligned}\vartheta(2t^2Mz) &= \vartheta\left(\frac{2t^2az + 2t^2b}{cz+d}\right) \\ &= \vartheta\left(\frac{2a(t^2z) + 2t^2b}{\frac{c}{t^2}(t^2z) + d}\right) \\ &= \vartheta(2M'(t^2z)), \text{ where } M' = \begin{pmatrix} a & t^2b \\ c/t^2 & d \end{pmatrix} \in \Gamma_0(4) \\ &= v_\vartheta(M')\left(\frac{c}{t^2}(t^2z) + d\right)^{1/2} \vartheta(2t^2).\end{aligned}$$

We now refer back to (3.3), to show that $v_\vartheta(M') = v_\vartheta(M)$. The case $c = 0$ is trivial, since we then have $v_\vartheta(M') = v_\vartheta(M) = d$. So, we assume $c \neq 0$:

$$\begin{aligned} v_\vartheta(M) &= v_\vartheta\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \left(\frac{c/2}{|d|}\right) e^{\pi i(d-1)/4}(-1)^{(sign(c)-1)/2}(-1)^{(sign(d)-1)/2} \\ &= \left(\frac{c/2}{|d|}\right)\left(\frac{2t}{|d|}\right)^2 e^{\pi i(d-1)/4}(-1)^{(sign(t^2c)-1)/2}(-1)^{(sign(d)-1)/2} \\ &= \left(\frac{4t^2c/2}{|d|}\right) e^{\pi i(d-1)/4}(-1)^{(sign(t^2c)-1)/2}(-1)^{(sign(d)-1)/2} \\ &= v_\vartheta(M'). \end{aligned}$$

Therefore, for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$,

$$\vartheta_{0,t}(2Mz) = v_\vartheta(M)(cz+d)^{1/2}\vartheta_{0,t}(2z).$$

This completes the proof of Theorem 3.2 in the case $r = 0$.

Case 2 Suppose $r = \frac{t}{2}$. Then,

$$\begin{aligned} \vartheta_{\frac{t}{2},t}(2z) &= \sum_{n \equiv \frac{t}{2}(t)} e^{2\pi i n^2 z} \\ &= \sum_n e^{2\pi i((tn+t/2)^2 z} \\ &= \sum_n e^{2\pi i(2n+1)^2(t/2)^2 z} \\ &= \sum_n \left(e^{2\pi i n^2(t/2)^2 z} - e^{2\pi i(2n)^2(t/2)^2 z}\right) \\ &= \vartheta(2(t/2)^2 z) - \vartheta(2t^2 z). \end{aligned}$$

We know that $\vartheta(2(t/2)^2z)$ and $\vartheta(2t^2z)$ are modular forms of weight $\frac{1}{2}$ on $\Gamma_0(4t^2)$. Therefore, if we show that these two modular forms both have multiplier system v_ϑ, we may conclude that $\vartheta_{\frac{t}{2},t}(2z)$ is itself a modular form on $\Gamma_0(4t^2)$, of weight $\frac{1}{2}$, with multiplier system v_ϑ.

From Case 1, it follows that

$$\vartheta_{\frac{t}{2},t}(2Mz) = \vartheta_{0,\frac{t}{2}}(2z) - \vartheta_{0,t}(2z).$$

Therefore, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0\left(4t^2\right)$,

$$\begin{aligned}
\vartheta_{-\frac{t}{2},t}(2Mz) &= \vartheta_{0,\frac{t}{2}}(2Mz) - \vartheta_{0,t}(2Mz) \\
&= v_\vartheta(M)(cz+d)^{1/2}\vartheta_{0,\frac{t}{2}}(2z) - v_\vartheta(M)(cz+d)^{1/2}\vartheta_{0,t}(2z) \\
&= v_\vartheta(M)(cz+d)^{1/2}\vartheta_{\frac{t}{2},t}(2z).
\end{aligned}$$

This completes the proof of Theorem 3.2 in the case $r = \frac{t}{2}$.

Case 3 Suppose $r \notin \{0, \frac{t}{2}\}$. We rewrite $\vartheta(2z)$ as follows:

$$\vartheta_{r,t}(2z) = \sum_n e^{2\pi i(nt+r)^2 z} = e^{2\pi i r^2 z} \sum_n e^{2\pi i (nt)^2 z} e^{4\pi i n r t z}.$$

To this sum, we apply Jacobi's triple-product identity:

$$\vartheta_{r,t}(2z) = \\ e^{2\pi i r^2 z} \prod_{m\geq 1} (1 - e^{2\pi i 2t^2 m z})(1 + e^{2\pi i((2m-1)t^2+2rt)z})(1 + e^{2\pi i((2m-1)t^2-2rt)z}).$$

Next, we rewrite $\vartheta_{r,t}(2z)$ in terms of generalized Dedekind η-products:

$$\begin{aligned}
&\vartheta_{r,t}(2z) \\
&= e^{2\pi i r^2 z} \prod_{m\geq 1} (1 - e^{2\pi i 2t^2 m z})(1 + e^{2\pi i((2m-1)t^2+2rt)z})(1 + e^{2\pi i((2m-1)t^2-2rt)z}) \\
&= e^{2\pi i(r^2-t^2/12)z}\eta(2t^2 z) \times \\
&\quad \frac{\prod_{m\geq 1}(1 - e^{2\pi i(4t^2 m-(2t^2-4rt))z})(1 - e^{2\pi i(4t^2 m-(2t^2+4rt))z})}{\prod_{m\geq 1}(1 - e^{2\pi i(2t^2 m-(t^2-2rt))z})(1 - e^{2\pi i(2t^2 m-(t^2+2rt))z})} \qquad (3.15) \\
&= e^{2\pi i(r^2-t^2/12)z}\eta(2t^2 z)\frac{e^{-2\pi i P_2\left(\frac{2t^2+4rt}{4t^2}\right)2t^2 z}\eta_{4t^2,2t^2+4rt}(z)}{e^{-2\pi i P_2\left(\frac{t^2+2rt}{2t^2}\right)t^2 z}\eta_{2t^2,t^2+2rt}(z)} \qquad (3.16) \\
&= e^{2\pi i z\left(r^2-\frac{t^2}{12}-P_2\left(\frac{1}{2}+\frac{r}{t}\right)t^2\right)}\frac{\eta_{4t^2,2t^2+4rt}(z)}{\eta_{2t^2,t^2+2rt}(z)}\eta(2t^2 z). \qquad (3.17)
\end{aligned}$$

To justify equation (3.16), we observe that $|r| < \frac{t}{2} \Longrightarrow t \pm 2r > 0$; therefore:

$$\begin{aligned} m \geq 1 &\implies 4t^2m - (2t^2 \pm 4rt) \geq 2t(t \mp 2r) > 0, \text{ and} \\ m \leq 0 &\implies 4t^2m - (2t^2 \pm 4rt) \leq -2t(t \pm 2r) < 0. \end{aligned}$$

Thus,

$$\{4t^2m - (2t^2 - 4rt) : m \in \mathbf{Z}^+\} = \{n \in Z^+ : n \equiv 2t^2 + 4rt(4t^2)\}$$

and

$$\{2t^2m - (t^2 - 2rt) : m \in \mathbf{Z}^+\} = \{n \in Z^+ : n \equiv t^2 + 2rt(2t^2)\},$$

and so the products on m in (3.15) run over the set of integers specified by the definitions of $\eta_{4t^2,2t^2+4rt}$ and η_{2t^2,t^2+2rt}, respectively.

Let us now evaluate the exponential term which appears in equation (3.17).

$$\begin{aligned} r^2 - \frac{t^2}{12} - t^2P_2\left(\frac{1}{2} + \frac{r}{t}\right) &= r^2 - \frac{t^2}{12} - t^2\left(\left(\frac{r}{t} + \frac{1}{2}\right)\left(\frac{r}{t} - \frac{1}{2}\right) + \frac{1}{6}\right) \\ &= r^2 - \frac{t^2}{12} - t^2\left(\frac{r^2}{t^2} - \frac{1}{4} + \frac{1}{6}\right) \\ &= r^2 - \frac{t^2}{12} - r^2 + \frac{t^2}{12} = 0. \end{aligned}$$

Thus, the exponential term vanishes, and so (3.17) becomes

$$\vartheta_{r,t}(2z) = \frac{\eta_{4t^2,2t^2+4rt}(z)}{\eta_{2t^2,t^2+2rt}(z)}\eta(2t^2z). \tag{3.18}$$

At this point, we note that $\vartheta_{r,t}(2z)$ is a product of functions which are *modular forms* on $\Gamma_1(4t^2)$; therefore, $\vartheta_{r,t}(2z)$ itself is a modular form on $\Gamma_1(4t^2)$. We now proceed to show that the multiplier system for $\vartheta_{r,t}(2z)$ turns out to be v_ϑ when $M \in \Gamma_0(3)$.

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(4t^2) \cap \Gamma_0(3)$. Note that $M \in \Gamma_1(4t^2) \Rightarrow$ $\gcd(a,2) = 1$, and $M \in \Gamma_0(3) \Rightarrow \gcd(a,3) = 1$; therefore, $\gcd(a,6) = 1$, so we may use (3.12) to determine the transformation law for $\eta_{4t^2,2t^2+4rt}(Mz)$:

$$\begin{aligned}
&\frac{\eta_{4t^2,2t^2+4rt}(Mz)}{\eta_{4t^2,2t^2+4rt}(z)} \\
&= \exp\left\{\pi i\left(4abt^2P_2\left(\frac{2t^2+4rt}{4t^2}\right) - \frac{ac}{24t^2} + (a-1)\left(\frac{2t^2+4rt}{4t^2} - \frac{1}{2}\right)\right)\right\} \\
&= \exp\left\{\pi i\left(4abt^2P_2\left(\frac{r}{t}+\frac{1}{2}\right) - \frac{ac}{24t^2} + (a-1)\left(\frac{r}{t}+\frac{1}{2}-\frac{1}{2}\right)\right)\right\} \\
&= \exp\left\{\pi i\left(4abt^2\left(\frac{r^2}{t^2}-\frac{1}{12}\right) - \frac{ac}{24t^2} + \frac{(a-1)r}{t}\right)\right\} \\
&= \exp\left\{\pi i\left(-\frac{4abt^2}{12} - \frac{ac}{24t^2}\right)\right\}. \qquad (3.19)
\end{aligned}$$

In (3.19), we note that $M \in \Gamma_1(4t^2) \Rightarrow 4t^2|(a-1)$. Therefore, we may state the transformation law for $\eta_{4t^2,2t^2+4rt}$ as follows:

$$\frac{\eta_{4t^2,2t^2+4rt}(Mz)}{\eta_{4t^2,2t^2+4rt}(z)} = \exp\left\{-\frac{\pi i a}{24}\left(8bt^2 + \frac{c}{t^2}\right)\right\}, \qquad (3.20)$$

for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(4t^2)$. Similarly,

$$\begin{aligned}
&\frac{\eta_{2t^2,t^2+2rt}(Mz)}{\eta_{2t^2,t^2+2rt}(z)} \\
&= \exp\left\{\pi i\left(2abt^2P_2\left(\frac{t^2+2rt}{2t^2}\right) - \frac{ac}{12t^2} + (a-1)\left(\frac{t^2+2rt}{2t^2} - \frac{1}{2}\right)\right)\right\} \\
&= \exp\left\{\pi i\left(2abt^2P_2\left(\frac{r}{t}+\frac{1}{2}\right) - \frac{ac}{12t^2} + (a-1)\left(\frac{r}{t}+\frac{1}{2}-\frac{1}{2}\right)\right)\right\} \\
&= \exp\left\{\pi i\left(2abt^2\left(\frac{r^2}{t^2}-\frac{1}{12}\right) - \frac{ac}{12t^2} + \frac{(a-1)r}{t}-\right)\right\} \\
&= \exp\left\{\pi i\left(-\frac{2abt^2}{12} - \frac{ac}{12t^2}\right)\right\},
\end{aligned}$$

and so we may write

$$\frac{\eta_{2t^2,t^2+2rt}(Mz)}{\eta_{2t^2,t^2+2rt}(z)} = \exp\left\{-\frac{\pi i a}{24}\left(4bt^2 + \frac{2c}{t^2}\right)\right\}, \qquad (3.21)$$

for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(4t^2)$. Since $\eta(z)$ is a modular form of weight $\frac{1}{2}$ on $\Gamma(1)$ with multiplier system v_η, it follows that, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2t^2)$:

$$\begin{aligned}
\eta(2t^2Mz) &= \eta\left(\frac{2t^2az + 2t^2b}{cz+d}\right) \\
&= \eta\left(\begin{pmatrix} a & 2t^2b \\ \frac{c}{2t^2} & d \end{pmatrix}(2t^2z)\right) \\
&= v_\eta\begin{pmatrix} a & 2t^2b \\ \frac{c}{2t^2} & d \end{pmatrix}\left(\frac{c}{2t^2}(2t^2z) + d\right)^{1/2}\eta(2t^2z).
\end{aligned}$$

Thus, for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4t^2)$ (so that $\frac{c}{2t^2}$ is even),

$$\eta(2t^2Mz) = v_\eta^{(t)}(M)(cz+d)^{1/2}\eta(2t^2z), \tag{3.22}$$

where

$$\begin{aligned}
&v_\eta^{(t)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&:= v_\eta\begin{pmatrix} a & 2t^2b \\ \frac{c}{2t^2} & d \end{pmatrix} \\
&= \left(\frac{c/2t^2}{d}\right)_* \exp\left\{\frac{\pi i}{12}\left[\frac{(a+d)c}{2t^2} - 2t^2bd\left(\frac{c^2}{(2t^2)^2} - 1\right) + 3d - 3 - \frac{3cd}{2t^2}\right]\right\} \\
&= \left(\frac{c/2t^2}{d}\right)_* \exp\left\{\frac{\pi i}{24}\left[\frac{(a-2d)c}{t^2} - bd\left(\frac{c^2}{t^2} - 4t^2\right) + 6(d-1)\right]\right\}.
\end{aligned}$$

We now combine (3.17), (3.20), (3.21) and (3.22) to derive a transformation law for $\vartheta_{r,t}$ on $\Gamma_1(4t^2)$. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(4t^2)$,

$$\begin{aligned}
&\vartheta_{r,t}(2Mz) \\
&= \frac{\eta_{4t^2,2t^2+4rt}(Mz)}{\eta_{2t^2,t^2+2rt}(Mz)}\eta(2t^2Mz) \\
&= \frac{\exp\left\{-\frac{\pi i a}{24}\left(8bt^2+\frac{c}{t^2}\right)\right\}\eta_{4t^2,2t^2+4rt}(z)}{\exp\left\{-\frac{\pi i a}{24}\left(4bt^2+\frac{2c}{t^2}\right)\right\}\eta_{2t^2,t^2+2rt}(z)}\left(\frac{c/2t^2}{d}\right)_* (cz+d)^{1/2}\eta(2t^2z)\times \\
&\qquad \exp\left\{\frac{\pi i}{24}\left[\frac{(a-2d)c}{t^2}-bd\left(\frac{c^2}{t^2}-4t^2\right)+6(d-1)\right]\right\} \\
&= \left(\frac{c/2t^2}{d}\right)_* (cz+d)^{1/2}\vartheta_{r,t}(2z)\times \\
&\qquad \exp\left\{\frac{\pi i a}{24}\left(-4bt^2+\frac{c}{t^2}\right)+\frac{\pi i}{24}\left[\frac{(a-2d)c}{t^2}-bd\left(\frac{c^2}{t^2}-4t^2\right)+6(d-1)\right]\right\} \\
&= \left(\frac{c/2t^2}{d}\right)_* (cz+d)^{1/2}\vartheta_{r,t}(2z)\times \\
&\qquad \exp\left\{\frac{\pi i}{24}\left(-4abt^2+\frac{ac}{t^2}+\frac{(a-2d)c}{t^2}-bd\left(\frac{c^2}{t^2}-4t^2\right)+6(d-1)\right)\right\} \\
&= \left(\frac{c/2t^2}{d}\right)_* (cz+d)^{1/2}\vartheta_{r,t}(2z)\times \\
&\qquad \exp\left\{\frac{\pi i}{24}\left(4bdt^2-4abt^2+\frac{2ac-2cd-bdc^2}{t^2}+6(d-1)\right)\right\} \\
&= \left(\frac{c/2t^2}{d}\right)_* e^{\frac{\pi i(d-1)}{4}}(cz+d)^{1/2}\vartheta_{r,t}(2z)\times \\
&\qquad \exp\left\{\frac{\pi i}{24}\left(4bt^2(d-a)+\frac{c}{t^2}(2a-2d-bcd)\right)\right\} \\
&= \left(\frac{c/2t^2}{d}\right)_* e^{\frac{\pi i(d-1)}{4}}(cz+d)^{1/2}\vartheta_{r,t}(2z)\times \\
&\qquad \exp\left\{\frac{\pi i}{24}\left(4bt^2(d-a)-\frac{2c}{t^2}(d-a)-\frac{bdc^2}{t^2}\right)\right\} \\
&= \left(\frac{c/2t^2}{d}\right)_* e^{\frac{\pi i(d-1)}{4}}(cz+d)^{1/2}\vartheta_{r,t}(2z)\exp\left\{\frac{\pi i}{24}\left(\left(4bt^2-\frac{2c}{t^2}\right)(d-a)-\frac{bdc^2}{t^2}\right)\right\}.
\end{aligned}$$

Therefore,

$$\frac{\vartheta_{r,t}(2Mz)}{(cz+d)^{1/2}\vartheta_{r,t}(2z)} = \begin{cases} \exp\left\{\frac{\pi i}{24}\left(\left(4bt^2 - \frac{2c}{t^2}\right)(d-a) - \frac{bdc^2}{t^2}\right)\right\} \times \\ \quad \left(\frac{c/2t^2}{|d|}\right)(-1)^{\frac{sign(c)-1}{2}\frac{sign(d)-1}{2}} e^{\frac{\pi i(d-1)}{4}}, & \text{if } c \neq 0, \\ \quad d e^{\frac{\pi i(d-1)}{4}}, & \text{if } c = 0. \end{cases}$$

We now make the simple observation

$$\left(\frac{c/2}{|d|}\right) = \left(\frac{\frac{c}{2t^2}t^2}{|d|}\right) = \left(\frac{c/2t^2}{|d|}\right)\left(\frac{t}{|d|}\right)^2 = \left(\frac{c/2t^2}{|d|}\right).$$

With this, we may use (3.3) to rewrite the above transformation law in the form

$$\vartheta_{r,t}(2Mz) = \tag{3.23}$$
$$v_\vartheta(M)\exp\left\{\frac{\pi i}{24}\left(\left(4bt^2 - \frac{2c}{t^2}\right)(d-a) - \frac{bdc^2}{t^2}\right)\right\}(cz+d)^{1/2}\vartheta_{r,t}(2z).$$

To prove the theorem, we must show that $48 \mid \left(\left(4bt^2 - \frac{2c}{t^2}\right)(d-a) + \frac{bdc^2}{t^2}\right)$. To this end, we note immediately that $4 \mid \left(4bt^2 - \frac{2c}{t^2}\right)$, $4t^2 \mid (d-a)$ and $4^2t^2 \mid \frac{c^2}{t^2}$; therefore, $\left(4bt^2 - \frac{2c}{t^2}\right)(d-a) - \frac{bdc^2}{t^2}$ is divisible by 4^2t^2. It remains to show, then, that $\left(4bt^2 - \frac{2c}{t^2}\right)(d-a) - \frac{bdc^2}{t^2}$ is divisible by 3.

If $3|t$, then the desired result follows, since (as noted in the preceding paragraph) $t \mid \left(\left(4bt^2 - \frac{2c}{t^2}\right)(d-a) - \frac{bdc^2}{t^2}\right)$.

We proceed, then, under the assumption that $3 \nmid t$. In this case we have $t^2 \equiv 1$ (3). We define $a', c', d' \in \mathbf{Z}$ as follows:

$$a' := \frac{a-1}{4t^2}, \quad d' := \frac{d-1}{4t^2}, \quad c' := \frac{c}{4t^2}. \tag{3.24}$$

Thus, $a = 4t^2a'+1, d = 4t^2d'+1$, and $c = 4t^2c$. We note that, since $M \in \Gamma_1(3)$, we must have $3|c'$. Further, by rewriting $ad - bc = 1$ in terms of these new variables, we derive the relationship:

$$\begin{aligned}(4t^2a'+1)(4t^2d'+1)-4t^2bc' &= 1\\ 16t^4a'd'+4t^2(a'+d') &= 4t^2bc'\\ \Longrightarrow a'd'+a'+d' &\equiv 0 \ \ (3)\\ \Longleftrightarrow (a'+1)(d'+1) &\equiv 1 \ \ (3)\\ \Longleftrightarrow a'+1\equiv d'+1 &\equiv \pm 1 \ \ (3)\\ \Longleftrightarrow a'\equiv d' &\equiv 0 \text{ or } 1 \ \ (3).\end{aligned} \tag{3.25}$$

In terms of a', c' and d', we have:

$$\begin{aligned}\left(4bt^2-\frac{2c}{t^2}\right)(d-a)-\frac{bdc^2}{t^2} &= (4bt^2-8c')4t^2(d'-a')-b(4t^2d'+1)16t^2c'^2\\ &\equiv b(d'-a') \ \ (3)\\ &\equiv 0 \ \ (3).\end{aligned}$$

The final step above follows from (3.25). This proves the theorem.

Corollary 3.2 *$\vartheta(2z;r,t)$ is an entire modular form on $\Gamma_1(4t^2)\cap\Gamma_0(3)$, of weight $\frac{1}{2}$, with multiplier system v_ϑ.*

Proof This follows from Lemma 3.1 and Theorem 3.2:

$$\begin{aligned}\vartheta(2Mz;r,t) &= \sum_{\substack{R=0\\ R^2\equiv r(t)}}^{t-1} \vartheta_{R,t}(2Mz)\\ &= \sum_{\substack{R=0\\ R^2\equiv r(t)}}^{t-1} v_\vartheta(M)(cz+d)^{1/2}\vartheta_{R,t}(2z)\\ &= v_\vartheta(M)(cz+d)^{1/2}\sum_{\substack{R=0\\ R^2\equiv r(t)}}^{t-1} \vartheta_{R,t}(2z)\\ &= v_\vartheta(M)(cz+d)^{1/2}\vartheta(2z;r,t).\end{aligned}$$

3.4 Sums of Squares

Definition 3.5

$$\vartheta^*(2z) := \sum_n e^{2\pi i(2n+1)^2 z}.$$

Remark 3.5

$$\vartheta^*(2z) = \vartheta(2z; 1, 2).$$

Remark 3.6 *It follows from Corollary 3.2 that $\vartheta^*(2z)$ is a modular form on $\Gamma_0(16)$.*

In the interest of simplifying notation later on, we define the functions f_s and g_s as follows:

Definition 3.6

$$f_s(z) := \vartheta(2z)^s, \qquad g_s(z) := \vartheta^*(2z)^s. \tag{3.26}$$

We now define $r_s(n)$ and $r_s^*(n)$ on $\{n \in \mathbf{Z} | n \geq 0\}$ in terms of their generating functions:

$$f_s(z) = \sum_{n \geq 0} r_s(n) e^{2\pi i n z}, \tag{3.27}$$

$$g_s(z) = \sum_{n \geq 0} r_s^*(n) e^{2\pi i n z}. \tag{3.28}$$

It follows that $r_s(n)$ and $r_s^*(n)$ have the following arithmetic properties:

$$r_s(n) = \#\{(x_1, ..., x_s) : x_i \in \mathbf{Z}, x_1^2 + ... + x_s^2 = n\}, \tag{3.29}$$

$$r_s^*(n) = \#\{(x_1, ..., x_s) : x_i \in \mathbf{Z}, 2 \not| x_i, x_1^2 + ... + x_s^2 = n\}. \tag{3.30}$$

Remark 3.7 *For* $n < 0$, $r_s(n) = r_s^*(n) = 0$.

Remark 3.8 *We wish to study the behavior of the ratio* $\dfrac{r_s(m)}{r_s^*(m)}$ *in arithmetic progressions on* m. *However, we note that* $r_s^*(m)$ *is nonzero if and only if* $m \equiv s\ (8)$ *and* $m \geq s$ *(since* $x^2 \equiv 1\ (8)$ *for all odd* x*). Therefore, we restrict ourselves to arithmetic progressions of the type* $m \equiv t\ (u)$ *with* $8|u$ *and* $t \equiv s\ (8)$.

Theorem 3.3 *Let* $s, t, u \in \mathbf{Z}$ *be given such that* $0 \leq t < u$, $8|u$, $s \geq u$ *and* $s \equiv t\ (8)$. *Put* $\mathcal{M} := \{m \in \mathbf{Z} : m \geq 0, m \equiv t\ (u)\}$. *Then,* $\dfrac{r_s(m)}{r_s^*(m)}$ *is not constant on* $\mathcal{M}$.

Remark 3.9 *The requirements* $8|u$, $m \equiv t\ (u)$ *and* $s \equiv t\ (8)$ *guarantee that* $m \equiv s\ (8), \forall m \in \mathcal{M}$. *Therefore,* $r_s^*(m) \neq 0$ *on* $\mathcal{M}$.

Remark 3.10 *This theorem is a generalization of the result of [3]. In particular, the cited result follows from this theorem if we put* $u = 8$ *and* $t = s$.

Proof We define the auxiliary function

$$F_s(z|t,u) := \sum_{\substack{m \equiv t(u) \\ m \geq s}} r_s(n) e^{2\pi i m z},$$

and note that

$$\begin{aligned}
&f_s(z;t,u) - F_s(z|t,u) \\
&= \sum_{\substack{m\equiv t(u)\\ m\geq 0}} r_s(m)e^{2\pi imz} - \sum_{\substack{m\equiv t(u)\\ m\geq t}} r_s(m)e^{2\pi imz} \\
&= \sum_{\substack{n\in Z\\ un+t\geq 0}} r_s(un+t)e^{2\pi i(un+t)z} - \sum_{\substack{n\in Z\\ un+t\geq s}} r_s(un+t)e^{2\pi i(un+t)z} \\
&= \sum_{n\geq\lceil\frac{-t}{u}\rceil} r_s(un+t)e^{2\pi i(un+t)z} - \sum_{n\geq\lceil\frac{s-t}{u}\rceil} r_s(un+t)e^{2\pi i(un+t)z} \\
&= \sum_{n\geq 0} r_s(un+t)e^{2\pi i(un+t)z} - \sum_{n\geq\lceil\frac{s-t}{u}\rceil} r_s(un+t)e^{2\pi i(un+t)z} \\
&= \sum_{0\leq n<\lceil\frac{s-t}{u}\rceil} r_s(un+t)e^{2\pi i(un+t)z}. \qquad (3.31)
\end{aligned}$$

Since $s \geq u$, the right-hand side of (3.31) is a finite, non-trivial polynomial in $e^{2\pi iz}$. For, since

$$\left\lceil\frac{s-t}{u}\right\rceil \geq \left\lceil\frac{u-t}{u}\right\rceil = 1,$$

at least one term must appear on the right-hand side of (3.31) – specifically, the term $r_s(t)e^{2\pi itz}$ (corresponding to $n = 0$) must appear.

Rather than similarly defining an auxiliary function to $g_s(z;t,u)$, we instead note that, since $r_s^*(m) = 0$ for all $m < s$,

$$\begin{aligned}
g_s(z;t,u) &= \sum_{\substack{m\equiv t(u)\\ m\geq s}} r_s^*(m)e^{2\pi imz} \\
&= \sum_{\substack{n\in Z\\ un+t\geq s}} r_s^*(un+t)e^{2\pi i(un+t)z} \\
&= \sum_{n\geq\lceil\frac{s-t}{u}\rceil} r_s^*(un+t)e^{2\pi i(un+t)z}.
\end{aligned}$$

Choose $c \in \mathbb{Q}$, independent of z, and put

$$\Psi_c(z; s, t, u) := f_s(z; t, u) - cg_s(z; t, u). \tag{3.32}$$

It follows from Theorems 1.1 and 3.1 that $f_s(z)$, $g_s(z)$, and (therefore) $\Psi_c(z; s, t, u)$ are modular form on $\Gamma_{0,u}(2u^2)$ of weight $\frac{s}{2}$ with multiplier system v_ϑ^s.

We combine (3.31) and (3.32) to obtain:

$$\begin{aligned}
&\Psi_c(z; s, t, u) \\
&= \sum_{0 \le n < \lceil \frac{s-t}{u} \rceil} r_s(un+t)e^{2\pi i(un+t)z} + F_s(z|t, u) - cg_s(z; t, u) \\
&= \sum_{0 \le n < \lceil \frac{s-t}{u} \rceil} r_s(un+t)e^{2\pi i(un+t)z} + \\
&\qquad \sum_{n \ge \lceil \frac{s-t}{u} \rceil} r_s(un+t)e^{2\pi i(un+t)z} - c \sum_{n \ge \lceil \frac{s-t}{u} \rceil} r_s^*(un+t)e^{2\pi i(un+t)z} \\
&= \sum_{0 \le n < \lceil \frac{s-t}{u} \rceil} r_s(un+t)e^{2\pi i(un+t)z} + \\
&\qquad \sum_{n \ge \lceil \frac{s-t}{u} \rceil} \left(r_s(un+t) - cr_s^*(un+t)\right) e^{2\pi i(un+t)z}. \\
&= \sum_{\substack{m \equiv t(u) \\ 0 \le m < u\lceil \frac{s-t}{u} \rceil + t}} r_s(m)e^{2\pi imz} + \sum_{\substack{m \equiv t(u) \\ m \ge u\lceil \frac{s-t}{u} \rceil + t}} \left(r_s(m) - cr_s^*(m)\right) e^{2\pi imz}.
\end{aligned} \tag{3.33}$$

Since $\Psi_c(z; s, t, u)$ is a nontrivial modular form on a subgroup of finite index in $\Gamma(1)$, it must have the real line as a natural boundary (see [6], p. 20, exercise 2); that is, the right-hand side of (3.33) may not be a finite polynomial in $e^{2\pi iz}$. Therefore, $r_s(m) - cr_s^*(m) \neq 0$ infinitely often for $m \in \mathcal{M}$, and so $\dfrac{r_s(m)}{r_s^*(m)}$ is not constant on $\mathcal{M}$. This completes the proof of Theorem 3.3.

We now prove a weaker, though still useful, result for the case $0 < s < u$.

Theorem 3.4 *Let $s, t, u \in Z$ be given such that $s > 0$, $0 \le t < u$, $8|u$ and $t \equiv s(8)$. Let $\mathcal{M} = \{m \in \mathbf{Z} : m \ge 0, m \equiv t\ (u)\}$. Suppose that there exists some $N \in \mathbf{Z}$, $N \ge 0$, for which $\frac{r_s(m)}{r_s^*(m)}$ is constant for $m \in \mathcal{M}, m \ge N$. Then, $\frac{r_s(m)}{r_s^*(m)}$ is constant on $\mathcal{M}$. In particular, $\frac{r_s(m)}{r_s^*(m)} = \frac{r_s(t)}{r_s^*(t)}$, for all $m \in \mathcal{M}$.*

Remark 3.11 *In other words: either $\frac{r_s(m)}{r_s^*(m)}$ is constant on* ***all*** *of $\mathcal{M}$, or there exists* ***no*** *positive integer N for which $\frac{r_s(m)}{r_s^*(m)}$ is constant in m for $m \in \mathcal{M}, m \ge N$.*

Remark 3.12 *This theorem provides an* ***experimental*** *approach to proving that the ratio $\frac{r_s(m)}{r_s^*(m)}$ is not constant on $\mathcal{M}$: if one can find* ***any*** *pair of integers $m_1, m_2 \in \mathcal{M}$ such that $\frac{r_s(m_1)}{r_s^*(m_1)} \neq \frac{r_s(m_2)}{r_s^*(m_2)}$, then the result of Theorem 3.4 follows.*

Proof Suppose that $\frac{r_s(m)}{r_s^*(m)} = c$ for all $m \in \mathcal{M}, m \ge N$. (Note: c is a rational number which depends on s, t, and u, but *not* on m.)

As in the proof of Theorem 3.3, we define

$$\Psi_c(z; s, t, u) := f_s(z; t, u) - c g_s(z; t, u),$$

and so

$$\begin{aligned} \Psi_c(z; s, t, u) &= \sum_{m \in \mathcal{M}} r_s(m) e^{2\pi i m z} - c \sum_{m \in \mathcal{M}} r_s^*(m) e^{2\pi i m z} \\ &= \sum_{m \in \mathcal{M}} (r_s(m) - c r_s^*(m)) e^{2\pi i m z}. \end{aligned} \tag{3.34}$$

Since $\Psi_c(z; s, t, u)$ is a modular form on $\Gamma_{0,u}(1024u^2)$, the right side of (3.34) may not be a nontrivial, finite polynomial in $e^{2\pi i z}$. Therefore, it must either be an infinite polynomial in $e^{2\pi i z}$, or *identically zero*. The former implies $\frac{r_s(m)}{r_s^*(m)} \neq c$ infinitely often in $\mathcal{M}$; the latter implies $\frac{r_s(m)}{r_s^*(m)} = c$ for all $m \in \mathcal{M}$. This completes the proof of Theorem 3.4.

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