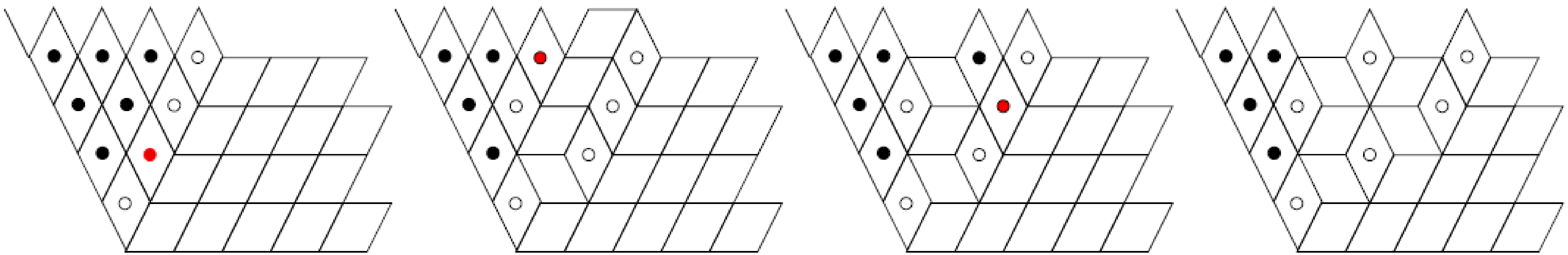
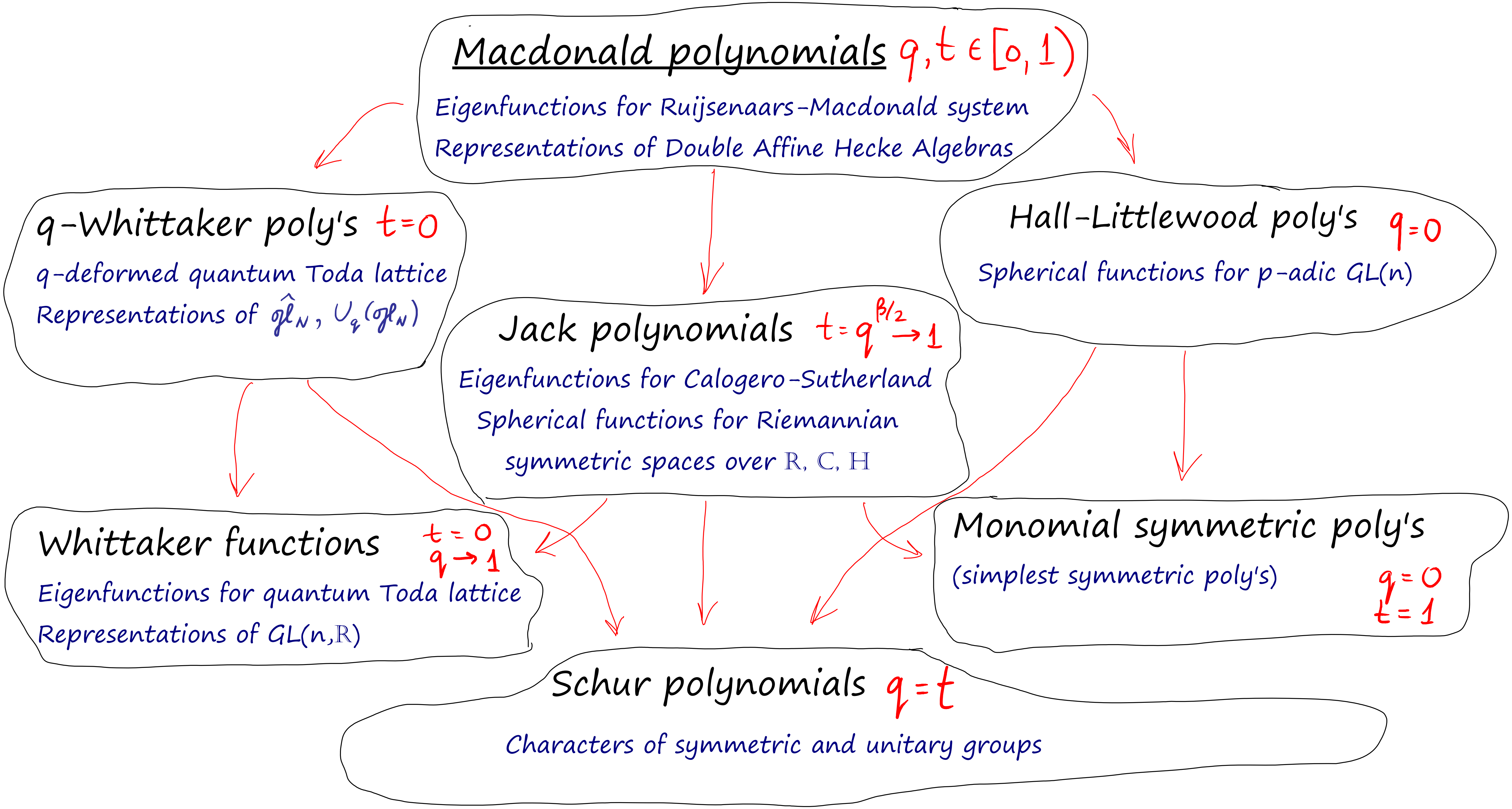
The push-block dynamics [B-Ferrari '08]

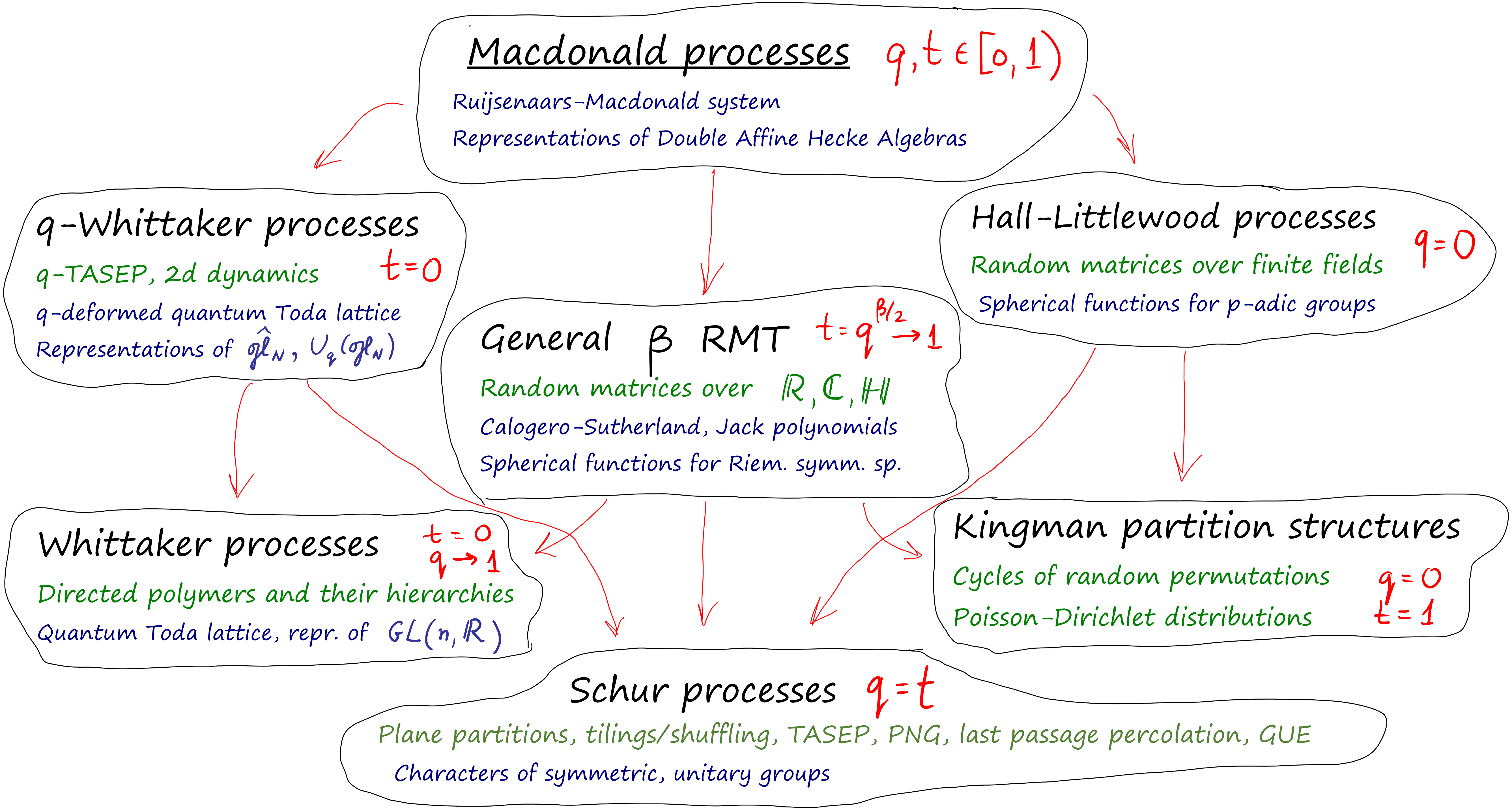
Each particle jumps to the right with rate 1. It is blocked by lower particles and it (short-range) pushes higher particles.



- Left-most particles form TASEP
- Right-most particles form PushTASEP

Previously studied asymptotics thus yields detailed information on large time behavior of these (2+1)d AKPZ and (1+1)d AKPZ models.


Macdonald polynomials $q,t \in [0,1)$
Eigenfunctions for Ruijsenaars-Macdonald system
Representations of Double Affine Hecke Algebras
q-Whittaker poly's $t=0$
q-deformed quantum Toda lattice
Representations of $\widehat{\mathfrak{gl}}_N$, $U_q(\mathfrak{gl}_N)$
Hall-Littlewood poly's $q=0$
Spherical functions for p-adic GL(n)
Jack polynomials $t=q^{\beta/2}\to 1$
Eigenfunctions for Calogero-Sutherland
Spherical functions for Riemannian
symmetric spaces over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$
Whittaker functions $t=0$, $q\to 1$
Eigenfunctions for quantum Toda lattice
Representations of $GL(n,\mathbb{R})$
Monomial symmetric poly's
(simplest symmetric poly's) $q=0$, $t=1$
Schur polynomials $q=t$
Characters of symmetric and unitary groups


Macdonald processes $q, t \in [0,1)$
Ruijsenaars-Macdonald system
Representations of Double Affine Hecke Algebras
q-Whittaker processes
q-TASEP, 2d dynamics
$t=0$
q-deformed quantum Toda lattice
Representations of $\widehat{\mathfrak{gl}}_N$, $U_q(\mathfrak{gl}_N)$
Hall-Littlewood processes
$q=0$
Random matrices over finite fields
Spherical functions for p-adic groups
General β RMT
$t=q^{\beta/2}\to 1$
Random matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$
Calogero-Sutherland, Jack polynomials
Spherical functions for Riem. symm. sp.
Whittaker processes
$t=0$
$q\to 1$
Directed polymers and their hierarchies
Quantum Toda lattice, repr. of $GL(n,\mathbb{R})$
Kingman partition structures
Cycles of random permutations
$q=0$
$t=1$
Poisson-Dirichlet distributions
Schur processes $q=t$
Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE
Characters of symmetric, unitary groups

Macdonald polynomials $P_\lambda(x_1,\dots,x_N) \in \mathbb{Q}(q,t)[x_1,\dots,x_N]^{S(N)}$ labelled by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0)$ form a basis in symmetric polynomials in N variables over $\mathbb{Q}(q,t)$. They diagonalize

$$\mathcal{D}_1 = \sum_{i=1}^{N} \Big(\prod_{a<b} (x_a - x_b)^{-1} \, T_{t,x_i} \prod_{a<b} (x_a - x_b) \Big) T_{q,x_i} = \sum_{i=1}^{N} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} \, T_{q,x_i}$$

with (generically) pairwise different eigenvalues $\quad (T_q f)(z) = f(qz)$

$$\mathcal{D}_1 P_\lambda = (q^{\lambda_1} t^{N-1} + q^{\lambda_2} t^{N-2} + \dots + q^{\lambda_N}) P_\lambda .$$

Macdonald polynomials have many remarkable properties that include orthogonality, simple reproducing kernel (Cauchy identity), Pieri and branching rules, index/variable duality, simple higher order Macdonald difference operators that commute with $\mathcal{D}_1$, etc.

Single level distributions

As in the Schur case, one can define probability measures via

$$\prod_{i=1}^{N} e^{\gamma(x_i-1)} = \sum_{\mu=(\mu_1\geq\ldots\geq\mu_N\geq 0)} \mathrm{Prob}_\gamma\{\mu\} \cdot \frac{P_\mu(x_1,\ldots,x_N)}{P_\mu(1,\ldots,1)}.$$

These are time γ distributions of the Markov chain with jump rates

$$L^{(N)}_{\text{Poisson}}(\mu\to\nu) = \sum_\nu \varphi_{\nu/\mu} \cdot \frac{P_\nu(1,\ldots,1)}{P_\mu(1,\ldots,1)}$$

replace Vandermondes

with $\varphi_{\nu/\mu}$ given by the Pieri rule (they are 0 or 1 for Schur)

$$(x_1+\ldots+x_N)\,P_\mu(x_1,\ldots,x_N) = \sum_\nu \varphi_{\nu/\mu}\, P_\nu(x_1,\ldots,x_N).$$

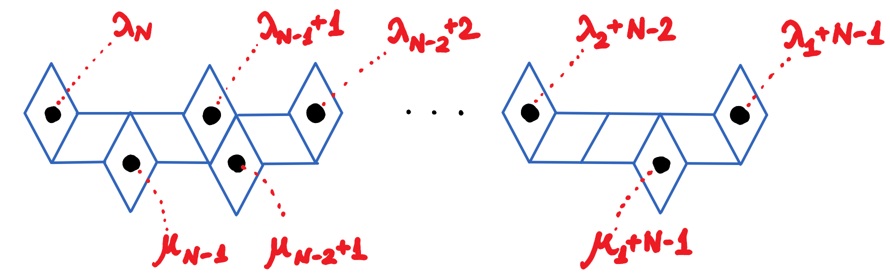
For $t=0$, $\varphi_{\mu+\vec{e}_j/\mu} = 1-q^{\mu_{j-1}-\mu_j}$.

This is a (q,t)-analog of the Dyson Brownian Motion.

Representation theoretic object: Quantum Random Walk.

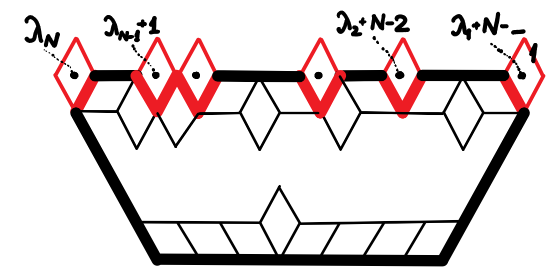
The (q,t)-Gibbs property

We define stochastic links Λ^N_{N-1} between N-tuples and $(N-1)$-tuples of integers using the branching rule



$$\frac{P_\lambda(x_1,\dots,x_{N-1},1)}{P_\lambda(1,\dots,1)} = \sum_{\mu \prec \lambda} \Lambda^N_{N-1}(\lambda \searrow \mu) \cdot \frac{P_\mu(x_1,\dots,x_{N-1})}{P_\mu(1,\dots,1)}.$$

<u>Def.</u> Random interlacing arrays $\lambda^{(1)} \prec \lambda^{(2)} \prec \dots \prec \lambda^{(N)}$ have the Macdonald-Gibbs property iff



$$\mathrm{Prob}\left\{(\lambda^{(1)},\dots,\lambda^{(N-1)}) \mid \lambda^{(N)}\right\} = \Lambda^N_{N-1}(\lambda^{(N)} \searrow \lambda^{(N-1)})\, \Lambda^{N-1}_{N-2}(\lambda^{(N-1)} \searrow \lambda^{(N-2)}) \dots \Lambda^2_1(\lambda^{(2)} \searrow \lambda^{(1)}).$$

$n!_q = 1 \cdot (1+q) \cdot (1+q+q^2) \cdots (1+q+\dots+q^{n-1})$

For $t=0$ the links are $\Lambda^N_{N-1}(\lambda \downarrow \mu) = \dfrac{P_\mu(1,\dots,1)}{P_\lambda(1,\dots,1)} \cdot \displaystyle\prod_{i=1}^{N-1} \frac{(\lambda_i - \lambda_{i+1})!_q}{(\lambda_i - \mu_i)!_q\, (\mu_i - \lambda_{i+1})!_q}.$

Macdonald processes

An (ascending) Macdonald process is a distribution on $\lambda^{(1)} \prec \lambda^{(2)} \prec \lambda^{(3)} \prec \cdots$ that is (q,t)-Gibbs (once can also use $(a_1, a_2, \ldots)$ instead of $(1,1, \ldots)$).

Example 1: Decompositions of $\prod_{i=1}^{N} e^{\gamma(x_i - 1)}$ correspond to the \`Plancherel specialization' (consistency with Gibbs is nontrivial).

Example 2: $t = q^{\theta} \to 1$, $a_j = t^j$ for $j \geq 1$, \`principal specialization'.
Single level measures converge to general $\beta = 2\theta$ Jacobi ensembles

$$\mathrm{const} \cdot \prod_{i<j} |y_i - y_j|^{\beta} \prod_i y_i^{s_0} (1 - y_i)^{s_1} \, dy_i, \qquad y_i \in (0,1).$$

Example 3: Plancherel specialization, $t=0$. Leads to local 2d dynamics, q-TASEP, q-PushASEP, random polymers in $(1+1)$d.
Will be our focus.

Macdonald operators

Macdonald's q-difference operators diagonalized by P_λ are

$$\mathcal{D}^{(k)} = \sum_{I\subset\{1,\dots,N\}} \prod_{\substack{i\in I\\ j\notin I}} \frac{t x_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i}, \qquad \mathcal{D}^{(k)} P_\lambda = e_k(q^{\lambda_1} t^{N-1}, \dots, q^{\lambda_N}) P_\lambda,$$

where $e_k(z_1, z_2, \dots) = \sum_{i_1<\dots<i_k} z_{i_1}\cdots z_{i_k}$. Using $\mathcal{D}\chi\big|_{x_j=1} = \sum_\lambda d_\lambda \operatorname{Prob}\{\lambda\} = \mathbb{E} d_\lambda$ with these operators gives many observables with explicit averages.

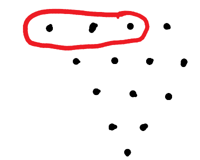
<u>Example 1:</u> For the Jacobi ensembles $\prod_{i<j} |y_i - y_j|^\beta \prod_i y_i^{s_0} (1-y_i)^{s_1}$ this gives averages of the powers sums $\sum_i y_i^k$ and of their products.

<u>Example 2:</u> For $t=0$ this gives averages of products of $q^{\lambda_N + \lambda_{N-1} + \dots + \lambda_{N-k+1}}$.

Integrals and scaling limits

For $t=0$ and Plancherel specialization (decomposition of $\prod_{i=1}^{N} e^{\gamma(x_i-1)}$),
turning Macdonald operator $\mathcal{D}_k$ into a contour integral gives

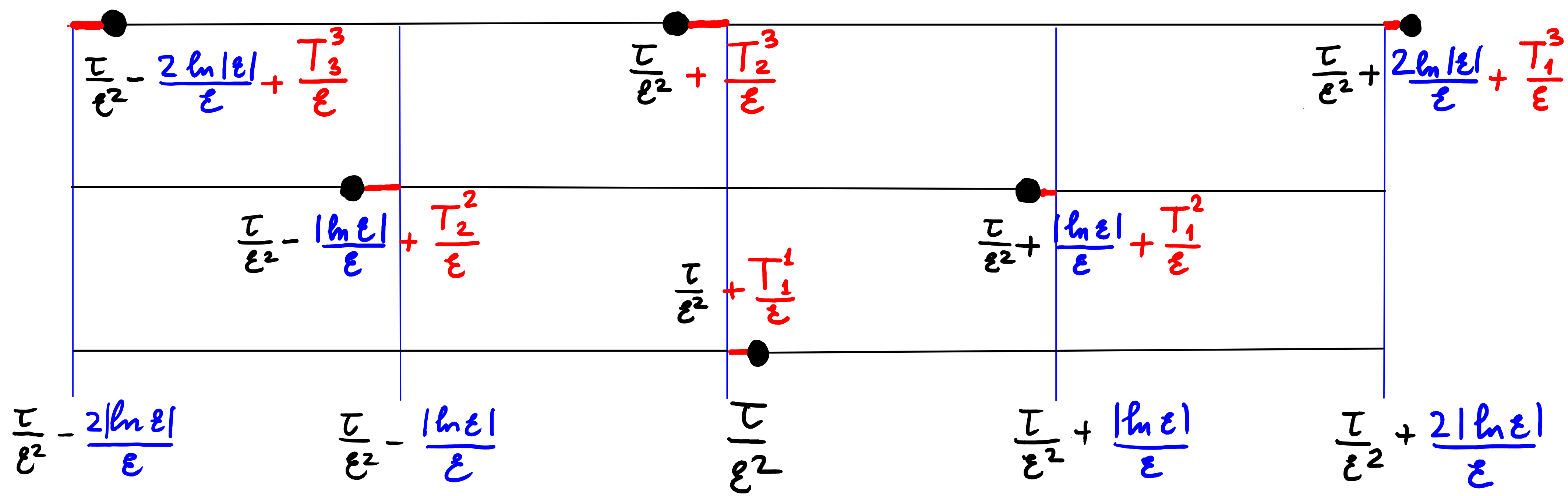
$$\mathbb{E}\, q^{\lambda_N^{(N)}+\dots+\lambda_{N-k}^{(N)}} = \frac{(-1)^{\frac{k(k+1)}{2}}}{(2\pi i)^k\, k!} \oint\cdots\oint_{\text{around } 1} \prod_{1\le A<B\le k} (z_A - z_B)^2 \prod_{j=1}^{k} \frac{e^{(q-1)\gamma z_j}}{(1-z_j)^N} \frac{dz_j}{z_j^k}$$



$N=5$
$k=3$

The RHS has a clear limit as $q=e^{-\varepsilon}\to 1$, $\gamma=\text{const}\cdot\varepsilon^{-1}$, z_j's unchanged.
This leads to a LLN $\lambda_j^{(N)} \sim c_j^{(N)}\cdot\varepsilon^{-1}$ and Gaussian fluctuations of size $\varepsilon^{-1/2}$.

A less obvious limit is $q=e^{-\varepsilon}\to 1$, $\gamma=\tau\cdot\varepsilon^{-2}$, $z_j = 1+\varepsilon w_j$ for $1\le j\le k$.
Then the RHS behaves as $e^{-\tau k \varepsilon^{-1}}\cdot \varepsilon^{k(k-N)}\cdot$ finite integral.
This suggests the following scaling behavior:



Theorem [B-Corwin '11] As $q = e^{-\varepsilon} \to 1$, $\gamma = \tau \cdot \varepsilon^{-2}$, under the scaling

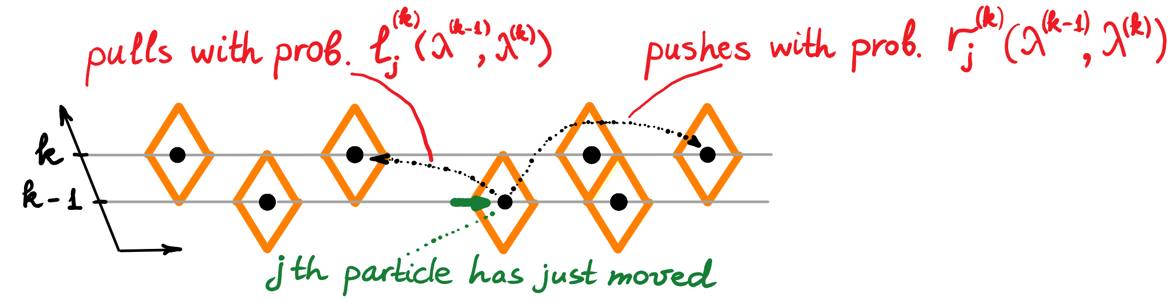
$$\lambda_j^{(N)} = \tau \varepsilon^{-2} - (N+1-2j)\frac{\ln \varepsilon}{\varepsilon} + T_j^N \varepsilon^{-1}$$

the t=0 Macdonald process with Plancherel specialization weakly converges to a probability distribution on real arrays $\{T_j^N\}$ (*the Whittaker process*).

Is there a probabilistic meaning behind the Whittaker process?

Back to Markov dynamics

The classification problem for the nearest neighbor Markov dynamics that preserve Gibbs measures and coincides with (q,t)-DBM on each level is (as for Schur) equivalent to a linear system of equations of the form [B–Petrov '13]



$$B_{j+1}^{(k)}\, r_{j+1}^{(k)} + B_j^{(k)}\, l_j^{(k)} + w_{j+1}^{(k)} = A_{j+1}^{(k)}$$

For $t=0$, the quantities $A_j^{(k)}$ and $B_j^{(k)}$ are local:

$$A_j^{(k)} = \frac{\left(1-q^{\lambda_{j-1}^{(k-1)}-\lambda_j^{(k)}}\right)\left(1-q^{\lambda_j^{(k)}-\lambda_{j+1}^{(k)}+1}\right)}{1-q^{\lambda_j^{(k)}-\lambda_j^{(k-1)}+1}}, \qquad B_j^{(k)} = \frac{\left(1-q^{\lambda_j^{(k-1)}-\lambda_{j+1}^{(k)}}\right)\left(1-q^{\lambda_{j-1}^{(k-1)}-\lambda_j^{(k-1)}+1}\right)}{1-q^{\lambda_j^{(k)}-\lambda_j^{(k-1)}+1}}.$$

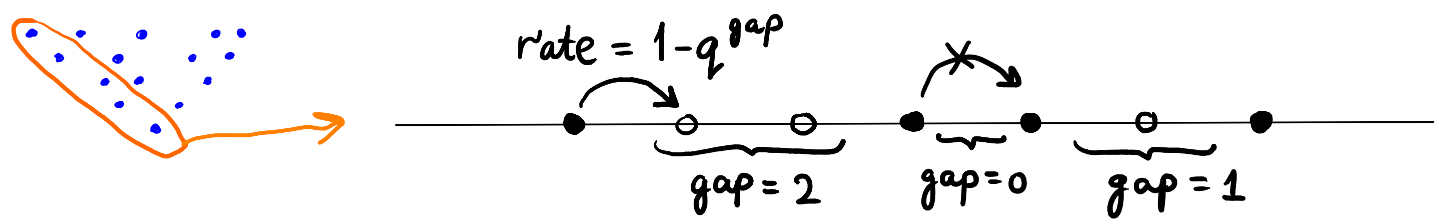
q-TASEP, q-PushTASEP, and 2d dynamics

There are many solutions. Imposing no pulling/pushing over distances >1 leads to the 2d local dynamics of [B-Corwin '11]:

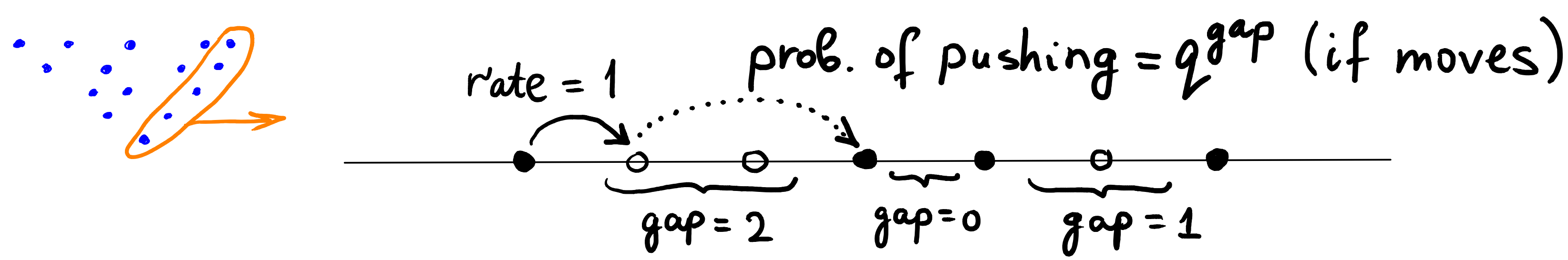
$$\ell_j = r_j \equiv 0, \qquad w_j^{(k)} = \frac{\left(1-q^{\lambda_{j-1}^{(k-1)}-\lambda_j^{(k)}}\right)\left(1-q^{\lambda_j^{(k)}-\lambda_{j+1}^{(k)}+1}\right)}{1-q^{\lambda_j^{(k)}-\lambda_j^{(k-1)}+1}}.$$

Simulation

Projecting to left-most particles of each row yields q-TASEP:



Imposing almost sure jump propagation $\ell_j + r_j \equiv 1$ and $w_j = \begin{cases} 1, & j=1 \\ 0, & j>1 \end{cases}$ and further projecting to right-most particles yields q-PushTASEP:



Semi-discrete Brownian directed polymers

Whittaker scaling on q-PushTASEP (and q-TASEP) yields

$$dT_1^N = dB_N + e^{T_1^{N-1} - T_1^N} d\tau, \qquad N \geq 1,$$

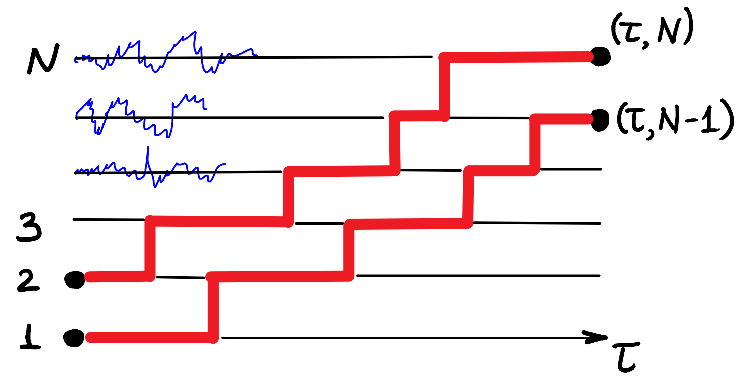
with independent Brownian motions $B_1, B_2, \ldots$ (same for $\{-T_N^N\}_{N\geq 1}$).

Solving gives $T_1^N = \log \int_{0<s_1<\cdots<s_{N-1}<\tau} e^{B_1(s_1) + (B_2(s_2) - B_2(s_1)) + \ldots + (B_N(\tau) - B_N(s_{N-1}))} ds_1 \cdots ds_{N-1}$.

<u>Theorem</u> [O'Connell '09], [B-Corwin '11]

Lebesgue

$$T_1^N + \ldots + T_k^N = \log \int \cdots \int e^{E(\phi_1) + \ldots + E(\phi_k)} d\phi_1 \cdots d\phi_k$$



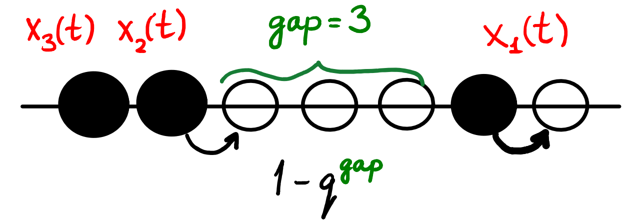
with integration over nonintersecting paths from (1,…,k) to (N-k+1,…,N). The measure is symmetric with respect to the flip $\{T_k^N \longleftrightarrow -T_{N-k+1}^N\}$.

$$E(\phi) = \int_\phi (\mathbb{R} \times \mathbb{Z})\text{-white noise}$$

$$= B_j(s_1) + (B_{j+1}(s_2) - B_{j+1}(s_1)) + \ldots$$

q-TASEP moments

We now focus on left-most particles (q-TASEP) and wish to study the asymptotics as N gets large.

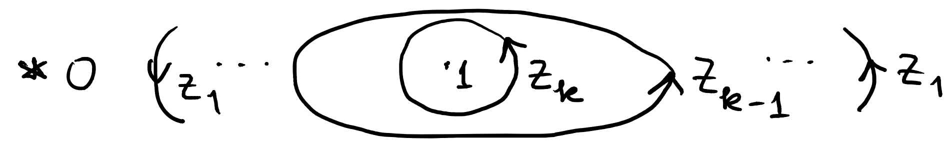


Theorem [B-Corwin '11], [B-C-Sasamoto '12], [B-C-Gorin-Shakirov '13]

For the q-TASEP with step initial data $\{x_n(0)=-n\}_{n\geq 1}$

$$\mathbb{E}\, q^{(x_{N_1}(t)+N_1)+\dots+(x_{N_k}(t)+N_k)} = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k}\oint\cdots\oint \prod_{A<B}\frac{z_A-z_B}{z_A-qz_B}\prod_{j=1}^{k}\frac{e^{(q-1)tz_j}}{(1-z_j)^{N_j}}\frac{dz_j}{z_j}$$

$(N_1\geq N_2\geq \dots \geq N_k)$



Proof. Consider the Macdonald process with Plancherel specialization and apply k first order Macdonald operators in $N_1, N_2, \dots, N_k$ variables. $\square$

Another proof via Quantum Integrable Systems will be given in Lecture 3.

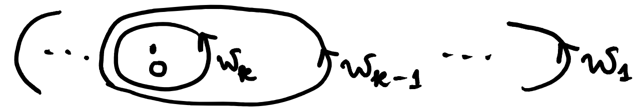
Polymer moments via nested integrals

By (formal) limit transitions:

For $$Z(N,\tau) = \int_{0<s_1<\dots<s_{N-1}<\tau} e^{B_1(s_1)+(B_2(s_2)-B_2(s_1))+\dots+(B_N(\tau)-B_N(s_{N-1}))}\, ds_1\cdots ds_{N-1}$$

$$\mathbb{E}\left[Z(N_1,\tau)\cdots Z(N_k,\tau)\right] = \frac{e^{\tau k/2}}{(2\pi i)^k}\oint\cdots\oint \prod_{1\le A<B\le k}\frac{w_A-w_B}{w_A-w_B-1}\prod_{j=1}^{k}\frac{e^{\tau w_j}}{w_j^k}\,dw_j$$

$N_1 \geq N_2 \geq \dots \geq N_k$



For $$Z(x,t) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\int :\exp:\left\{\int_0^t \dot{W}(s,b(s))\,ds\right\} db$$

Brownian bridge $b:[0,t]\to\mathbb{R}$, $b(0)=0$, $b(t)=x$

2d white noise

$$\frac{\partial Z}{\partial t} = \frac{1}{2}\frac{\partial^2 Z}{\partial x^2} + \dot{W}Z \qquad \text{SHE}$$

$$\frac{\partial \ln Z}{\partial t} = \frac{1}{2}\frac{\partial^2 \ln Z}{\partial x^2} + \left(\frac{\partial \ln Z}{\partial x}\right)^2 + \dot{W} \qquad \text{KPZ}$$

$$\mathbb{E}\left[Z(x_1,t)\cdots Z(x_k,t)\right] = \int_{\alpha_1-i\infty}^{\alpha_1+i\infty} dz_1 \int_{\alpha_2-i\infty}^{\alpha_2+i\infty} dz_2 \cdots \int_{\alpha_k-i\infty}^{\alpha_k+i\infty} \prod_{1\le A<B\le k}\frac{z_A-z_B}{z_A-z_B-1}\prod_{j=1}^{k} e^{\frac{t}{2}z_j^2 + x_j z_j}$$

$x_1 \leq x_2 \leq \dots \leq x_k$

$\alpha_1 > \alpha_2 + 1 > \dots > \alpha_k + (k-1)$

Is this sufficient for determining the distributions of Z's?

Intermittency

Polymer partition functions Z are intermittent. Higher moments are dominated by higher peaks and do not determine the distrib. This is measured by moment Lyapunov exponents $\gamma_p = \lim_{t\to\infty} \frac{\ln \mathbb{E}\, Z^p(t)}{t}$.

$\frac{\gamma_p}{p} \neq \text{const}$ means intermittency [Zeldovitch et al. '87].

By steepest descent in nested integrals one shows:

Semi-discrete: $\gamma_p = H_p(z_c)$, where (for $N=\tau$) z_c is the crit. point of

[B-Corwin '12]

$$H_p(z) = \frac{p^2}{2} + pz - \log\left(\frac{\Gamma(z+p)}{\Gamma(z)}\right) \text{ on } (0,+\infty).$$

Continuous: $\gamma_p = \frac{p^3 - p}{24}$.

[Kardar '87], [Bertini-Cancrini '95]

The speed of growth of Lyapunov exp's does not predict fluctuation exponents!

Replica trick

In its simplest incarnation, ignoring intermittency, replica trick analytically continues moments off positive integers and uses

$$\log Z = \lim_{p\to 0} \frac{Z^p - 1}{p}, \qquad \lim_{t\to\infty} \frac{\ln Z}{t} = \lim_{p\to 0} \lim_{t\to\infty} \frac{1}{t} \frac{e^{t\gamma_p} - 1}{p} = \lim_{p\to 0} \frac{\gamma_p}{p}$$

to predict the almost sure behavior. This gives correct LLN values:

Semi-discrete: $\lim_{p\to 0} \frac{1}{p}\left(\frac{p^2}{2} + pz - \log \frac{\Gamma(z+p)}{\Gamma(z)} \right) = z - \left(\log \Gamma(z)\right)'$, take value at crit. point on $(0,+\infty)$

Proved: [O'Connell-Yor '01], [Moriarty-O'Connell '07]

Continuous: $\lim_{p\to 0} \frac{1}{p} \cdot \frac{p^3 - p}{24} = -\frac{1}{24}$.

Proved: [Amir-Corwin-Quastel '10], [Sasamoto-Spohn '10]

More elaborate treatment of moments gives limiting fluctuations [Dotsenko '10+], [Calabrese-Le Doussal-Rosso '10+]. WHY?

q-TASEP moments and contour deformation

The distribution of q^{x_n} for q-TASEP particles is NOT intermittent.

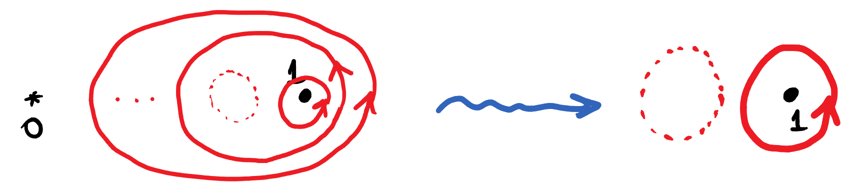
We can find the distribution and then take the limit to polymers.

But nested contours are not suited for very large moments.

Lemma

$$\frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \underbrace{\oint \cdots \oint}_{\text{nested}} \prod_{1\le A<B\le k} \frac{z_A - z_B}{z_A - q z_B} \, \frac{f(z_1)\cdots f(z_k)}{z_1\cdots z_k} \, dz_1 \cdots dz_k \qquad f(z) = \frac{e^{(q-1)tz}}{(1-z)^N}$$

$$= k!_q \sum_{\substack{\lambda = (\lambda_1 \ge \dots \ge \lambda_\ell \ge 0) \\ \lambda_1 + \dots + \lambda_\ell = k \\ \lambda = 1^{m_1} 2^{m_2} \dots}} \frac{1}{m_1! \, m_2! \cdots} \, \frac{(1-q)^k}{(2\pi i)^\ell} \underbrace{\oint \cdots \oint}_{\text{small}} \det\left[\frac{1}{w_i q^{\lambda_i} - w_j}\right]_{i,j=1}^{\ell} \prod_{j=1}^{\ell} f(w_j) f(q w_j) \cdots f(q^{\ell-1} w_j) \, dw_j$$



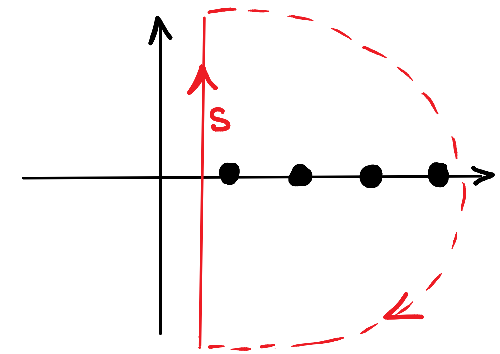
This formula plays a key role in spectral analysis of Quantum Integrable Systems in Lecture 3. The dets are similar to inverse squared normes of Bethe eigenstates.

Laplace transforms

It is convenient now to take the generating function $\sum_{k\geq 0} \mathbb{E}\left(q^{x_N}\right)^k \frac{\zeta^k}{k!_q}$.

Replace the sum over ordered cluster sizes by that over unordered unrestricted integers $n_1, n_2, \dots$ (removes the combinatorial factor), and use the Mellin-Barnes transform

$$\sum_{n\geq 1} g(q^n)\,\zeta^n = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \underbrace{\Gamma(-s)\,\Gamma(1+s)}_{=\frac{\pi}{\sin \pi s}}(-\zeta)^s\, g(q^s)\,ds$$



The result admits direct term-wise limit to polymers:

$$\mathbb{E}\, e^{-u Z(N,\tau)} = 1 + \sum_{\ell\geq 0} \frac{1}{\ell!\,(2\pi i)^{2\ell}} \oint_{|v_j|=1/4}\cdots\oint dv_1\cdots dv_\ell \int_{3/4-i\infty}^{3/4+i\infty}\cdots\int ds_1\cdots ds_\ell$$

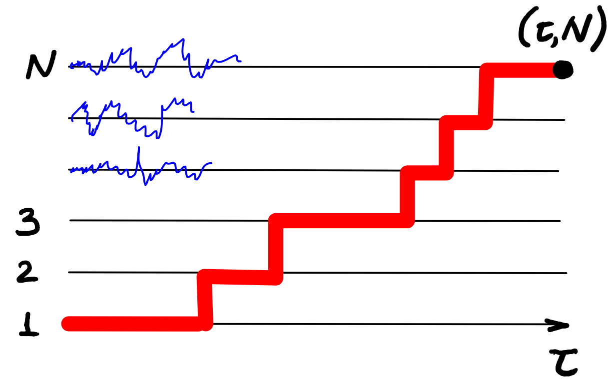
$$\times \prod_{j=1}^{\ell} \frac{\pi}{\sin \pi s_j}\, e^{\Phi(s_j+v_j)-\Phi(v_j)} \cdot \det\left[\frac{1}{s_i+v_i-v_j}\right]_{i,j=1}^{\ell}$$

$$\Phi(z) = \frac{\tau}{2} z^2 + z\cdot \ln u - N \ln \Gamma(z)$$

Limit theorem

Theorem [B-Corwin '11, B-Corwin-Ferrari '12] For any $\varkappa > 0$

$$\lim_{N\to\infty} \mathbb{P}\left\{ \frac{Z(N,\varkappa N) - f_\varkappa \cdot N}{g_\varkappa \cdot N^{1/3}} \le r \right\} = F_{GUE}(r)$$



The proof is by steepest descent analysis of the last expression.

The Tracy-Widom GUE distribution arises as

$$F_{GUE}(g_\varkappa r) = 1 + \sum_{\ell \ge 1} \frac{1}{\ell!\,(2\pi i)^{2\ell}} \int \cdots \int da_1 \cdots da_\ell \int \cdots \int db_1 \cdots db_\ell$$

$$\times \prod_{j=1}^{\ell} \frac{1}{a_j - b_j} \, \frac{\exp\left(-\frac{g_\varkappa^3}{3} a_j^3 + r a_j\right)}{\exp\left(-\frac{g_\varkappa^3}{3} b_j^3 + r a_j\right)} \cdot \det\left[\frac{1}{b_i - a_j}\right]_{i,j=1}^{\ell}.$$

Back to the replica trick

q-Laplace transform for q-TASEP = Generating series of q-TASEP moments

converges

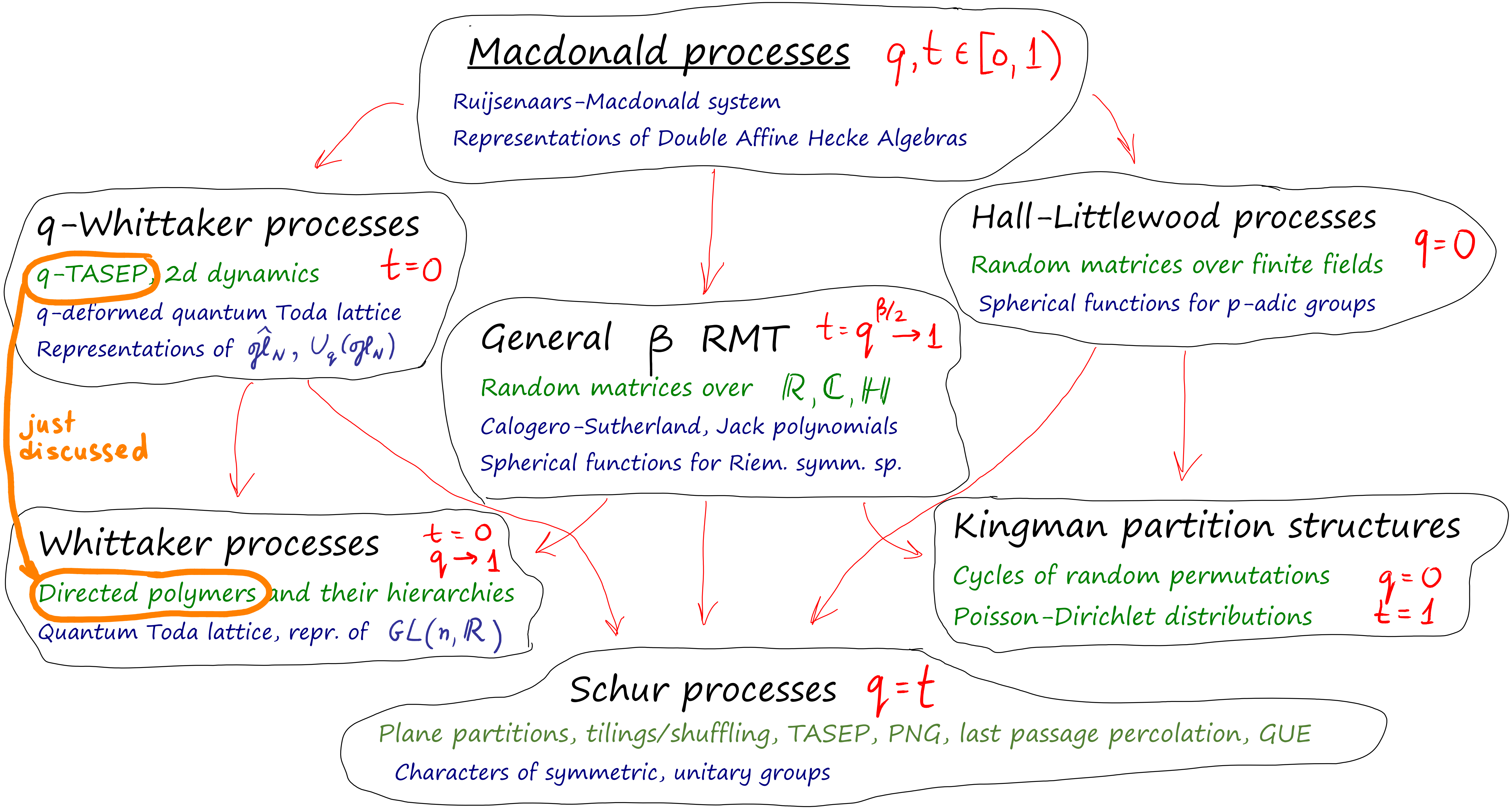
series diverges

converge

Laplace transform of polymer partition function ...... Generating series of polymer moments

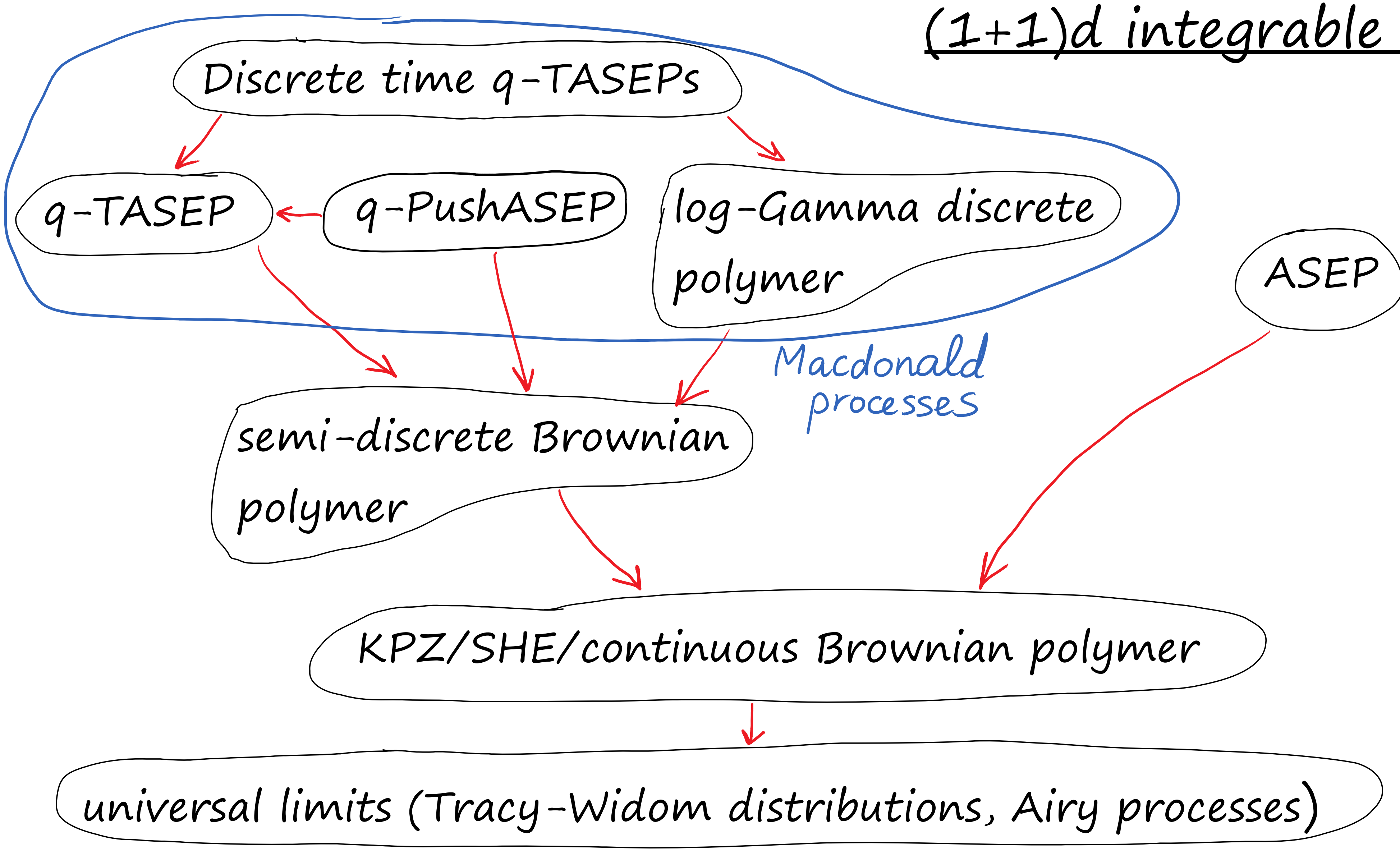
replica trick

The bona fide argument on the q-level is the only currently available explanation of why the replica trick works in this case. This will be extended in Lecture 3.



Macdonald processes $q, t \in [0, 1)$
Ruijsenaars–Macdonald system
Representations of Double Affine Hecke Algebras
q-Whittaker processes $t=0$
q-TASEP, 2d dynamics
q-deformed quantum Toda lattice
Representations of $\widehat{\mathfrak{gl}}_N$, $U_q(\mathfrak{gl}_N)$
just discussed
Hall-Littlewood processes $q=0$
Random matrices over finite fields
Spherical functions for p-adic groups
General β RMT $t = q^{\beta/2} \to 1$
Random matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$
Calogero-Sutherland, Jack polynomials
Spherical functions for Riem. symm. sp.
Whittaker processes $t=0$ $q \to 1$
Directed polymers and their hierarchies
Quantum Toda lattice, repr. of $GL(n, \mathbb{R})$
Kingman partition structures
Cycles of random permutations $q=0$
Poisson-Dirichlet distributions $t=1$
Schur processes $q=t$
Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE
Characters of symmetric, unitary groups

(1+1)d integrable KPZ systems



Aiming at accessing other integrable KPZ systems and more general initial conditions, Lecture 3 will present a different approach.