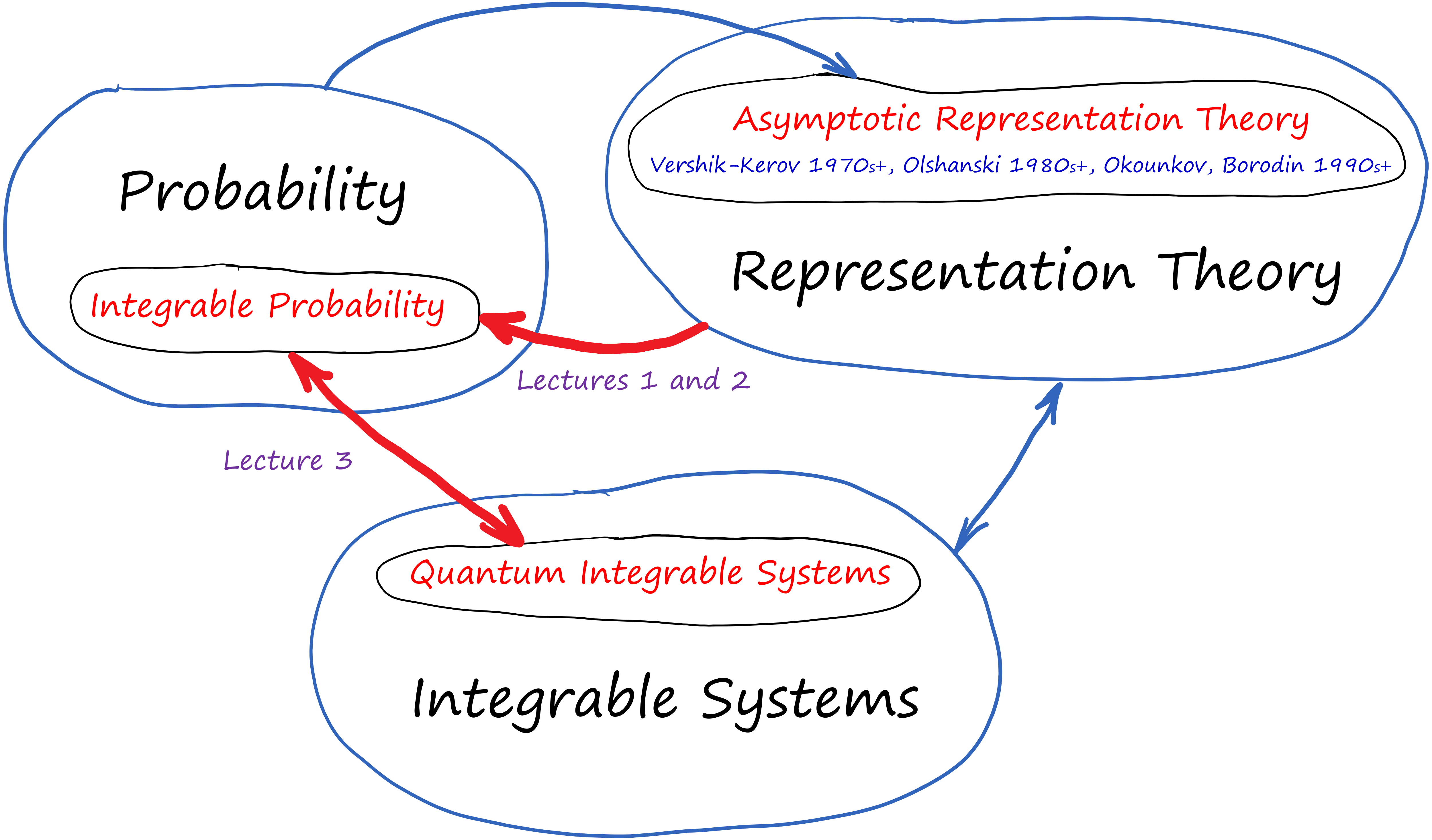
Macdonald processes

Alexei Borodin



Probability

Integrable Probability

Asymptotic Representation Theory

Vershik-Kerov 1970s+, Olshanski 1980s+, Okounkov, Borodin 1990s+

Representation Theory

Lectures 1 and 2

Lecture 3

Quantum Integrable Systems

Integrable Systems

Probabilistic objectives

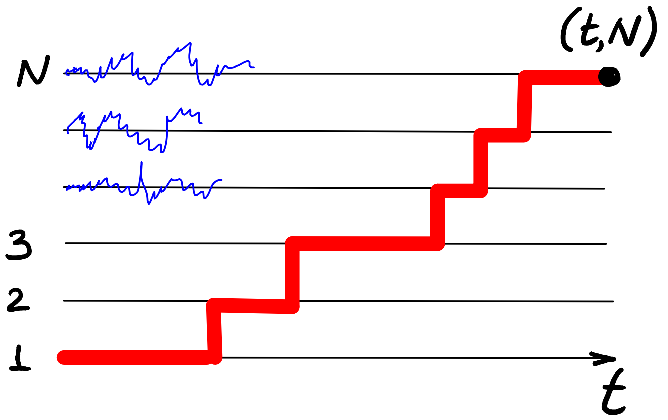
We wish to establish law of large numbers and fluctuations behaviour for a (growing) variety of integrable probabilistic models that have an additional algebraic structure, like

- Random matrix ensembles with rotational symmetry
- Exclusion processes in (1+1)d: TASEP, ASEP, PushASEP, q-versions, etc.
- Special directed random polymers in (1+1)d
- Special tiling (or dimer) models in 2d
- Random growth of discretized interfaces in (2+1)d

Universality principles suggest that same fluctuations hold in broad universality classes (Wigner matrices, KPZ, general dimers)

Example 1: Semi-discrete Brownian polymer

$$F_t^N = \log \int_{0<s_1<\dots<s_{N-1}<t} e^{B_1(0,s_1)+B_2(s_1,s_2)+\dots+B_N(s_{N-1},t)} \, ds_1 \dots ds_{N-1}$$



$B_1, \dots, B_N$ are independent Brownian motions

$$B_k(\alpha,\beta) := B_k(\beta) - B_k(\alpha) = \int_\alpha^\beta \dot{B}_k(x)\,dx$$

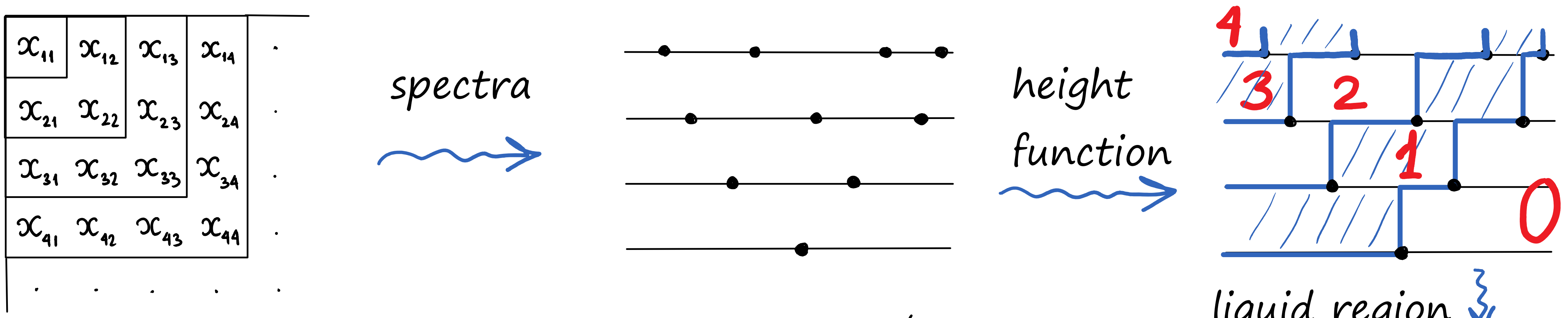
Theorem [B-Corwin '11, B-Corwin-Ferrari '12] For any $\varkappa > 0$

$$\lim_{N\to\infty} \mathbb{P}\left\{ \frac{F^N_{\varkappa N} - f_\varkappa \cdot N}{g_\varkappa \cdot N^{1/3}} \leq r \right\} = F_{GUE}(r)$$

Tracy-Widom limit distribution for the largest eigenvalue of large Hermitian random matrices

- $f_\varkappa$ conjectured in [O'Connell-Yor '01], proved in [Moriarty-O'Connell '07]
- [Spohn '12] matched the result with (1+1)d KPZ scaling conjecture

Example 2: Corners of random matrices

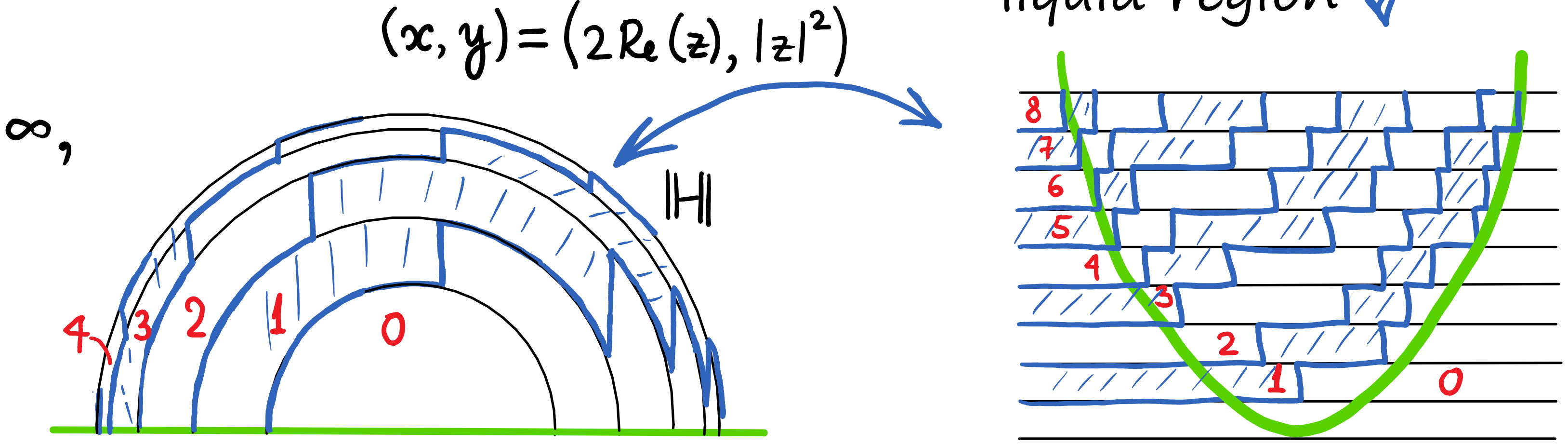


Theorem As $z \mapsto L^{-1} z$, $L \to \infty$,

Fluctuations $\Rightarrow$

Gaussian (massless)

Free Field on $\mathbb{H}$



- GUE: Implicit in [B-Ferrari, 2008], related to AKPZ in (2+1)d
- GUE/GOE type Wigner matrices : [B, 2010]
- General beta, classical weights : [B-Gorin, 2013]

Two characteristic properties

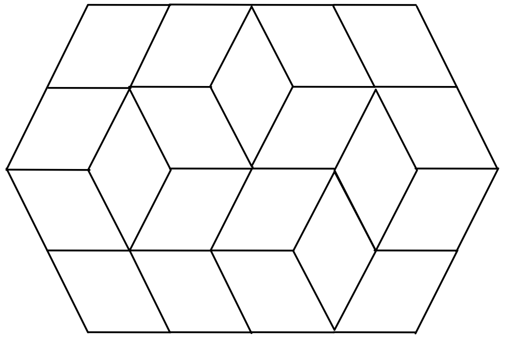
Integrable probabilistic models typically share two key features:

- There is a large family of observables whose averages are explicit and asymptotically tractable;
- There is a natural Markov evolution that acts nicely.

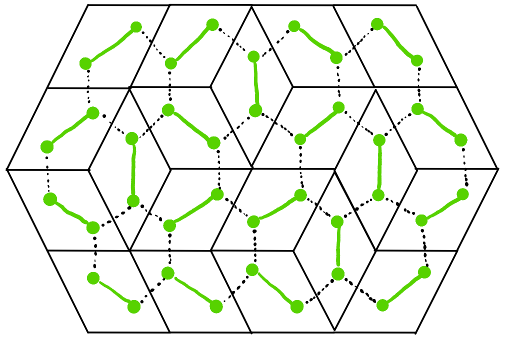
Representation theory is helpful in identifying both.

Let us illustrate on lozenge tilings.

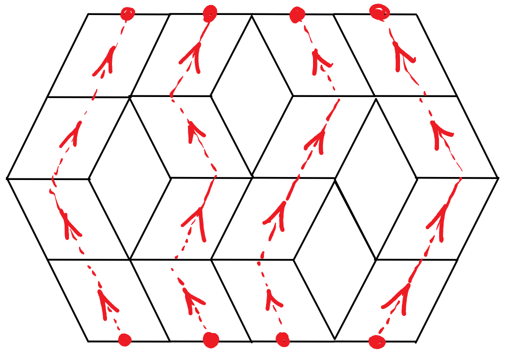
From probability to representation theory



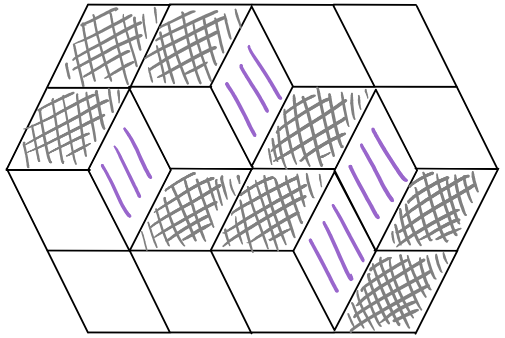
Lozenge tilings are...



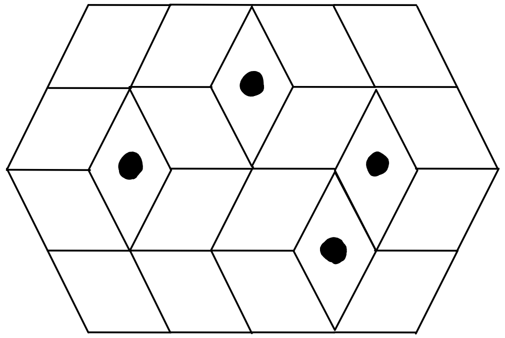
dimers on hexagonal lattice



nonintersecting Bernoulli paths



stepped surfaces



interlacing particle configurations

But they are also labels for Gelfand-Tsetlin bases of irreps of $U(N)$ or $GL(N, \mathbb{C})$.

Finite-dim representations of unitary groups (H. Weyl, 1925-26)

A representation of $U(N)$ is a group homomorphism $T: U(N) \to GL(V)$.

It is irreducible if V has no invariant subspaces.

Every (finite-dimensional) representation is a direct sum of irreps.

Fact: T is uniquely determined by the (diagonalizable) action of the abelian subgroup H of diagonal matrices.

$$V = \bigoplus_{i=1}^{\dim V} \mathbb{C} v_i, \qquad T\left(\begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_N \end{bmatrix} \right) = \begin{bmatrix} t_1(z_1,\dots,z_N) & & \\ & \ddots & \\ & & t_{\dim V}(z_1,\dots,z_N) \end{bmatrix}$$

$$t_j : S^1 \times \dots \times S^1 \to \mathbb{C}^\times, \qquad t_j(z_1,\dots,z_N) = z_1^{k_1} \cdots z_N^{k_N}, \qquad (k_1,\dots,k_N) \in \mathbb{Z}^N.$$

$\uparrow$

weight

Finite-dim representations of unitary groups (H. Weyl, 1925-26)

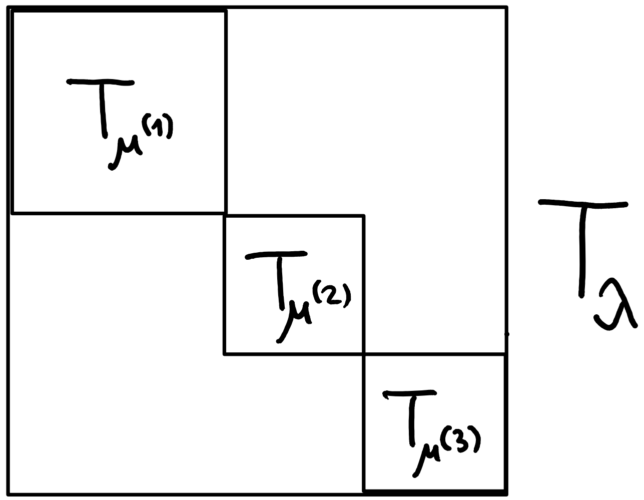
Theorem Irreducible representations are parametrized by their highest weights $\lambda = (\lambda_1 \geqslant \dots \geqslant \lambda_N) \in \mathbb{Z}^N$. The corresponding generating function of all weights has the form

$$\sum_{\text{weights of } T_\lambda} z_1^{k_1} \cdots z_N^{k_N} = \mathrm{Trace}\left(T_\lambda\left(\begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_N \end{bmatrix}\right)\right) = \frac{\det\left[z_i^{\lambda_j + N - j}\right]_{i,j=1}^{N}}{\underset{\text{Vandermonde det.}}{\det\left[z_i^{N-j}\right]_{i,j=1}^{N}}}.$$

These are the characters of the corresponding representations, also known as the Schur polynomials.

Branching and lozenges

Reducing the symmetry group from U(N) to U(N−1) may lead to a split of an irrep into a direct sum of those for the smaller group.

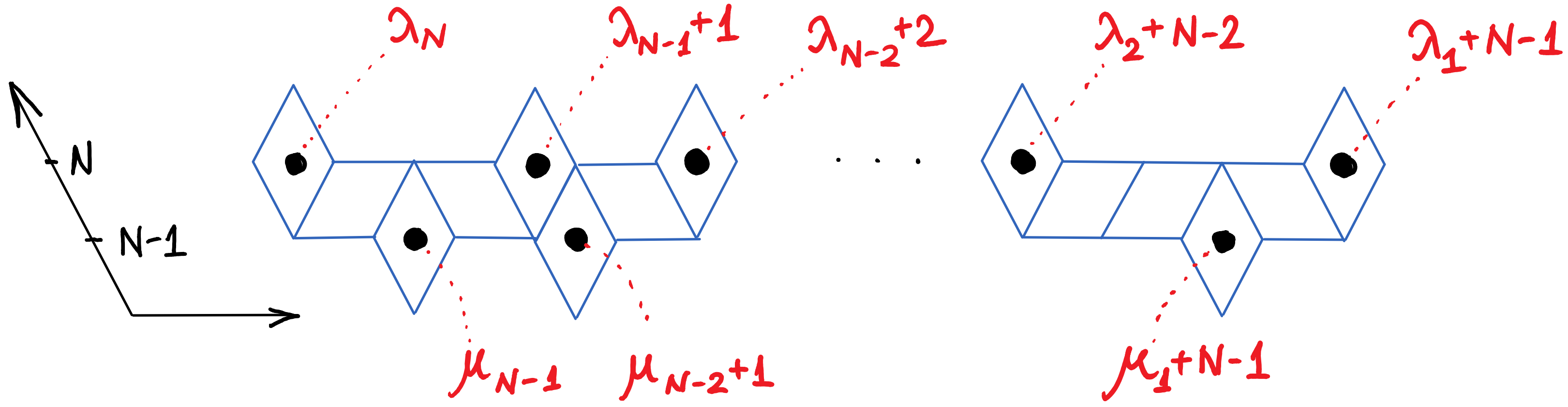


This is encoded by Schur polynomials:

$$S_\lambda(z_1,\dots,z_{N-1},1) = \sum_{\mu\prec\lambda} S_\mu(z_1,\dots,z_{N-1})$$

where μ interlaces λ : $\lambda_N \le \mu_{N-1} \le \lambda_{N-1} \le \dots \le \lambda_2 \le \mu_2 \le \lambda_1$,

or pictorially:

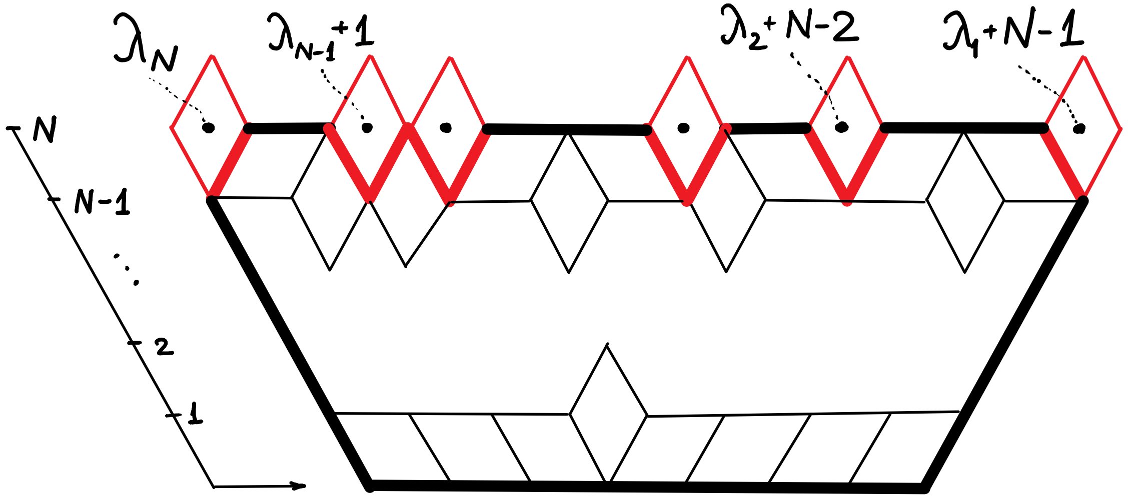


Gelfand-Tsetlin basis

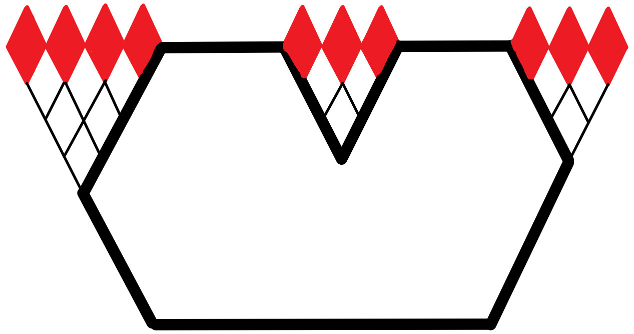
Reducing the symmetry all the way down the tower

$$U(N) \supset U(N-1) \supset \dots \supset U(2) \supset U(1)$$

yields a basis in T_λ labelled by lozenge tilings of specific domains:

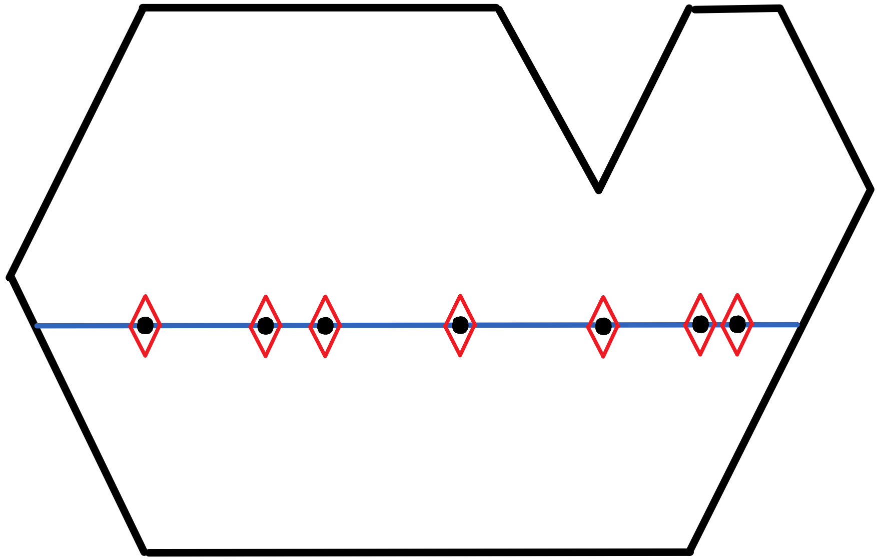


An example:



[Gelfand-Tsetlin, 1950] used this basis to explicitly write down the action of generators.

Back to probability



Consider the uniform measure on tilings. How to describe its projection to a horizontal section of the polygon? Equivalently, how to decompose a known irrep of U(N) on irreps of U(k) ⊂ U(N)?

This is a problem of noncommutative harmonic analysis. In terms of characters (Schur polynomials):

$$\chi(z_1,\dots,z_k)=\sum_{\mu=(\mu_1\geq\dots\geq\mu_k)} \mathrm{Prob}\{\mu\}\,\frac{S_\mu(z_1,\dots,z_k)}{S_\mu(1,\dots,1)},\qquad \chi(z_1,\dots,z_k)=\frac{S_\lambda(\overbrace{z_1,\dots,z_k,1,\dots,1}^{N})}{S_\lambda(1,\dots,1)}.$$

Classical harmonic analysis

The (abelian) group $\mathbb{R}$ acts on $L^2(\mathbb{R})$ by shifting the argument.

The irreps are all 1-dim of the form $p \mapsto$ multiplication by e^{-ipx}.

For

$$\chi(x) = \int_{-\infty}^{+\infty} e^{-ipx}\, m(dp)$$

there are (at least) two ways to extract information about m.

Inverse Fourier transform: $$\frac{m(dp)}{dp} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx}\chi(x)\,dx$$ (hard)

Differential operators: $$\int_{-\infty}^{\infty} p^n\, m(dp) = \left(i\frac{d}{dx}\right)^n \chi(x)\Big|_{x=0}$$ (simple)

The observables

If

$$\chi(z_1,\dots,z_k) = \sum_{\mu=(\mu_1\geq\dots\geq\mu_k)} \mathrm{Prob}\{\mu\} \, \frac{S_\mu(z_1,\dots,z_k)}{S_\mu(1,\dots,1)}$$

and $\mathcal{D} S_\mu = d_\mu S_\mu$, then $\mathcal{D}\chi\Big|_{z_1=\dots=z_k=1} = \sum_{\mu} d_\mu \, \mathrm{Prob}\{\mu\} = \mathbb{E}\, d_\mu$.

The **Casimir-Laplace operator** (generates circular Dyson BM)

$$C_2 = \frac{1}{\prod_{i<j}(z_i-z_j)} \circ \sum_{i=1}^{k} \left(z_i \frac{\partial}{\partial z_i}\right)^2 \circ \prod_{i<j}(z_i-z_j).$$

As $S_\mu(z) = \det\left[z_i^{\mu_j+k-j}\right] / \prod_{i<j}(z_i-z_j)$, $\quad C_2 S_\mu = \sum_{i=1}^{k} (\mu_i+k-i)^2 \cdot S_\mu$.

d_μ

A q-analog: Replace $\left(z\frac{\partial}{\partial z}\right)^2$ by $(T_q f)(z) = f(qz)$. Then $C^{(q)} S_\mu = \sum_{i=1}^{k} q^{\mu_i+k-i} \cdot S_\mu$.

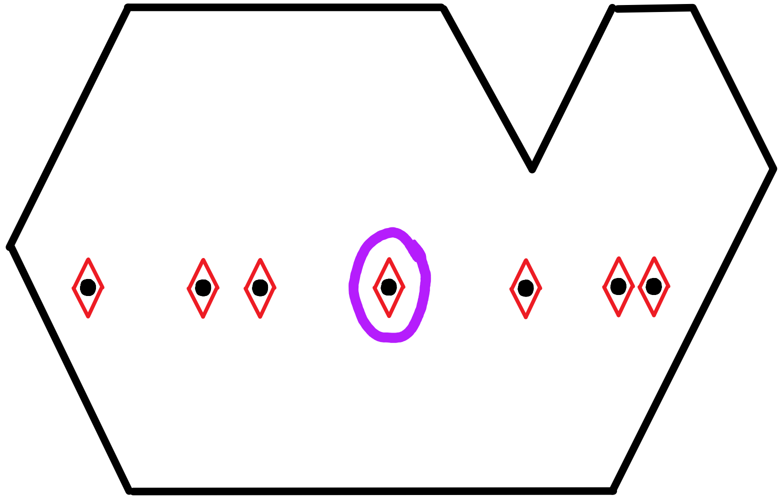
Correlation functions

First correlation function:

$$\varrho_1(m,k) = \mathrm{Prob}\left\{ m \in \{\mu_j + k - j\}_{j=1}^{k} \right\} =$$

$$= \text{coeff. of } q^m \text{ in } \mathbb{E}\left(\sum q^{\mu_j+k-j}\right)$$

$$= \text{coeff. of } q^m \text{ in } C^{(q)}\chi\big|_{z_1=\dots=z_k=1}.$$



Higher correlation functions require products $C^{(q_1)}\cdots C^{(q_n)}$.

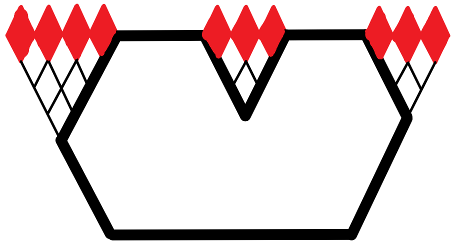
If χ factorizes, $\chi(z_1,\dots z_k) = \varphi(z_1)\cdots\varphi(z_k)$,

$$\text{coeff. of } q^m \text{ in } C^{(q)}\chi\big|_{z_j\equiv 1} = \frac{1}{(2\pi i)^2} \underset{\text{around } 0}{\oint} \frac{dv}{v} \underset{\text{around } 1}{\oint} dw \; \frac{\varphi(v)\,(v-1)^k\, v^{-m}}{\varphi(w)\,(w-1)^k\, w^{-m}} \, \frac{1}{v-w}.$$

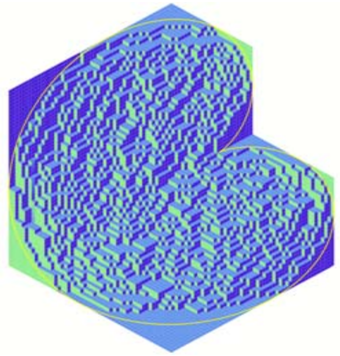
For the n-point correlation function the integral is 2n-fold.

Asymptotics

For \`infinitely tall polygons' (corresponding to characters of $U(\infty)$, example on next slide), χ indeed factorizes, and steepest descent yields limit shapes, bulk (discrete sine), edge (GUE, Airy, Pearcey), and global (free field) fluctuations [B-Kuan '07], [B-Ferrari '08].



For ordinary polygons in our class, the factorization is only approximate, yet same formulas can be used to prove similar results [Petrov '12], [Gorin-Panova '13].

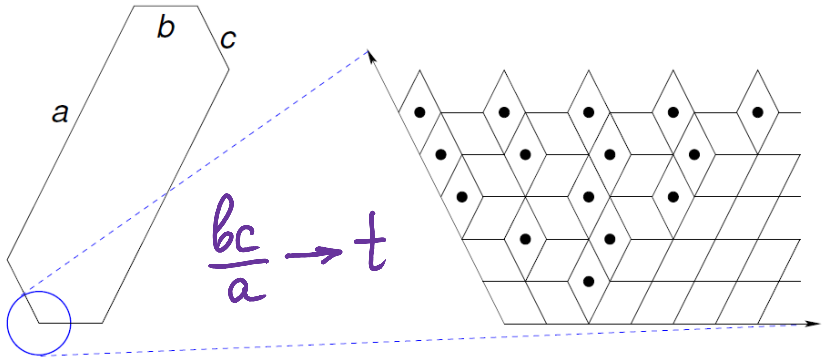


More general limit shapes were obtained by [Kenyon-Okounkov '05], who also conjectured the rest.

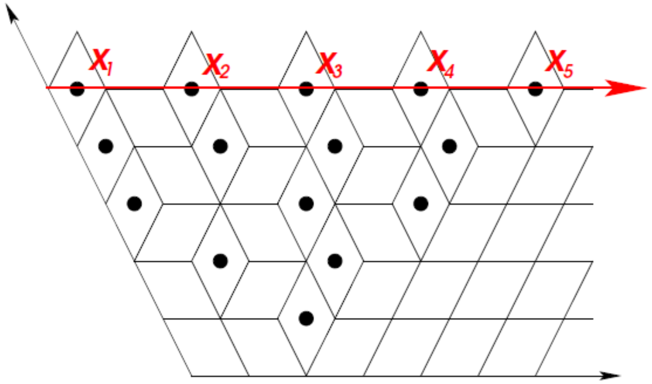
Markov evolution

We focus on $\chi(z_1,\dots z_k)=\prod_{i=1}^{k} e^{t(z_i-1)}$, $t\geq 0$.

This corresponds to a limit of hexagons:



On a fixed horizontal slice, the coordinates of vertical lozenges are distributed as



$$\mathrm{Prob}\left\{(x_1,\dots,x_k)\in\mathbb{Z}^{k}_{\geq 0}\right\}=\text{const}\cdot\prod_{i<j}(x_i-x_j)^2\prod_{i=1}^{k}\frac{t^{x_i}}{x_i!}.$$

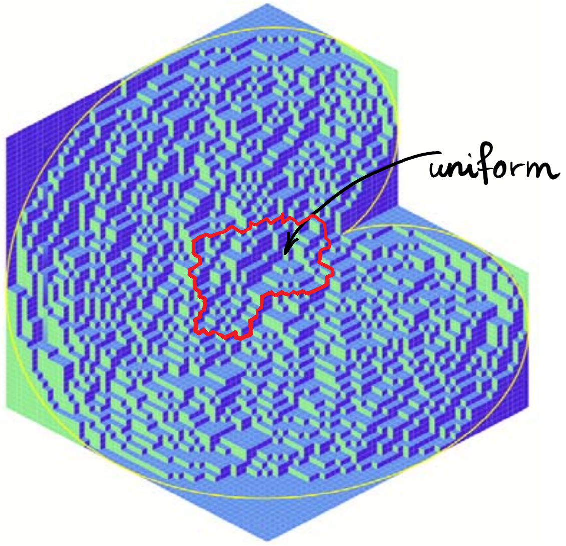
This is time t distribution of the Markov chain with generator

$$L^{(k)}_{\text{Poisson}}=\left(\prod_{i<j}(x_i-x_j)\right)^{-1}\circ\sum_{i=1}^{k}\nabla_{x_i}\circ\left(\prod_{i<j}(x_i-x_j)\right),\qquad (\nabla f)(x)=f(x+1)-f(x),$$

which can also be viewed as k conditioned 1d Poisson processes.

The Gibbs property

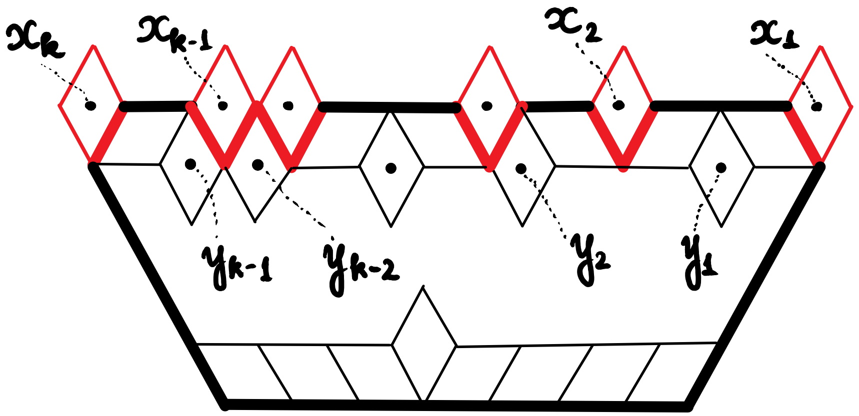
Uniformly distributed tilings obviously enjoy the Gibbs property: Given a boundary condition, the distribution in any subdomain is also uniform.



Apply to bottom k rows:

$$\mathrm{Prob}\{y \mid x\} = \frac{\#\text{ of height } (k-1) \text{ tilings with top row } y}{\#\text{ of height } k \text{ tilings with top row } x}$$

$$= (k-1)!\,\frac{\prod_{1\le i<j\le k-1}(y_i-y_j)}{\prod_{1\le i<j\le k}(x_i-x_j)} =: \Lambda^{k}_{k-1}(x \searrow y)$$



These stochastic links intertwine \`perpendicular' Markov chains along (k−1)st and k-th rows with generators $L^{(k-1)}_{\mathrm{Poisson}}$ and $L^{(k)}_{\mathrm{Poisson}}$.

Two-dimensional Markov evolution: Axiomatics

Inspired by two ad hoc constructions (RSK and [O'Connell '03+]; `stitching' of intertwined Markov chains [Diaconis-Fill '90], [B-Ferrari '08]), we look for Markov chains on tilings that satisfy:

I. For each $k \geq 1$, the evolution of the bottom k rows $(\lambda^{(1)} \prec \lambda^{(2)} \prec \dots \prec \lambda^{(k)})$ is independent of the higher rows.

II. For each $k \geq 1$, the evolution preserves the Gibbs property on the bottom k rows:

$$m(\lambda^{(k)}) \Lambda^{k}_{k-1}(\lambda^{(k)} \searrow \lambda^{(k-1)}) \cdots \Lambda^{2}_{1}(\lambda^{(2)} \searrow \lambda^{(1)}) \xrightarrow{\text{time } t} \tilde{m}(\lambda^{(k)}) \Lambda^{k}_{k-1}(\lambda^{(k)} \searrow \lambda^{(k-1)}) \cdots \Lambda^{2}_{1}(\lambda^{(2)} \searrow \lambda^{(1)})$$

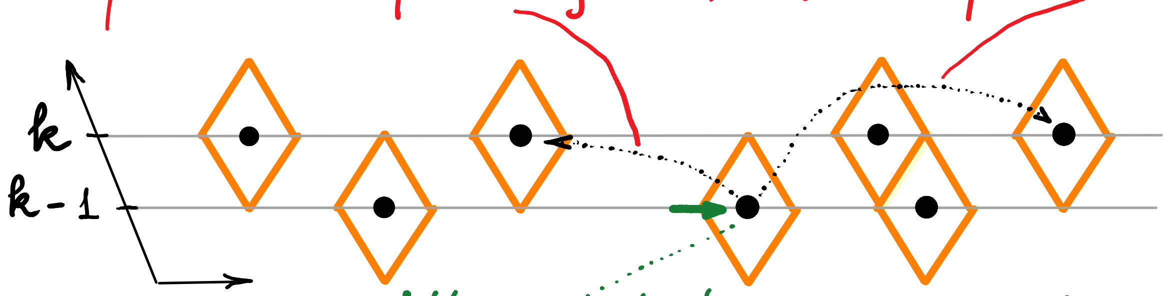
III. For each $k \geq 1$, the map $m \mapsto \tilde{m}$ is the time t evolution of the Markov chain with generator $L^{(k)}_{\text{Poisson}}$.

Nearest neighbor interaction

- Each particle jumps to the right by 1 independently, with exp. distributed waiting time; rate $w_j^{(k)}(\lambda^{(k-1)};\lambda^{(k)})$ for j-th particle on level k.
- A move of any particle may instantaneously trigger moves of its top-left (pulling) and top-right (pushing) neighbors.

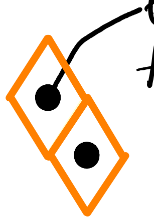
pulls with prob. $\ell_j^{(k)}(\lambda^{(k-1)};\lambda^{(k)})$

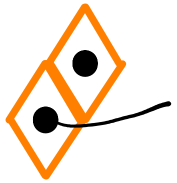
pushes with prob. $r_j^{(k)}(\lambda^{(k-1)},\lambda^{(k)})$



j th particle has just moved

`No-nonsense': (a) If a particle is blocked from the bottom, its jump rate is 0, and when pushed it donates the move to its right neighbor; (b) If a particle is blocked from the top, $r_j = 1$.


blocked from bottom


blocked from top

Classification of nearest neighbor dynamics

Theorem [B-Petrov '13] A nearest neighbor Markov evolution satisfies I-III (independence of bottom rows, preservation of Gibbs, horizontal sections evolve according to $L^{(k)}_{\mathrm{Poisson}}$) if and only if for any k ≥ 1 and any j ≥ 0 such that (j+1)st particle on level k is not blocked from the bottom,

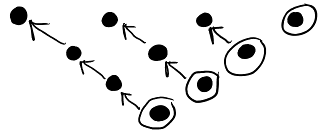
$$r^{(k)}_{j+1} + \ell^{(k)}_{j} + w^{(k)}_{j+1} = 1$$

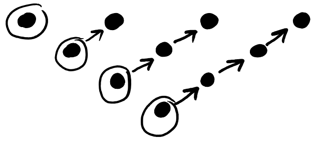
no Vandermondes!

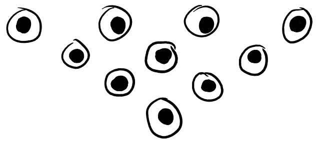
with nonexisting parameters at edges set to 0.

There are many solutions, all act the same on the Gibbs measures!

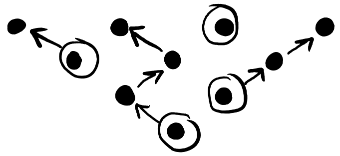
- $\ell_j \equiv 1,\ r_j \equiv 0,\ w_j = \begin{cases} 1, & j=1 \\ 0, & j>1 \end{cases}$ gives row RSK
- $\ell_j \equiv 0,\ r_j \equiv 1,\ w_j = \begin{cases} 1, & j \text{ maximal} \\ 0, & \text{otherwise} \end{cases}$ gives column RSK
- $\ell_j = r_j \equiv 0,\ w_j \equiv 1$ gives push-block dynamics





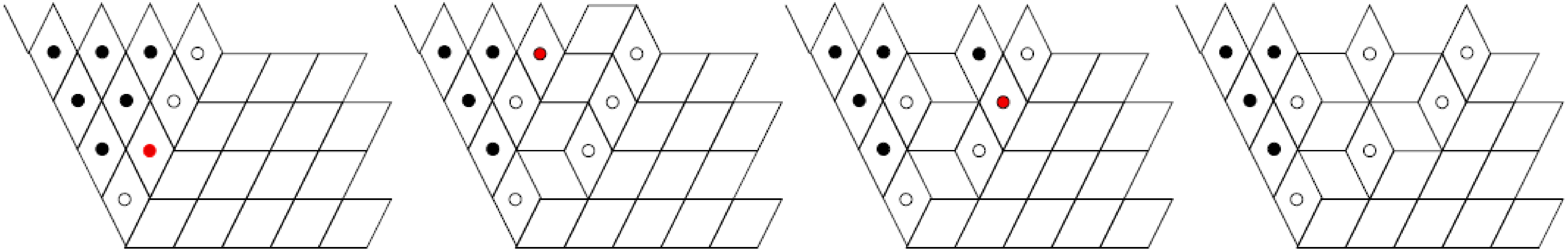


Many other possibilities, e.g.

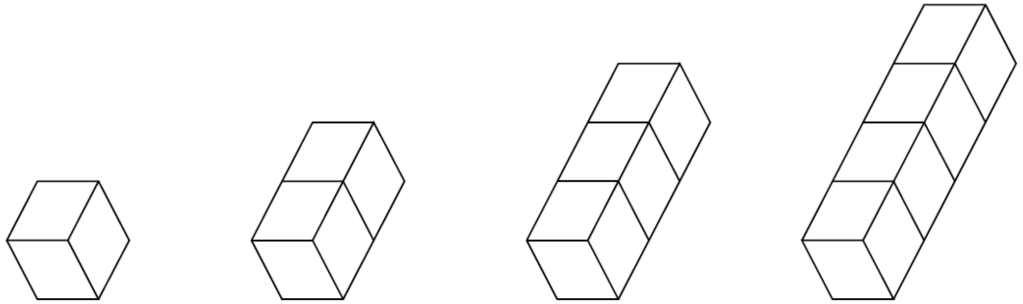


The push-block dynamics [B-Ferrari '08]

Each particle jumps to the right with rate 1. It is blocked by lower particles and it (short-range) pushes higher particles.



In 3d, this can be viewed as adding directed columns



Column deposition - Animation