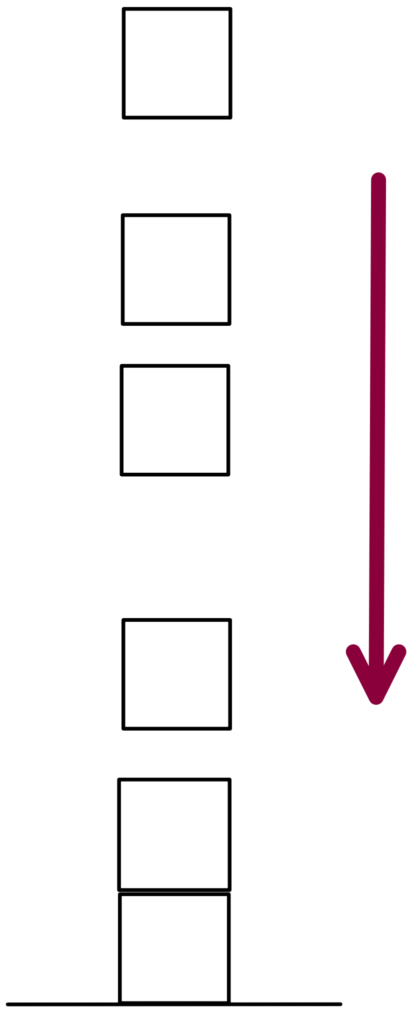
Integrable Probability

Alexei Borodin

What is integrable probability?



Imagine you are building a tower out of standard square blocks that fall down at random time moments.

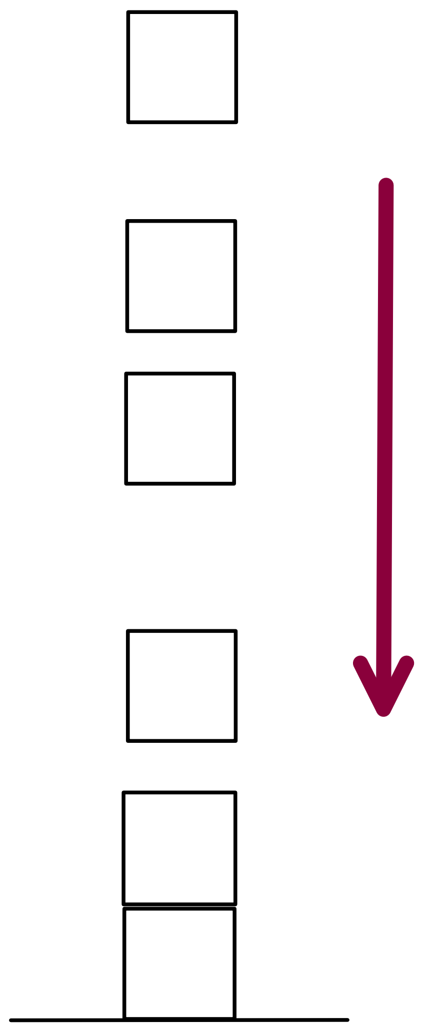
How tall will it be after a large time T?

It is natural to expect that

Height = const · T + random fluctuations

What can one say about the fluctuations?

What is integrable probability?



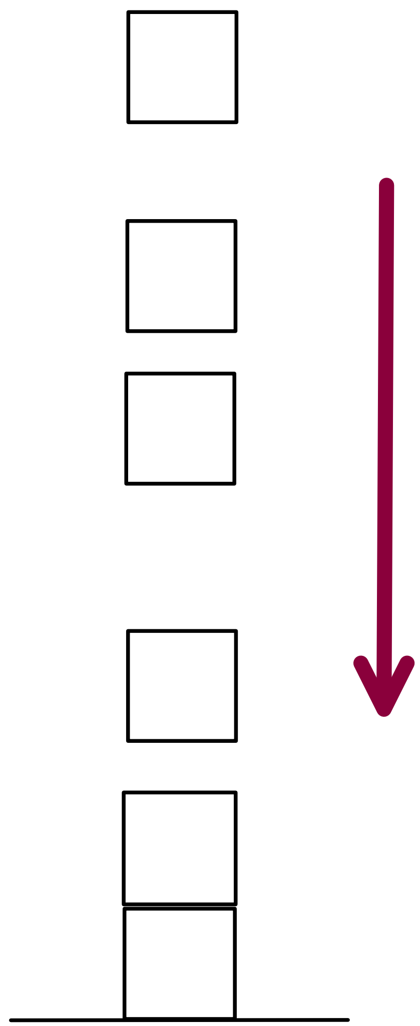
A simple integrable example: The time is discrete, each second a new block falls with probability 1/2 (independently of what happened before). Then

$$\mathrm{Prob}\{\text{height} = n \ \text{after time } T\} = \frac{1}{2^T} \frac{T!}{n!\,(T-n)!}$$

<u>Theorem</u> *[De Moivre 1738], [Laplace 1812]*

$$\lim_{T\to\infty} \mathrm{Prob}\left\{\text{height}(T) \le \frac{T}{2} + \frac{s}{2}\cdot T^{1/2}\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-x^2/2}\, dx$$

What is integrable probability?



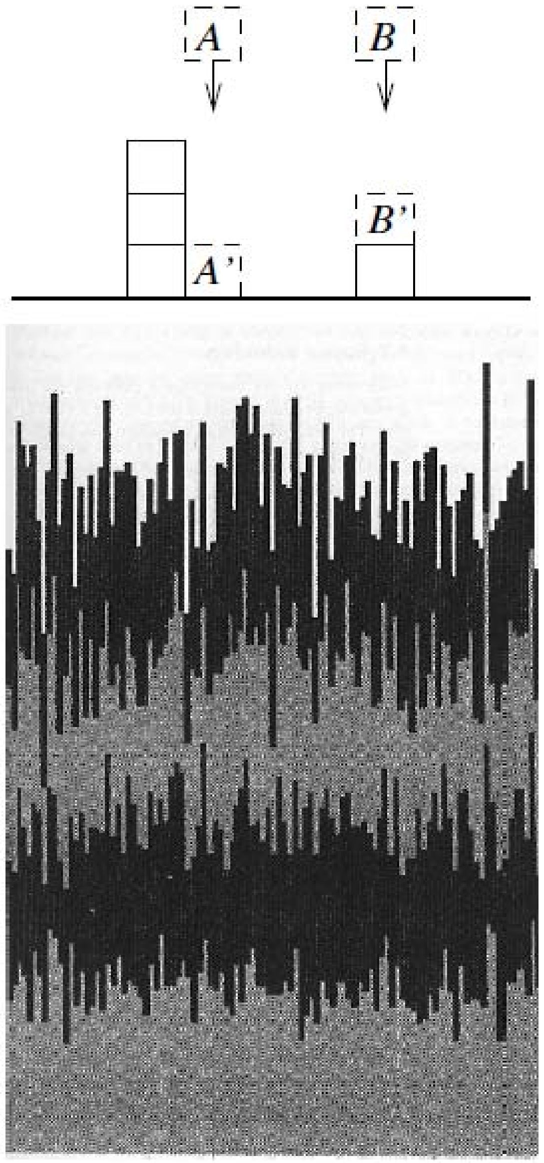
Universality principle: For a broad class of different possibilities for randomness, the size and distribution of the fluctuations must be the same (up to scaling constants).

This is the Central Limit Theorem, proved by Chebyshev-Markov-Lyapunov in 1887-1901.

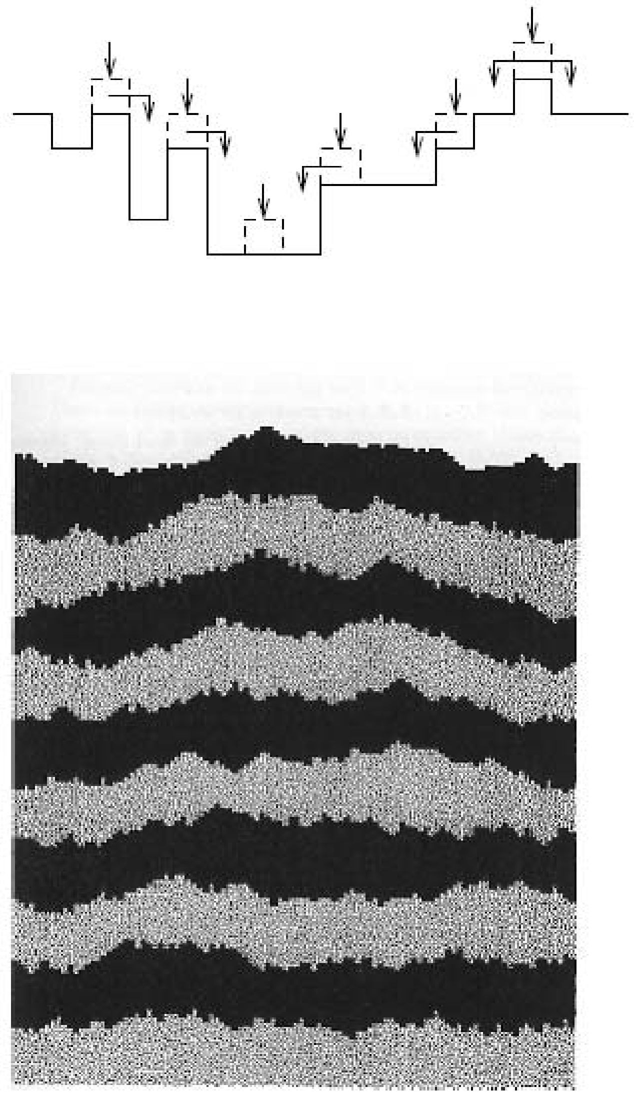
An integrable example predicts the behavior of the whole universality class.

Three models of random interface growth in (1+1)d

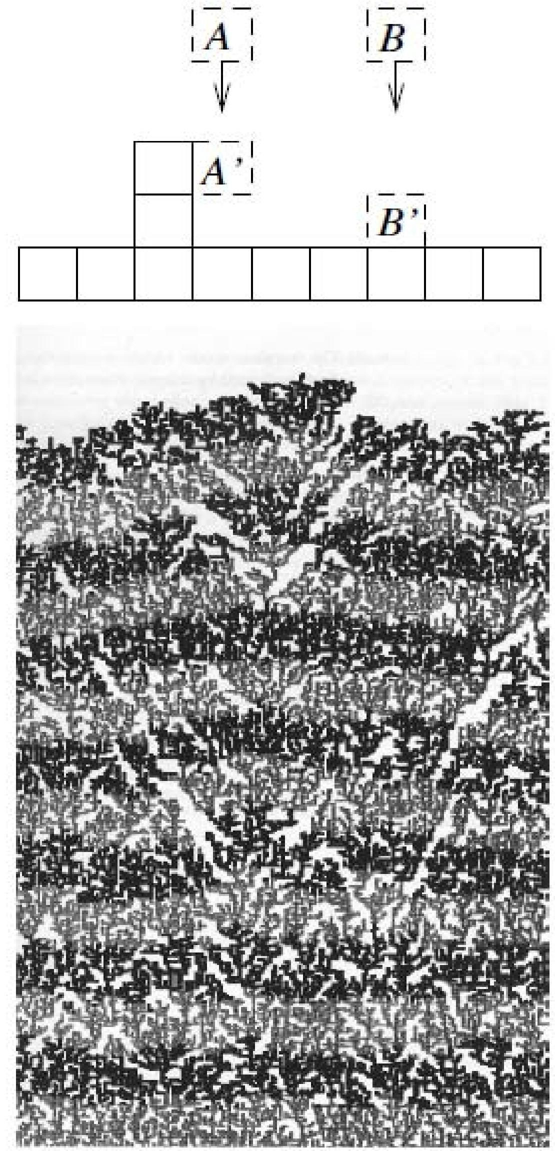
Random deposition



Random deposition with relaxation



Ballistic deposition

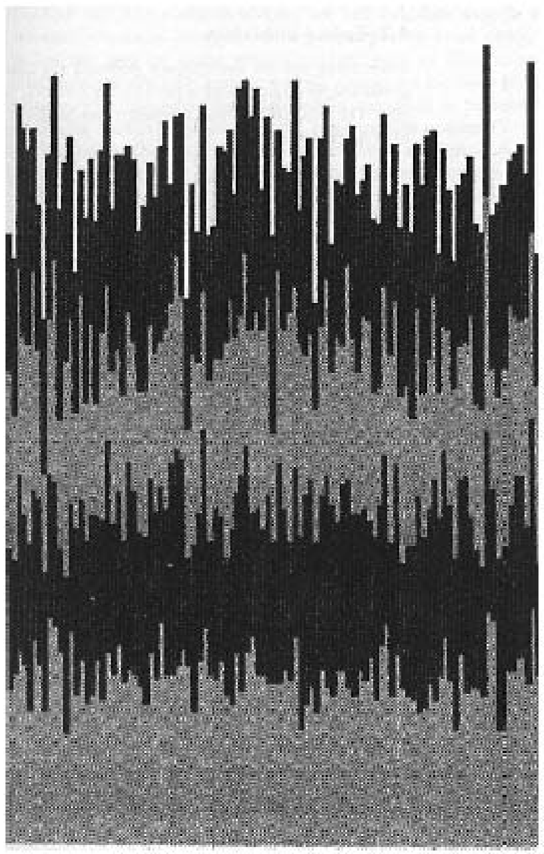


Three models of random interface growth in (1+1)d

Classical CLT

$\partial_t h = \eta(x,t)$

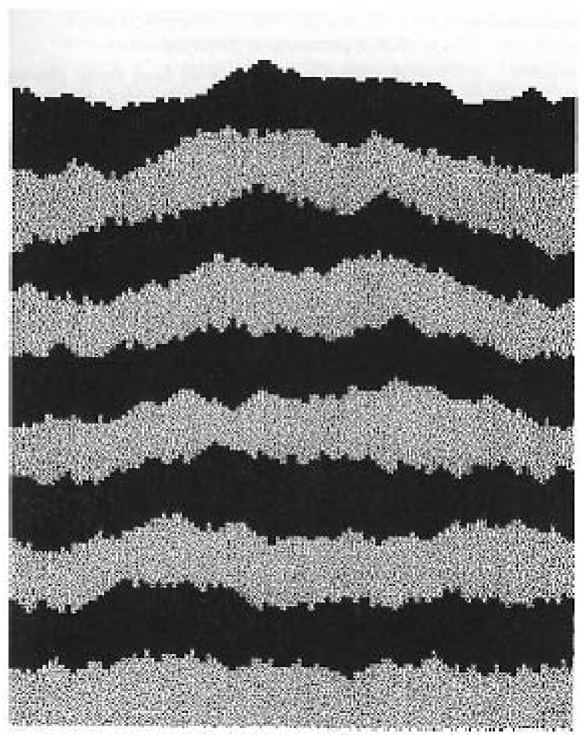
$t^{\frac{1}{2}}$ fluctuations



Edwards-Wilkinson eq.

$\partial_t h = \nu\, \partial_x^2 h + \eta(x,t)$

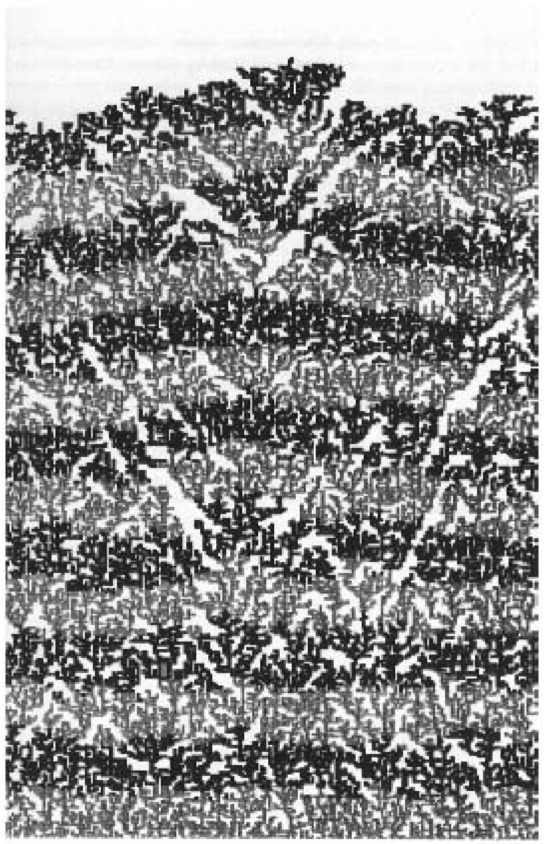
$t^{\frac{1}{4}}$ fluctuations



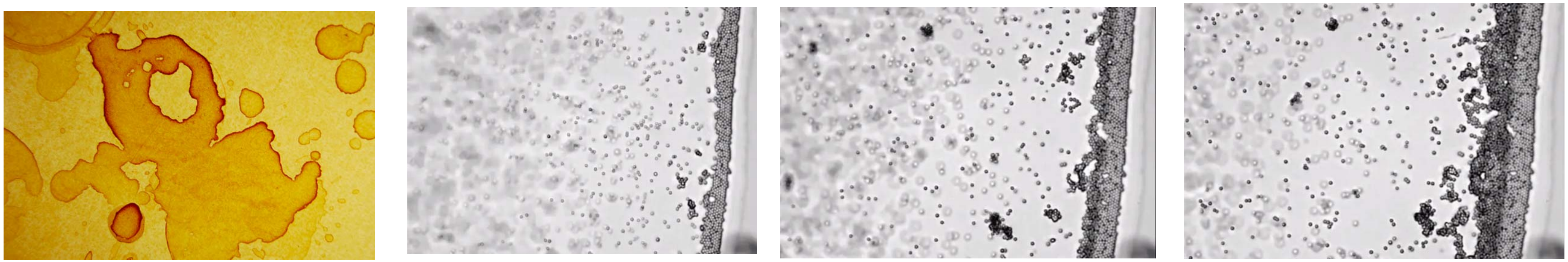
Kardar-Parisi-Zhang eq.

$\partial_t h = \nu\, \partial_x^2 h + \lambda(\partial_x h)^2 + \eta$

$t^{\frac{1}{3}}$ fluctuations

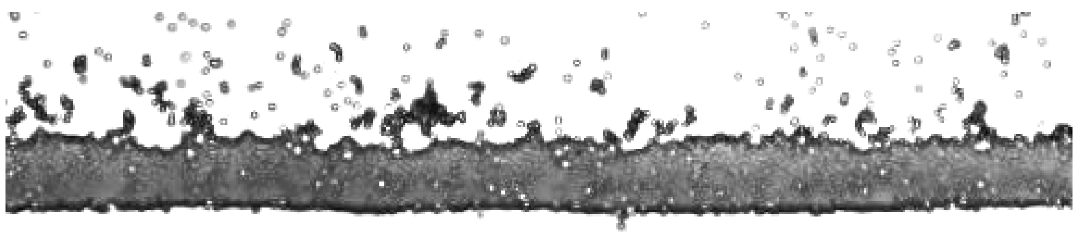


Experimental example: coffee ring effect



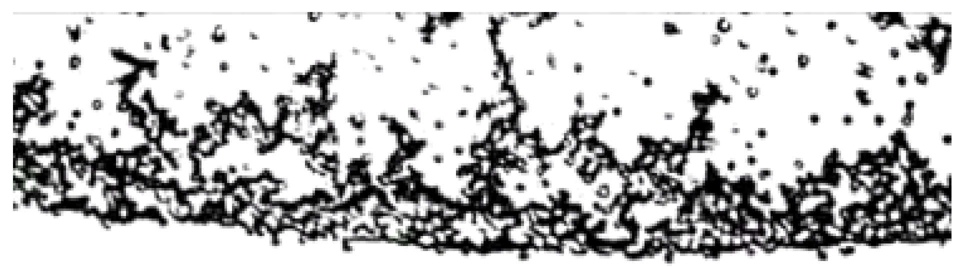
Perfectly round particles:

$t^{1/2}$ fluctuations, CLT statistics



Slightly elongated particles:

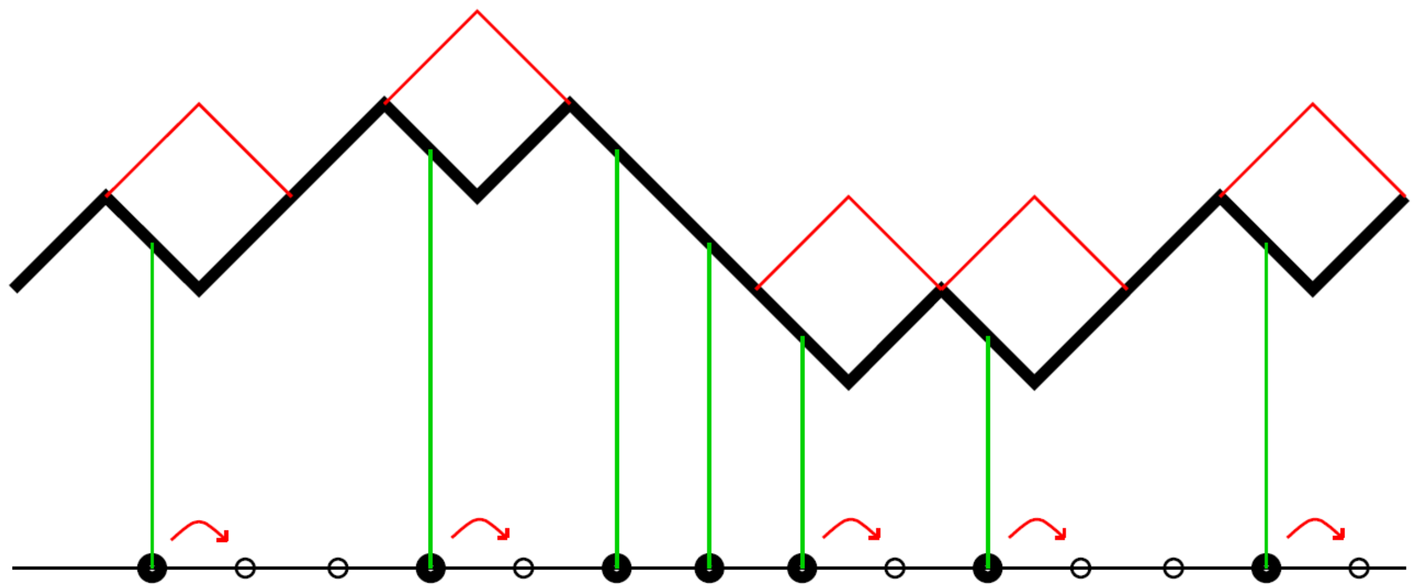
$t^{1/3}$ fluctuations, KPZ statistics



[Yunker-Lohr-Still-Borodin-Durian-Yodh, PRL 2013]

Totally Asymmetric Simple Exclusion Process (TASEP):

An integrable random interface growth model



[TASEP animation](#)

by Patrik Ferrari

Red boxes are added independently at rate 1. Equivalently, particles with no right neighbor jump independently with waiting time distributed as $e^{-x}dx$.

TASEP - hydrodynamic limit

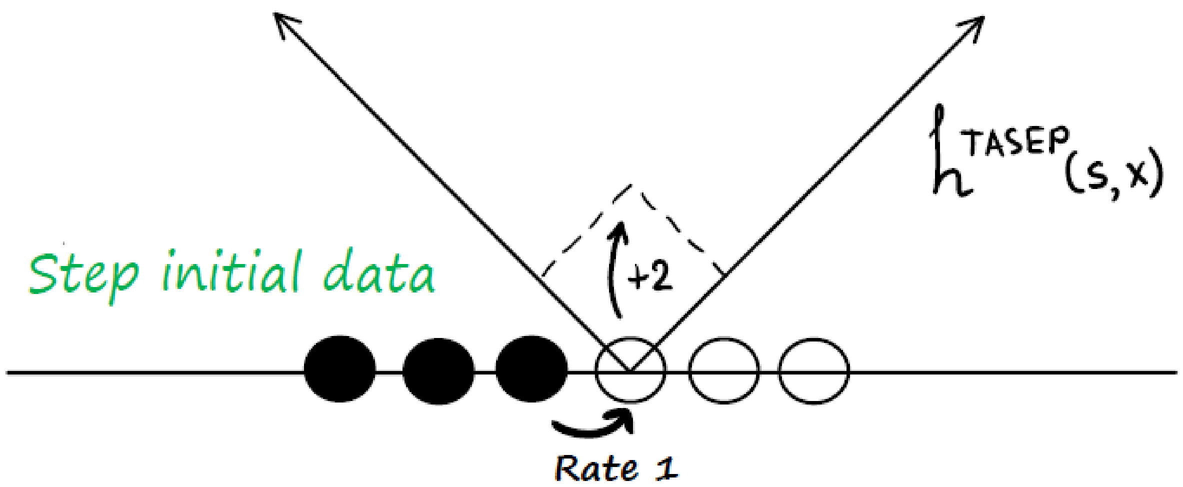
At large time, over distances comparable with time (macroscopic or hydrodynamic limit), the evolution of the interface is deterministic with high probability.

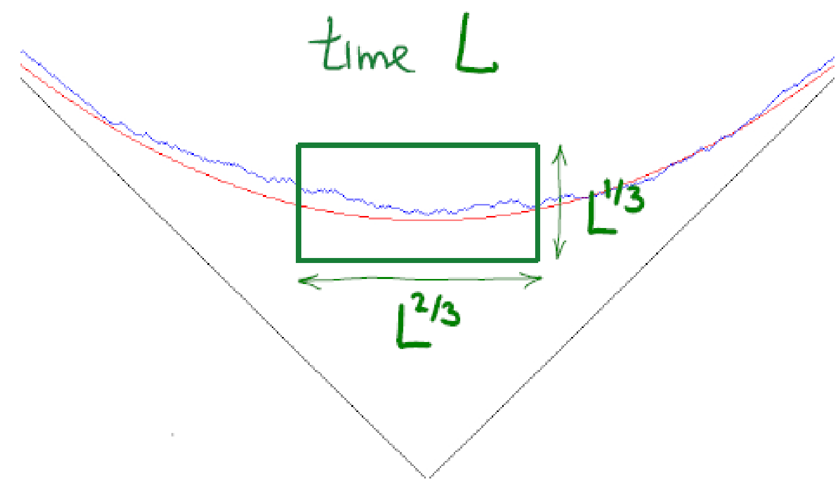
In terms of the average density of particles ρ it is described by the inviscid Burgers equation

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\left(\rho(1-\rho)\right)$$

It is known to develop shocks that correspond to traffic jams of the particle system.

TASEP – fluctuations





$$h_L(t,x) := \frac{1}{L^{1/3}}\, h^{TASEP}(Lt, L^{2/3}x) - L^{2/3}\,\frac{t}{2}$$

Theorem *[Johansson, 1999]* For TASEP with step initial data

$$\lim_{L\to\infty} P\{h_L(1,0) \geq -s\} = F_{GUE}(s)$$

Tracy-Widom limit distribution for the largest eigenvalue of large Hermitian matrices

Edge scaling limit of random matrices

Gaussian Unitary Ensemble (GUE) consists of Hermitian $N \times N$ matrices $H = H^*$ distributed as $\text{const} \cdot e^{-\text{Trace}(H^2)} dH$ [Wigner, 1955].

Theorem [Tracy-Widom, 1993]

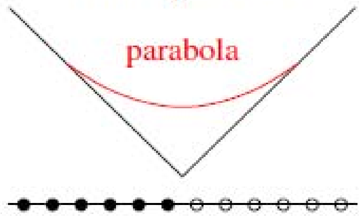
$$\lim_{N \to \infty} \text{Prob}\left\{ \frac{(\text{top eigenvalue of } H) - \sqrt{2N}}{N^{-1/6}} \le x \right\} = F_2(x)$$

where $u = \sqrt{-(\log F_2)''}$ satisfies $u'' = xu + 2u^3$ with appropriate initial conditions.

For real symmetric matrices one similarly defines the Gaussian Orthogonal Ensemble (GOE) and $F_1(x)$.

TASEP – fluctuations depend on hydrodynamic profile

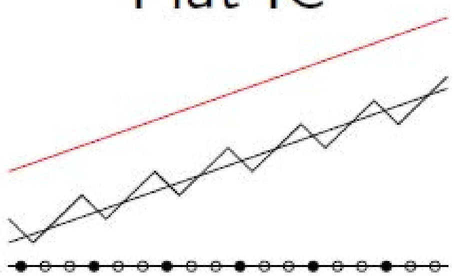
Step IC



$(t^{2/3}, t^{1/3})$-scaling leads to $Airy_2$ process; one-point fluctuations are those of the edge of GUE

Johansson 1999, 2002
Prähofer-Spohn 2002

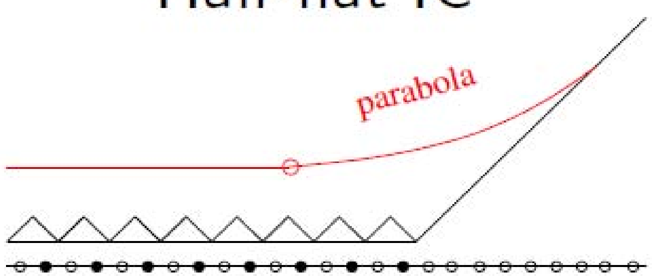
Flat IC



$Airy_1$ process and edge of GOE

Sasamoto 2005
Borodin-Ferrari-Prähofer-Sasamoto 2006, 2007

Half-flat IC



$Airy_1$ in flat part,
$Airy_2$ in curved part,
$Airy_{1\to 2}$ in between.

Borodin-Ferrari-Sasamoto 2007

Kardar-Parisi-Zhang (KPZ) universality class

TASEP is an integrable representative of the (conjectural) KPZ universality class of growth models in 1+1 dimensions that is characterized by

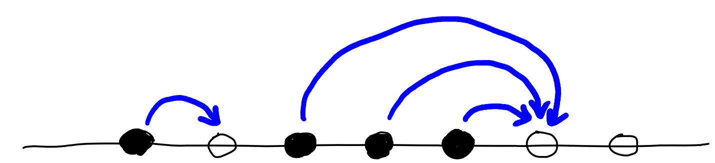
- Locality of growth (no long-range interaction)
- A smoothing mechanism (a.k.a. relaxation, no fractal structures)
- Lateral growth (speed of growth depends on the slope)

If the speed of growth does not depend on the slope than the model is in the Edwards-Wilkinson class with different fluctuations ($t^{1/4}$ instead of $t^{1/3}$ and Gaussian distributions)

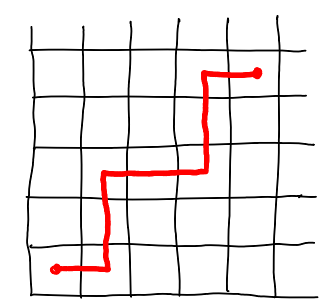
TASEP is one of a few growth models in the KPZ class that can be analyzed via the techniques of determinantal point processes (or free fermions, nonintersecting paths, Schur processes).

Other examples include

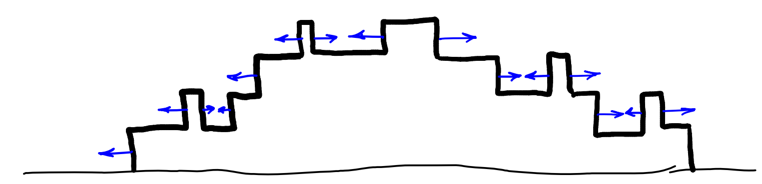
- Discrete time TASEPs with sequential/parallel update
- PushASEP or long range TASEP



- Directed last passage percolation in 2d with geometric/Bernoulli/exponential weights



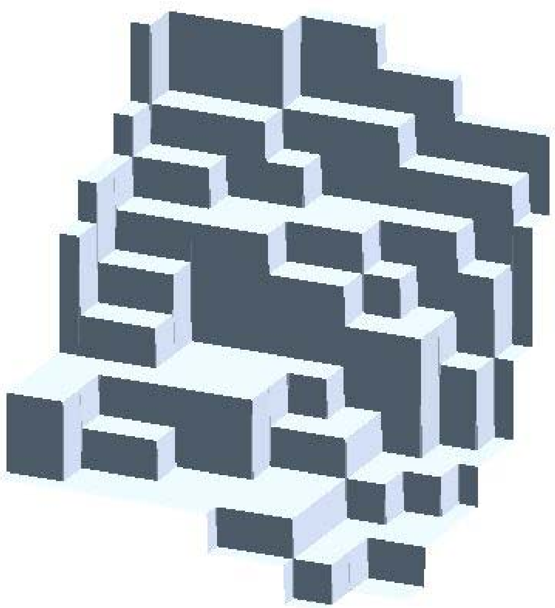
- Polynuclear growth processes

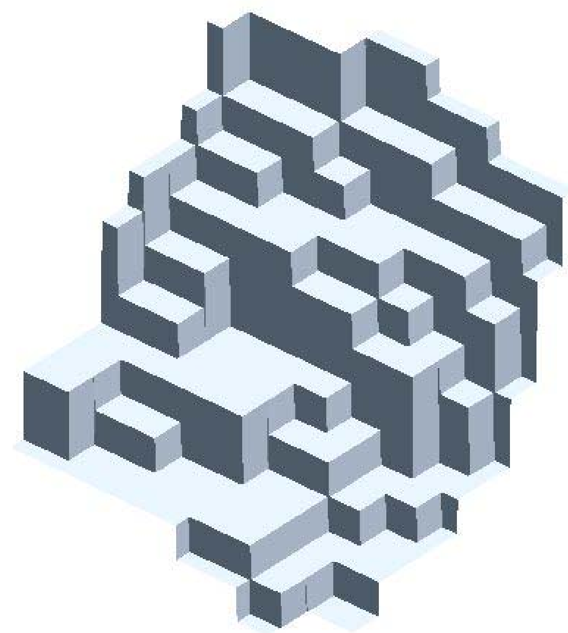


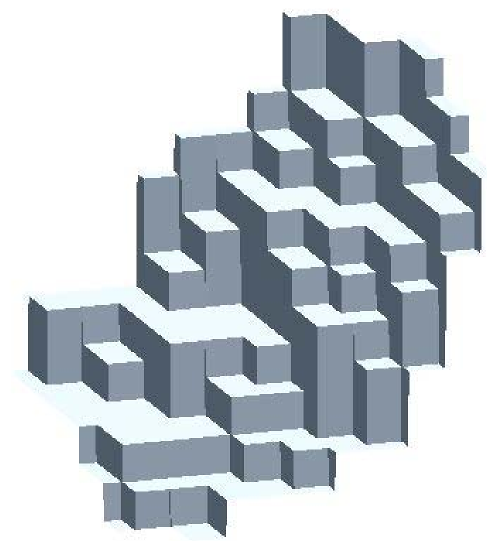
More recently, a variety of non-determinantal, yet still integrable models have been analyzed in the large time limit:

- ASEP [Tracy-Widom, 2007-09], [Borodin-Corwin-Sasamoto, 2012]
- KPZ equation or continuous Brownian polymer
  [Amir-Corwin-Quastel, 2010], [Sasamoto-Spohn, 2010], [Dotsenko, 2010+], [Calabrese-Le Doussal-Rosso, 2010+], [Borodin-Corwin-Ferrari, 2012]
- q-TASEP [Borodin-Corwin, 2011+], [Ferrari-Veto, 2013]
- (Semi)-Discrete directed polymers with special weights
  [O'Connell, 2009], [Borodin-Corwin, 2011], [Borodin-Corwin-Ferrari, 2012], [Corwin-O'Connell-Seppalainen-Zygouras, 2011], [Borodin-Corwin-Remenik, 2012]
- Stochastic six-vertex (square ice) model
  [Borodin-Corwin-Gorin, 2014]

How about random interfaces in the 3d space?

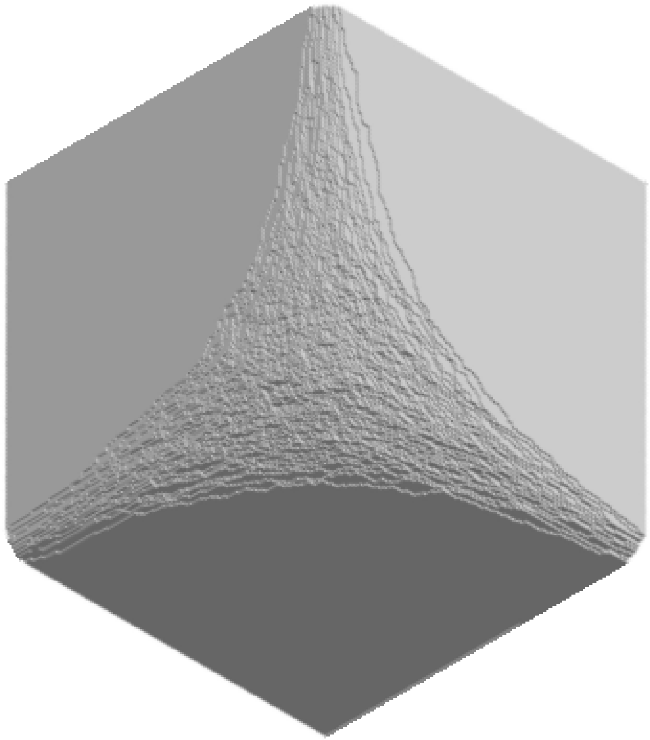




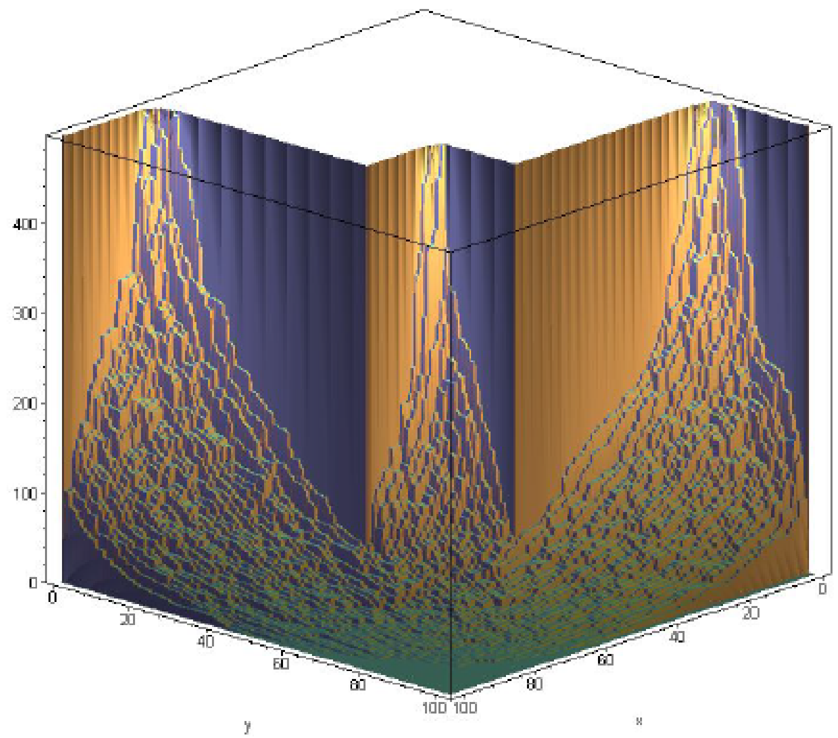


We will consider stepped surfaces built from 1x1x1 cubes.

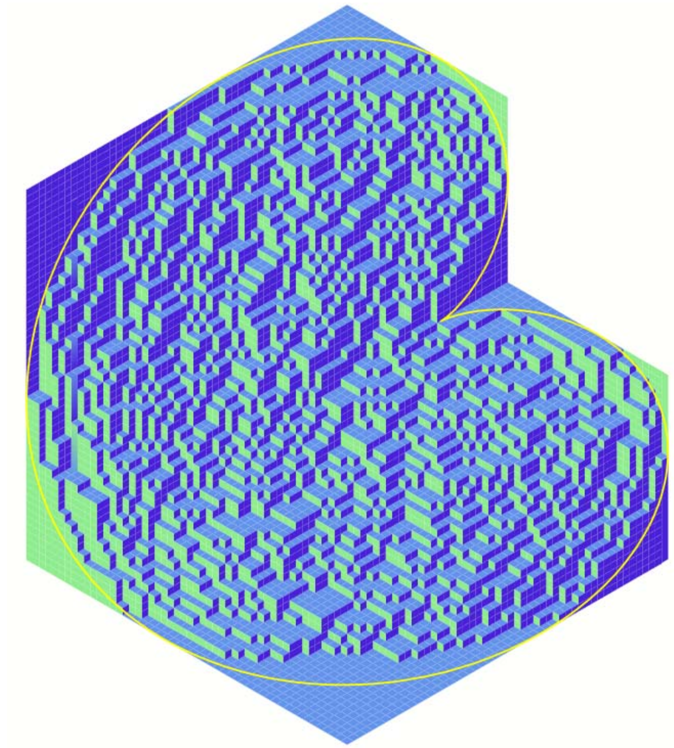
Uniformly random stepped surfaces produce beautiful algebraic limit shapes that vary depending on the boundary conditions



corner of a room



a room with several corners



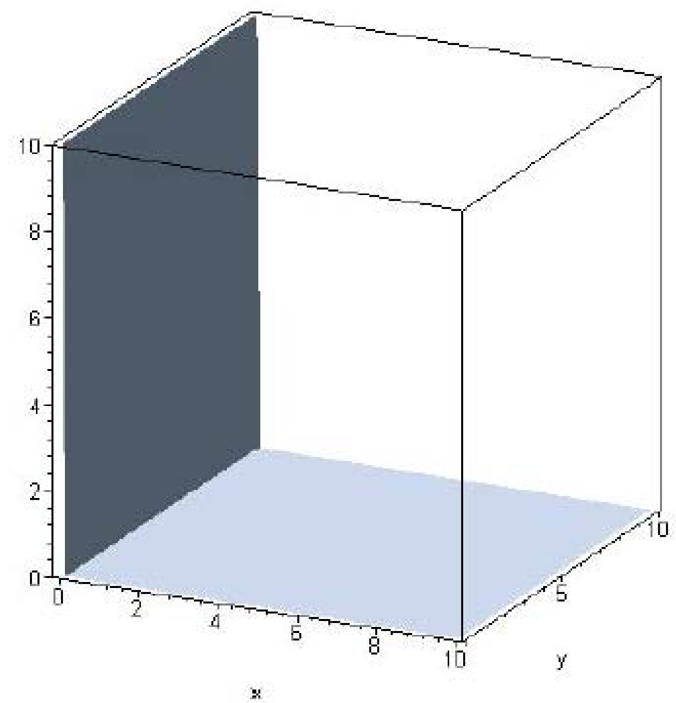
a cardioid

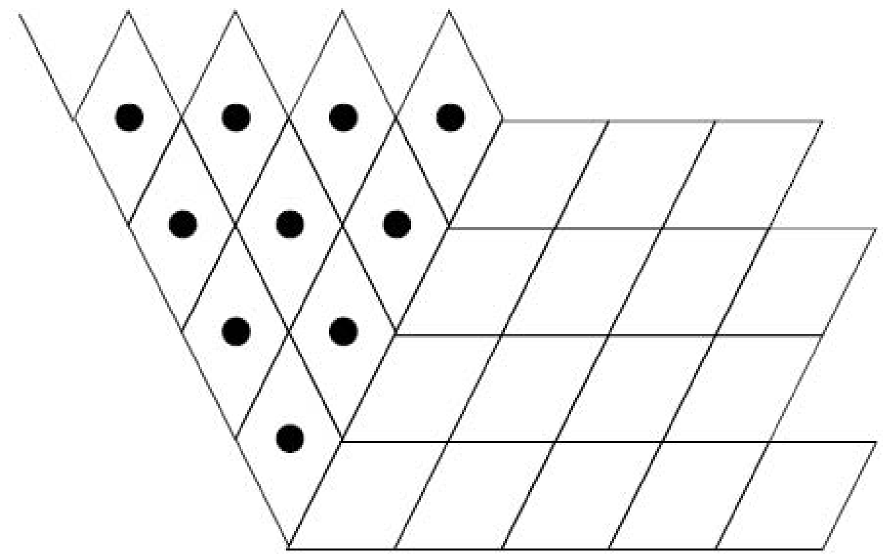
[Okounkov-Reshetikhin, 2001, 2005], [Kenyon-Okounkov, 2006]

Can one grow them?

An integrable random growth model [Borodin-Ferrari, 2008]

Consider the `empty' initial condition

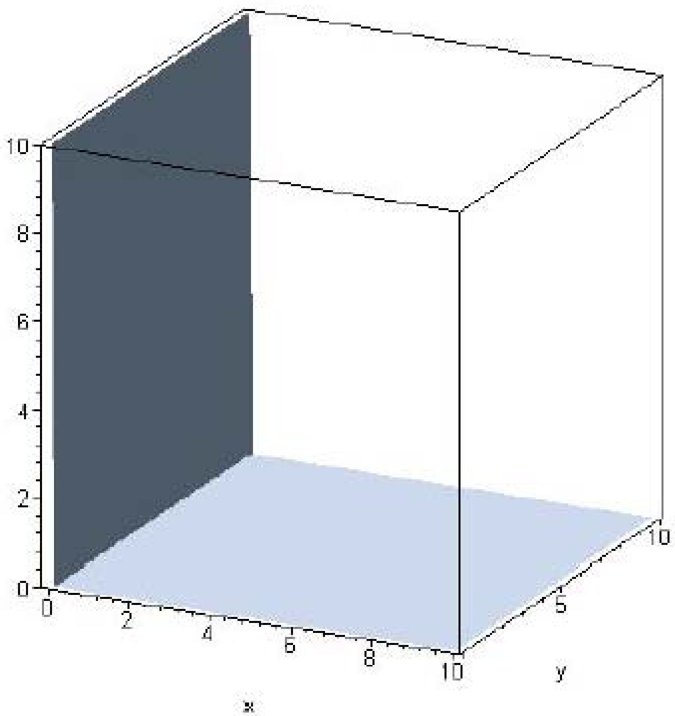


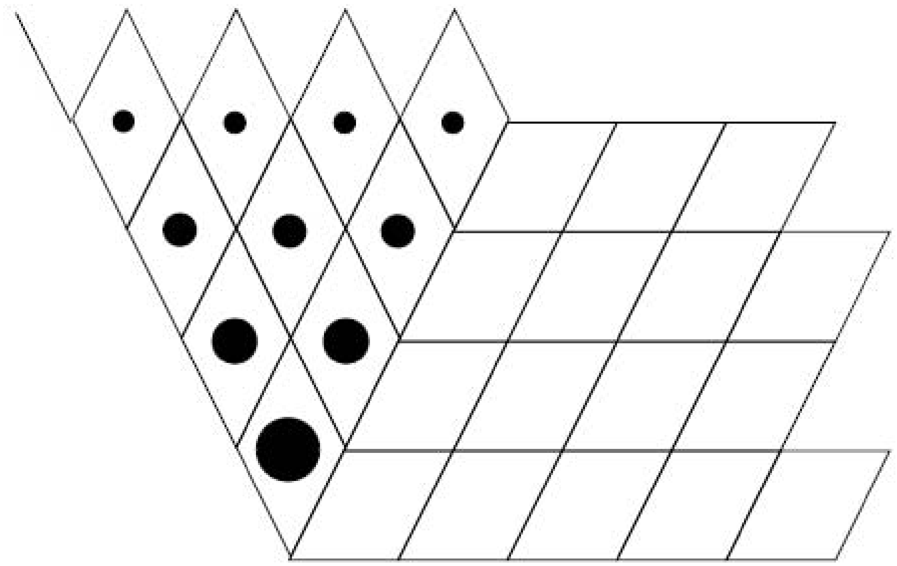


It is often convenient to represent a stepped surface as a flat tiling by rhombi of three types (a.k.a. `lozenges').
Place particles inside rhombi of a fixed type.

An integrable random growth model

Consider the `empty' initial condition



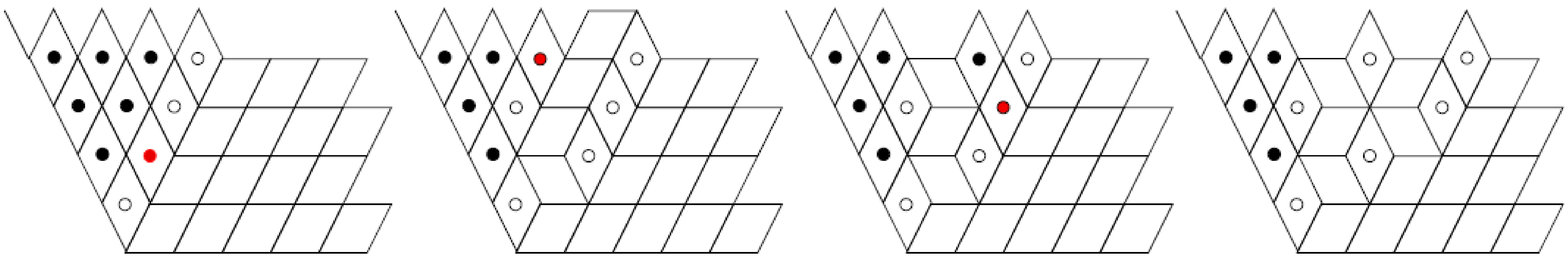


Imagine that particles have weights that decrease upwards.

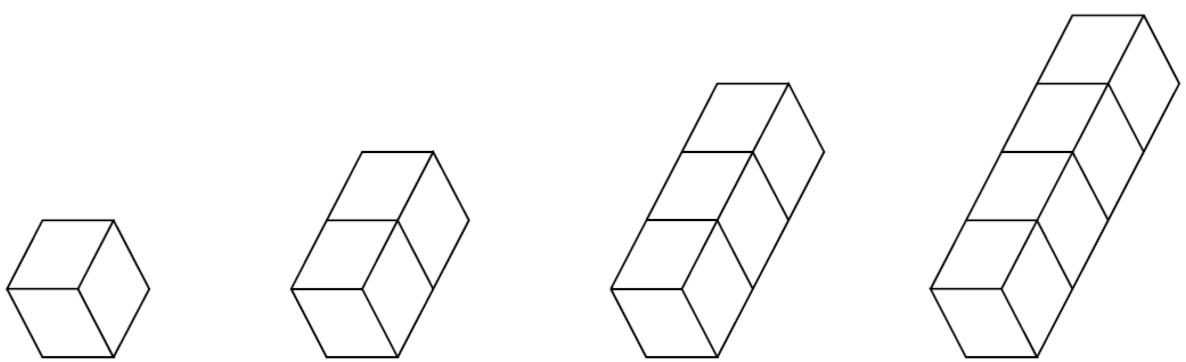
An integrable random growth model

Each particle jumps to the right independently with rate 1.

It is blocked by heavier particles and it pushes lighter particles.



In 3d, this can be viewed as adding directed columns



Column deposition - Animation

by Patrik Ferrari

An integrable random growth model

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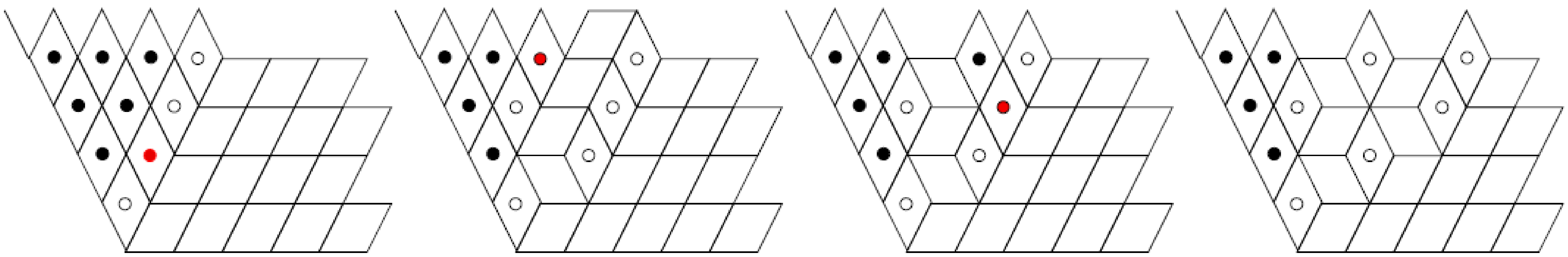
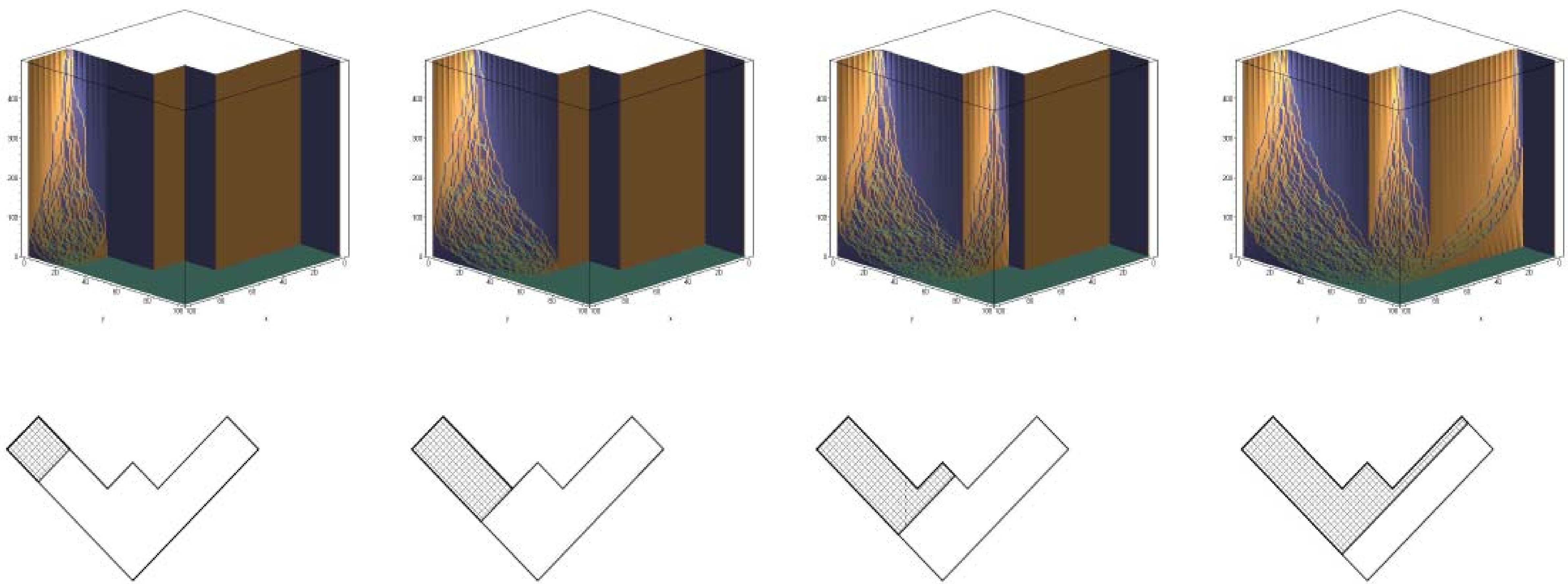


- Left-most particles form TASEP
- Right-most particles form PushTASEP
- Large time (diffusive) limit of the evolution of n particles on the n-th horizontal level is Dyson's Brownian motion for GUE

Growing plane partitions [Borodin-Gorin, 2009], [Borodin, 2010]

A very similar stochastic evolution can be used to grow stepped random interfaces with given boundary conditions by growing their support:

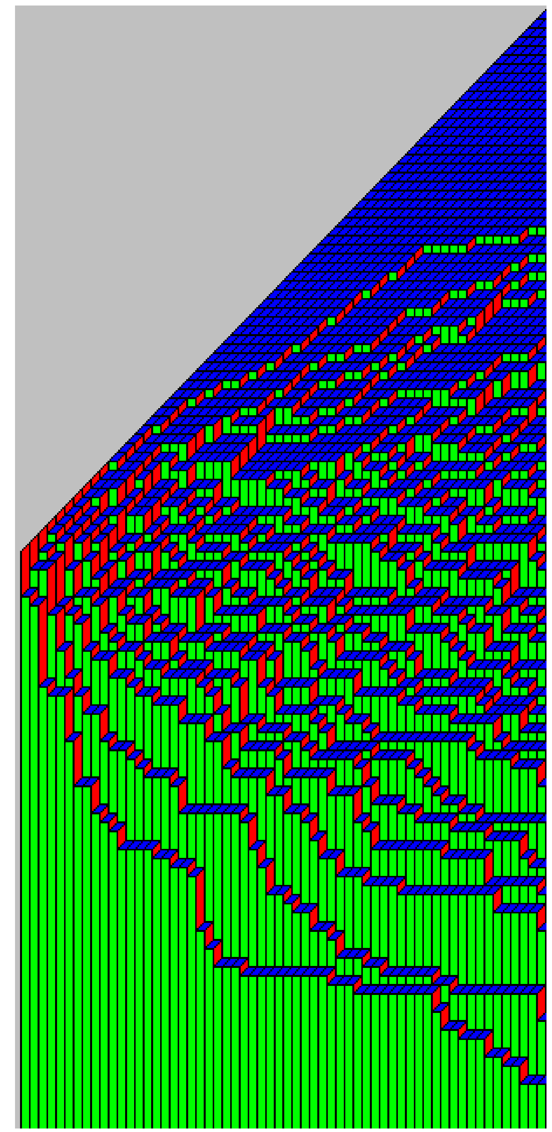
Simulation by Vadim Gorin



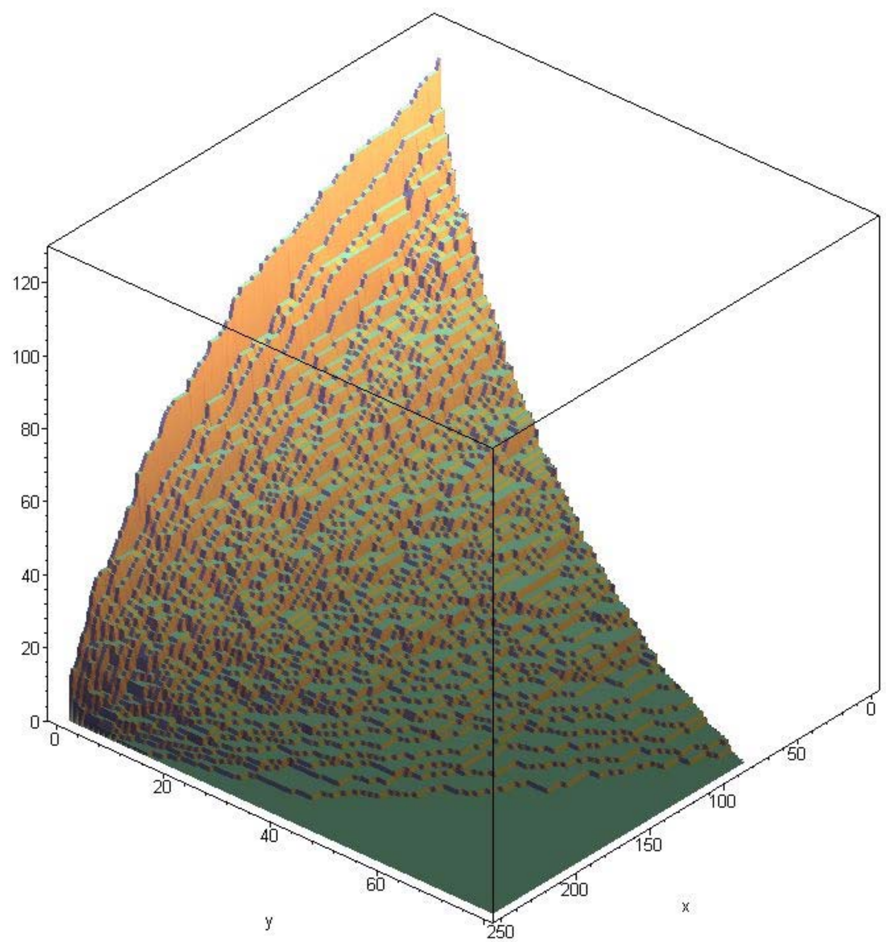
Large time behaviour

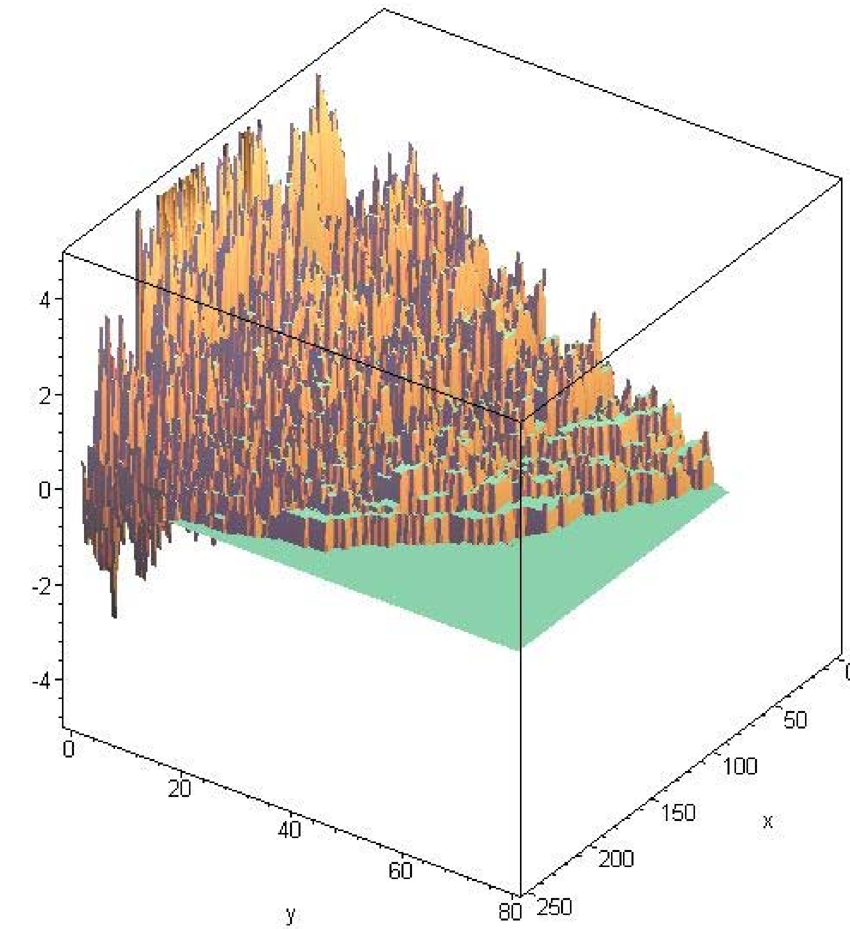
The following are broadly expected and proved in a few cases:

- In the hydrodynamic scaling, a deterministic limit shape arises. It is described by $\partial_t h = f(x, \nabla h)$.
- The models belong to the anisotropic KPZ universality class associated with the (formal) equation $\partial_t h = \Delta h + (\partial_x h)^2 - (\partial_y h)^2 + \text{white noise}$.
- One-point fluctuations in the bulk are Gaussian with log(t) variance (predicted in [Wolf, 1991])
- Multi-point fluctuation are described by the two-dimensional Gaussian Free Field.



Column deposition – multipoint fluctuations





The (unscaled) fluctuations are two rough to have values at given locations.

Column deposition – multipoint fluctuations

Under a bijection $\Omega : \{\text{limit shape}\} \to \mathbb{H} = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$ the fluctuations converge to the Gaussian generalized function

$$GFF(\Omega) = \sum_k \xi_k \cdot \frac{\varphi_k(\Omega)}{\sqrt{\lambda_k}}$$

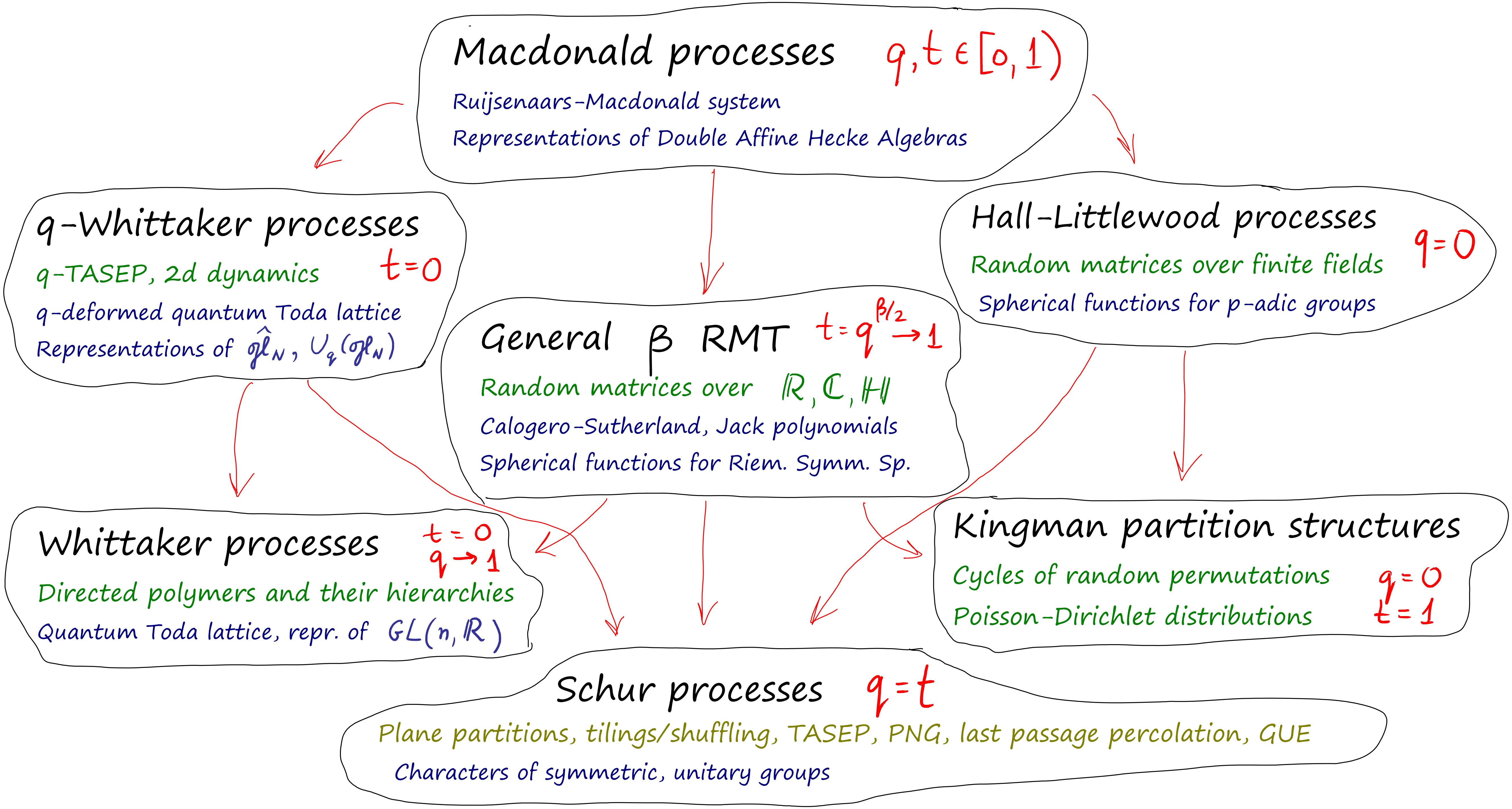
where φ_k's are the eigenfunctions of $-\Delta$ on $\mathbb{H}$ with zero boundary conditions, λ_k is the corresponding eigenvalue, and ξ_k's are i.i.d. standard Gaussians.

The GFF was first obtained, and hence conjectured to be universal for (2+1)d anisotropic KPZ growth, via a rigorous analysis of the integrable column deposition model.

The approach that leads to large time/space analysis of the integrable probabilistic models is largely algebraic.

The hierarchy of integrable models shadows that of multivariate special functions that originate from representation theory and integrable systems as characters/zonal spherical functions for Lie groups/symmetric spaces over real/complex, finite, and p-adic fields, and as eigenfunctions for integrable quantum many body systems.

Representation theoretic tools are essential in our approach.


Macdonald processes
$q, t \in [0,1)$
Ruijsenaars–Macdonald system
Representations of Double Affine Hecke Algebras
q–Whittaker processes
$t=0$
q–TASEP, 2d dynamics
q–deformed quantum Toda lattice
Representations of $\widehat{gl}_N$, $U_q(gl_N)$
Hall–Littlewood processes
$q=0$
Random matrices over finite fields
Spherical functions for p–adic groups
General β RMT
$t=q^{\beta/2} \to 1$
Random matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$
Calogero–Sutherland, Jack polynomials
Spherical functions for Riem. Symm. Sp.
Whittaker processes
$t=0$
$q \to 1$
Directed polymers and their hierarchies
Quantum Toda lattice, repr. of $GL(n, \mathbb{R})$
Kingman partition structures
Cycles of random permutations
Poisson–Dirichlet distributions
$q=0$
$t=1$
Schur processes
$q=t$
Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE
Characters of symmetric, unitary groups