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REGULARITY OF A CLASS OF WEAK SOLUTIONS TO THE MONGE-AMPÈRE EQUATION

A Dissertation
Submitted to
the Temple University Graduate Board

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in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

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by
David Hartenstine
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ABSTRACT

REGULARITY OF A CLASS OF WEAK SOLUTIONS TO THE MONGE-AMPÈRE EQUATION

David Hartenstine
DOCTOR OF PHILOSOPHY

Temple University, August, 2001

Professor Cristian Gutiérrez, Chair

In this dissertation we examine the regularity properties of Aleksandrov solutions to the Monge-Ampère equation $\det D^2u = \mu$, where the Borel measure μ satisfies a weak condition, D_ϵ, on the sections of u. The condition referred to is actually a family of conditions, indexed by $\epsilon \in (0, 1]$. The case $\epsilon = 1$ corresponds to a doubling property. The doubling condition implies D_ϵ for every ϵ. We show that when the function u is globally defined and its Monge-Ampère measure Mu is D_ϵ, then Mu is actually doubling, so that the conditions are equivalent in this case. We then explore the regularity properties of functions u defined on bounded domains for which Mu is D_ϵ. These results are an extension of the regularity theory available when $0 < \lambda \leq \mu \leq \Lambda$ to a wider class of measures.

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To my wife Mary Beth and my parents.

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CHAPTER 1

INTRODUCTION

This dissertation is concerned with determining properties of convex solutions to the Monge-Ampère equation. In its classical formulation, this is the fully nonlinear equation $\det D^2u = f$, where D^2u is the Hessian of the function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. When f is non-negative, it makes sense to consider convex solutions, since if u is C^2 and convex, then $\det D^2u \geq 0$. For every continuous function u we can construct a Borel measure, called the Monge-Ampère measure associated with u, see Theorem 2.1.2. This idea allows us (following Aleksandrov) to define a notion of weak solution for the Monge-Ampère equation. Given a Borel measure ν on $\Omega \subset \mathbb{R}^n$, we say that the convex function $u \in C(\Omega)$ is a weak (or Aleksandrov) solution to $\det D^2u = \nu$ if the Monge-Ampère measure associated to u, denoted Mu, equals ν. Notice that these weak solutions are always convex, and so automatically possess certain properties. In this dissertation, we are primarily interested in establishing what further properties are satisfied by convex functions whose Monge-Ampère measures satisfy a specific condition.

This condition is actually a family of conditions, indexed by a parameter $\epsilon \in (0, 1]$. When $\epsilon = 1$, this condition becomes a doubling property. When this condition is satisfied by a measure μ, we write μ is D_ϵ or $\mu \in D_\epsilon$. If the measure $\mu \in D_{\epsilon_0}$, then $\mu \in D_\epsilon$ for all $\epsilon \leq \epsilon_0$. In particular, if $\mu \in D_1$, then μ is D_ϵ for all ϵ. The definition of D_ϵ is stated in terms of properties satisfied by the cross-sections of the function u (see Definition 2.2.1), and the normalized distance to the boundary of a convex set. The cross-sections (or simply sections) of u are the level sets of $u(x) - l(x)$ where $l(x)$ is a supporting hyperplane to u. These sets are convex. The idea of studying these sets in order to analyze u is due to Caffarelli, and is described and further developed in [1], [2] and [3], among others. The notion of normalized distance to the boundary of a convex set was introduced by Jerison in [5] and appears below as Definition 2.3.1. In spite of the fact that it seems like we may be considering many conditions, it turns out that the actual value of ϵ for a given measure is largely irrelevant. None of the results we prove are dependent in a qualitative way on the value of ϵ.

Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Suppose $u \in C(\bar{\Omega})$ is a weak solution of $\det D^2u = \mu$, where $0 < \lambda \leq \mu \leq \Lambda$, and is zero on $\partial\Omega$. Then u is strictly convex and is $C^{1,\alpha}$ for some α in the interior of Ω. For proofs of these statements see [1], [2] and [3]. We show that these results also hold with the weaker hypotheses of D_ϵ.

We conclude this Introduction with a summary of the main results and an outline of what follows. Chapter 2 contains background material and preliminary results needed for the statements and proofs of the main results that are in Chapter 3. The first section of Chapter 3 compares the doubling condition and D_ϵ. We prove that if u is defined on all of $\mathbb{R}^n$ and satisfies the D_ϵ condition for some $0 < \epsilon < 1$, then the doubling condition is also satisfied, so in this case the conditions are equivalent. An example shows that this is not true if the domain of definition is bounded. The rest of the dissertation is concerned with properties of solutions satisfying the D_ϵ condition on bounded convex domains. The next section contains the proof of strict convexity for nontrivial solutions to the Dirichlet problem with zero boundary data. This is a consequence of a Caffarelli-type extremal points theorem (Theorem 3.2.1). The situation of non-constant boundary data is also analyzed in this section. Section 3.3 is devoted to a rather technical selection or compactness result (Lemma 3.3.1) for functions whose Monge-Ampère measures satisfy the D_ϵ condition. This lemma is employed in the proof of a useful property about sections with base point lying at least a certain distance from the boundary of the domain. As a consequence of this property, we obtain a result first claimed by Jerison in [5]: if $u \in C(\bar{\Omega})$ is convex, zero on $\partial\Omega$ and $Mu \in D_\epsilon$, then u is $C^{1,\alpha}$ in the interior of Ω. Both of these results are discussed in Section 3.4. Finally, in Chapter 4, we show that analogs of Jerison's estimates (which are used repeatedly in proving the results mentioned above) are true for parabolically convex solutions of the parabolic Monge-Ampère equation $-u_t \det D_x^2 u = f$ on bowl-shaped domains.

CHAPTER 2

PRELIMINARY MATERIAL

2.1 Weak Solution

The weak solution considered for the Monge-Ampère equation $\det D^2u(x) = f(x)$ is that of Aleksandrov, and requires the notions of the normal mapping of a function $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$, and the Monge-Ampère measure associated to u. We begin by explaining these topics. This material comes from Chapter 1 of [3].

Definition 2.1.1 *The normal mapping (or sub-differential) of u is the set-valued function $\partial u : \Omega \to \mathcal{P}(\mathbb{R}^n)$, whose value at a point x_0 is the set*

$$\partial u(x_0) = \{p \in \mathbb{R}^n : u(x) \geq u(x_0) + p \cdot (x - x_0),\ x \in \Omega\}.$$

If $E \subset \Omega$, the normal map of E is $\partial u(E) = \bigcup_{x \in E} \partial u(x)$.

Given a function u, $\partial u(x_0)$ is the set of points p that determine supporting hyperplanes (affine functions $l(x)$ such that $l(x_0) = u(x_0)$ and $l(x) \leq u(x)$ for all $x \in \Omega$) to u at x_0. It is possible for $\partial u(x_0)$ to be empty. Indeed, if u is strictly concave on Ω then the normal mapping will be empty at every point of the domain. When the function u is convex, $\partial u(x)$ is nonempty at every point. If u is differentiable at x_0 and $\partial u(x_0) \neq \emptyset$ then $\partial u(x_0) = Du(x_0)$. Since convex functions are differentiable at almost every point, the normal mapping provides a substitute for the gradient for these functions.

Theorem 2.1.2 *(Theorem 1.1.13 in [3]) If Ω is open and $u \in C(\Omega)$ then the family of sets*

$$S = \{E \subset \Omega : \partial u(E) \text{ is Lebesgue measurable}\}$$

is a Borel σ-algebra. The map $Mu : S \to \bar{\mathbb{R}}$ defined by $Mu(E) = |\partial u(E)|$ (where the notation $|S|$ indicates the Lebesgue measure of the set S) is a measure, finite on compact subsets, called the Monge-Ampère measure associated with the function u.

We are now in a position to define the weak solution.

Definition 2.1.3 *Let ν be a Borel measure defined in Ω, an open and convex subset of $\mathbb{R}^n$. The convex function $u \in C(\Omega)$ is an Aleksandrov or weak solution to the Monge-Ampère equation $\det D^2u = \nu$ if the Monge-Ampère measure Mu associated with the function u equals ν.*

This notion of weak solution is reasonable, since if $u \in C^2(\Omega)$ is convex and is a classical solution of $\det D^2u(x) = f(x)$, then

$$Mu(E) = \int_E \det D^2u(x)\, dx = \int_E f(x)\, dx$$

holds for any Borel set $E \subset \Omega$. This means that u is an Aleksandrov solution of $\det D^2u = f(x)\, dx$.

We now state a lemma that will be useful in some of the convergence arguments below.

Lemma 2.1.4 *(Lemma 1.2.3 in [3]) If u_k are convex functions in Ω such that $u_k \to u$ uniformly on compact subsets of Ω then the associated Monge-Ampère measures Mu_k converge to Mu weakly. In other words,*

$$\int f(x)\, dMu_k \to \int f(x)\, dMu$$

for every $f \in C_0^0(\Omega)$.

Note that these solutions are always convex. The main objective of this thesis is to determine what further regularity properties are satisfied by solutions when the measure on the right-hand side satisfies a certain condition explained below.

Before introducing this condition, we state the fundamental existence and uniqueness result for Aleksandrov solutions of the Monge-Ampère equation.

Theorem 2.1.5 *(Theorem 1.6.2 in [3]) If Ω is open bounded and strictly convex, μ is a finite Borel measure on Ω, and $g \in C(\partial\Omega)$, then there exists a unique $u \in C(\bar{\Omega})$ which satisfies $Mu = \mu$ in Ω and $u = g$ on $\partial\Omega$.*

Finally, to end this section, we mention a comparison principle that will be exploited in a barrier argument in the proof of strict convexity of solutions to the Dirichlet problem with nonzero boundary data.

Theorem 2.1.6 *(Theorem 1.4.6 in [3]) Let $u, v \in C(\bar{\Omega})$ be convex functions such that for every Borel set $E \subset \Omega$,*

$$|\partial u(E)| \leq |\partial v(E)|.$$

Then

$$\min_{\bar{\Omega}}\{u - v\} = \min_{\partial\Omega}\{u - v\}.$$

2.2 Cross-Sections and Normalization

The approach to studying regularity of solutions to this equation involves analyzing the properties of the sections of the solution u.

Definition 2.2.1 *The sections of u are the (convex) sets*

$$S(x_0, p, t) = \{x \in \Omega : u(x) < u(x_0) + p \cdot (x - x_0) + t\},$$

where $p \in \partial u(x_0)$ and $t > 0$.

These sets are formed by taking a supporting hyperplane to u at x_0, sliding it up by a height of t, and looking at the values of $x \in \Omega$ for which the graph of u lies below this raised hyperplane. Roughly speaking, the main idea of this approach is that if the properties of these sections are known, this should provide valuable information about the function u itself. We make the assumption that the sections of u are bounded sets. This means that the graph of u does not contain any rays. Furthermore, when the function u is defined on a bounded set Ω, we only consider values of the parameter t for which either $\overline{S(x_0, p, t)} \subset \Omega$ or if $\overline{S(x_0, p, t_0)} \cap \partial\Omega \neq \emptyset$, then $\overline{S(x_0, p, t)} \subset \Omega$ for $t < t_0$.

We now introduce some notation and terminology. For any bounded convex set S, its center of mass will be denoted $c(S)$, and this point is given by

$$c(S)_i = \frac{1}{|S|} \int_S x_i \, dx \text{ for } i = 1, \ldots n.$$

Secondly, αS for $\alpha > 0$ refers to the dilation of S with respect to $c(S)$ by a factor of α. In other words,

$$\alpha S = \{\alpha(x - c(S)) + c(S) : x \in S\}$$

Various conditions can be imposed on the measure appearing on the right-hand side of the equation. One case is to consider those measures that are bounded between two positive constants, i.e. $\lambda|E| \leq \mu(E) \leq \Lambda|E|$. In this case, many regularity results are known: see [1], [2] and [3]. This thesis is concerned with two generalizations of this condition, which are stated in terms of properties satisfied by the sections of u. The first is a doubling condition: The measure μ is said to be doubling on the sections of u if $\mu(S) \leq C\mu(\frac{1}{2}S)$ for all sections S and some positive constant C. If $\lambda \leq \mu \leq \Lambda$, then μ is doubling, since

$$\mu(S) \leq \Lambda|S| = 2^n \Lambda|\frac{1}{2}S| = 2^n \frac{\Lambda}{\lambda} \lambda |\frac{1}{2}S| \leq 2^n \frac{\Lambda}{\lambda} \mu(\frac{1}{2}S).$$

The second generalization of the measure being bounded between two constants requires a little more background, and will be introduced in Section 2.3 below.

Definition 2.2.2 *A convex set Ω is said to be normalized if $c(\Omega) = 0$ and $B_{\alpha_n}(0) \subset \Omega \subset B_1(0)$, where α_n is a dimensional constant.*

The following result of Fritz John plays a critical role in many of the arguments below.

Theorem 2.2.3 *Let $\Omega \subset \mathbb{R}^n$ be open bounded and convex. Then there exists an invertible affine transformation T such that $T(\Omega)$ is normalized.*

The proof of this theorem uses the following result concerning ellipsoids of minimum volume.

Lemma 2.2.4 *(Lemma 1.8.1 in [3]) Let $S \subset \mathbb{R}^n$ be bounded and convex. Suppose also that S has nonempty interior. Then:*

(a) Let $x_0 \in S$. Consider the class F_0 of ellipsoids centered at x_0 that contain S. Then there exists $E_0 \in F_0$ such that $|E_0| \leq |E|$ for all $E \in F_0$. We say that E_0 is an ellipsoid of minimum volume for S centered at x_0.

(b) Consider the class F_1 of all ellipsoids that contain S. Then there exists $E_1 \in F_1$ such that $|E_1| \leq |E|$ for all $E \in F_1$. E_1 is called an ellipsoid of minimum volume for S.

We conclude this section with a useful formula for changing variables by an affine transformation (See p. 47 of [3]). Let T be an invertible affine transformation, $Tx = Ax + b$ for some nonsingular matrix A and some $b \in \mathbb{R}^n$. Suppose $u : \mathbb{R}^n \to \mathbb{R}$ and $v(y) = \lambda^{-1}u(T^{-1}y)$ where $\lambda > 0$. The affine function $l(x) = u(x_0) + p \cdot (x - x_0)$ is a supporting hyperplane to u at x_0 if and only if $\bar{l}(y) = v(Tx_0) + \lambda^{-1}(A^{-1})^t p \cdot (y - Tx_0)$ is a supporting hyperplane to v at Tx_0. This means that if $S = S_u(x_0, p, t)$ is a section of u, then $T(S)$ is a section of v. More precisely, $T(S) = S_v(Tx_0, \lambda^{-1}(A^{-1})^t p, \frac{t}{\lambda})$. This also implies that

$$\frac{1}{\lambda}(A^{-1})^t(\partial u(E)) = \partial v(TE),$$

and hence that

$$Mv(TE) = \frac{1}{\lambda^n}|\det A^{-1}|Mu(E) \tag{2.1}$$

for any Borel set E. From this formula and the fact that for any section S, $T(\alpha S) = \alpha T(S)$, we get that if Mu is doubling (or D_ϵ introduced in the next section), then so is Mv, with the same constant.

2.3 Normalized Distance to the Boundary

Given a bounded open and convex set S, we can define, following Jerison (p.31 in [5]), a dimensionless, normalized distance from an interior point to the boundary of S.

Definition 2.3.1 *The normalized distance from $x \in S$ to the boundary of the convex set S is*

$$\delta(x, S) = \min \frac{|x - x_1|}{|x - x_2|}$$

where x_1 and x_2 are in ∂S and the three points x, x_1 and x_2 are collinear.

In other words, consider a line through the point x. This line intersects ∂S in exactly two points x_1 and x_2. We can then form the fraction appearing in the definition. Then take the minimum over all lines passing through x, to determine $\delta(x, S)$.

Note that this quantity is always less than or equal to 1. Other properties of the normalized distance appear below.

We are now ready to define the second condition that will be imposed on the measure on the right-hand side of the equation. This condition was first considered in [5].

Definition 2.3.2 *Let ϵ satisfy $0 < \epsilon \leq 1$. The measure μ is said to be D_ϵ or $\mu \in D_\epsilon$, if*

$$\int_S \delta(x, S)^{1-\epsilon} d\mu \leq C\mu\left(\frac{1}{2}S\right)$$

for all sections S.

Notice that the doubling condition is included in this family of conditions, and corresponds to $\epsilon = 1$. Since $\delta(x, S) \leq 1$ for any x and any S, if the measure

μ is doubling, then it satisfies D_ϵ for any ϵ. More generally, if $\mu \in D_{\epsilon_0}$, then $\mu \in D_\epsilon$ for every $\epsilon < \epsilon_0$. When we need to specify the constant appearing in the definition of D_ϵ, we use the notation $\mu \in D_\epsilon(C)$ to denote that μ satisfies the D_ϵ condition with the constant C.

Two important properties of the normalized distance $\delta(\cdot,\cdot)$ are contained in the next result.

Lemma 2.3.3 *(a)The normalized distance is invariant under affine transformations. If T is an affine transformation, then $\delta(x, S) = \delta(Tx, T(S))$.*

(b) When the set S is normalized, $\delta(x, S) \approx dist(x, \partial S)$.

Remark These properties taken together provide a justification for calling $\delta(\cdot,\cdot)$ the "normalized distance to the boundary". Let S be any open, bounded, convex set in $\mathbb{R}^n$. By Theorem 2.2.3, there exists an affine transformation T that normalizes S. Using the two properties listed above, we see that $\delta(x, S) = \delta(Tx, T(S)) \approx dist(Tx, \partial T(S))$.

Proof To see why *(a)* holds, simply notice that for any affine transformation T,

$$\frac{|x_1 - x|}{|x_2 - x|} = \frac{|Tx_1 - Tx|}{|Tx_2 - Tx|},$$

and recall that T maps straight lines to straight lines.

The proof of *(b)* requires a little more work. Let S be a normalized convex set. In addition to the usual notion of $dist(x, \partial S)$ we also consider the radial distance from $x \neq 0$ to the boundary ∂S. This is given by $|x - cx|$, where $c > 0$ and $cx \in \partial S$. We denote this radial distance by $dist_r(x, \partial S)$. We will show that $\delta(x, S) \geq C_1 dist(x, \partial S)$ and that $\delta(x, S) \leq C_2 dist_r(x, \partial S)$ for all $x \in S$. The claim will then follow since $dist(\cdot, \partial S) \approx dist_r(\cdot, \partial S)$. For any $x \in S$, we have that

$$\delta(x, S) = \min \frac{|x - x_1|}{|x - x_2|} \geq \frac{dist(x, \partial S)}{diam(S)} \geq \frac{1}{2} dist(x, \partial S).$$

Now given $x \in S$ $(x \neq 0)$, let $cx \in \partial S$ be the point for which $|x - cx| = dist_r(x, \partial S)$. Let $\bar{x}$ denote the antipodal point to cx. In other words, $\bar{x} \in \partial S$ and the three points x, cx and $\bar{x}$ are collinear. Then

$$\delta(x, S) \leq \frac{|x - cx|}{|x - \bar{x}|}.$$

The segment from x to $\bar{x}$ contains the segment from 0 to $\bar{x}$ which is larger than α_n since $B_{\alpha_n}(0) \subset S$. Therefore, $\delta(x, S) \leq \frac{1}{\alpha_n} dist_r(x, S)$.

The claim will follow if the radial distance to the boundary and the usual distance to the boundary are comparable. Clearly, $dist_r(x, \partial S) \geq dist(x, \partial S)$. Let $dist(x, \partial S) = \epsilon$ and suppose $dist(x, \partial S) \neq dist_r(x, \partial S)$. Then $B_\epsilon(x) \subset S$. Construct a line l through cx and any point $y \in \partial S$, where $dist(x, \partial S)$ is attained. We now construct two similar triangles. The first one has vertices x, y and cx. The second has vertices at 0 and at cx, the angle at 0 is the same as the angle at x for the first triangle, and the third side of the second triangle is the line l. Note that by convexity, the third vertex of the second triangle lies outside of S, so $\alpha \geq \alpha_n$, where α is the length of the side of the second triangle that connects 0 to the line l. This side corresponds to the side of the first triangle that connects x to y. Then, by similarity,

$$\frac{|cx - x|}{|x - y|} = \frac{|cx - x|}{\epsilon} = \frac{|cx - 0|}{\alpha} = \frac{|cx|}{\alpha} \leq \frac{1}{\alpha_n}|cx| \leq \frac{1}{\alpha_n}.$$

Therefore, $|cx - x| \leq \alpha_n^{-1}\epsilon = \alpha_n^{-1} dist(x, \partial S)$, and these two notions of distance to the boundary are comparable with constants depending only on dimension. This completes the proof. □

Lemma 2.3.4 *Let S be a bounded convex domain. The normalized distance to the boundary $\delta(x, S)$ is continuous.*

Proof Fix $x \in S$. In a neighborhood of each point of ∂S, the boundary is the graph of a Lipschitz function. Cover the compact set ∂S with finitely many of these neighborhoods. Let K be the largest of the Lipschitz constants associated with these neighborhoods, and let $\epsilon < \frac{dist(x, \partial S)}{2(K+1)}$. Let $\bar{x} \in B_\epsilon(x)$, and let l be any line through x. Denote the points where l crosses ∂S by x_1 and x_2. Now translate l in a parallel fashion until it passes through $\bar{x}$. Call this line $\bar{l}$, and denote the points where $\bar{l}$ intersects ∂S by $\bar{x}_1$ and $\bar{x}_2$. Then since the boundary of S is locally Lipschitz and $dist(l, \bar{l}) < \epsilon$, $|x_1 - \bar{x}_1|$ and $|x_2 - \bar{x}_2|$ are both smaller than $K\epsilon$.

Then we have the inequalities:

$$|\bar{x} - \bar{x}_1| \leq |x - \bar{x}| + |x - x_1| + |\bar{x}_1 - x_1| \leq |x - x_1| + (K + 1)\epsilon$$

and

$$|\bar{x} - \bar{x}_2| \geq |x - x_2| - |\bar{x}_2 - x_2| - |x - \bar{x}| \geq |x - x_2| - (K + 1)\epsilon.$$

Therefore, we see that

$$\frac{|\bar{x} - \bar{x}_1|}{|\bar{x} - \bar{x}_2|} \leq \frac{|x - x_1| + (K + 1)\epsilon}{|x - x_2| - (K + 1)\epsilon}.$$

It follows that

$$\frac{|\bar{x} - \bar{x}_1|}{|\bar{x} - \bar{x}_2|} - \frac{|x - x_1|}{|x - x_2|} \leq \frac{(K + 1)\epsilon + |x - x_1|}{|x - x_2| - (K + 1)\epsilon} - \frac{|x - x_1|}{|x - x_2|}$$

$$= \frac{(K + 1)\epsilon(|x - x_2| + |x - x_1|)}{|x - x_2|^2 - (K + 1)\epsilon|x - x_2|} \leq \frac{2(K + 1)\epsilon diam(S)}{dist(x, \partial S)(1 - (K + 1)\epsilon)}.$$

Hence, we observe that for any line l and any $\bar{x} \in B_\epsilon(x)$, the quantity

$$\left|\frac{|\bar{x} - \bar{x}_1|}{|\bar{x} - \bar{x}_2|} - \frac{|x - x_1|}{|x - x_2|}\right|$$

can be made arbitrarily small by choosing ϵ small enough, i.e. $\delta(x, S)$ is continuous. □

2.4 Three Important Estimates

Two pointwise estimates play a crucial role in what follows. Both describe a relationship between the size of u at an interior point of the domain and the "distance" the point lies from the boundary of the domain. A third estimate establishes a connection between a function's minimum and the integral of the normalized distance to the boundary of the domain over the whole domain with respect to the Monge-Ampère measure.

Theorem 2.4.1 ***Aleksandrov's Estimate*** *(Theorem 1.4.2 in [3]) If* $\Omega \subset \mathbb{R}^n$ *is bounded, open and convex and* $u \in C(\bar{\Omega})$ *is convex and* $u|_{\partial\Omega} = 0$, *then*

$$|u(x_0)|^n \leq C(n)(diam\Omega)^{n-1}dist(x_0, \partial\Omega)Mu(\Omega)$$

for all $x_0 \in \Omega$.

When $Mu \in D_\epsilon$ for $0 < \epsilon < 1$, $Mu(\Omega)$ can be infinite, making this estimate not useful, but there is a substitute:

Theorem 2.4.2 ***Aleksandrov-Jerison*** *(Lemma 7.3 in [5]) If* Ω *is convex and normalized,* $u \in C(\bar{\Omega})$ *is convex and* $u|_{\partial\Omega} = 0$, *and* $0 < \epsilon \leq 1$,

$$|u(x_0)|^n \leq C(n, \epsilon)\delta(x_0, \Omega)^\epsilon \int_\Omega \delta(x, \Omega)^{1-\epsilon}\, dMu$$

for all $x_0 \in \Omega$.

Proof Without loss of generality, multiply u by a positive constant so that $u(x_0) = -1$. Choose positive constants s and β small enough so that $\beta n \leq \epsilon$ and $\sum_{k=1}^{\infty} s_k \leq \frac{1}{2}$ where $s_k = s2^{-k\beta}$. Let A denote $\delta(x_0, \Omega)^\epsilon \int_\Omega \delta(x, \Omega)^{1-\epsilon} dMu(x)$. We need to show that $A \geq C = C(s)$, since s depends on ϵ.

Let $\Omega_k = \{x \in \Omega : u(x) \leq \lambda_k \equiv -1+s_1+s_2+\cdots+s_k\}$ and $\delta_k = dist(\Omega_k, \partial\Omega)$ for $k = 0, 1, \ldots$

We claim that $\delta_k \not\to 0$ as $k \to \infty$. If this is not true, then we could select a sequence $\{y_k\}$ with $y_k \in \Omega_k$, such that $dist(y_k, \partial\Omega) \to 0$. From this sequence we can choose a convergent subsequence, also denoted $\{y_k\}$. Then $0 = \lim_{k\to\infty} u(y_k) \leq -1 + \sum s_k \leq -\frac{1}{2}$, a clear contradiction.

Hence there exists a smallest k for which $\delta_{k+1} > \frac{1}{2}\delta_k$. Let $x_k \in \partial\Omega_k$ be a point closest to $\partial\Omega$, i.e. $dist(x_k, \partial\Omega) = \delta_k$.

Then we have that $dist(x_k, \partial\Omega_{k+1}) < \frac{1}{2}\delta_k$. To see this note that the segment L from x_k to $\partial\Omega$ with length δ_k intersects $\partial\Omega_{k+1}$ at some point, say x_{k+1}. Then $dist(x_{k+1}, \partial\Omega) \geq dist(\partial\Omega_{k+1}, \partial\Omega) = \delta_{k+1}$. Now

$$\delta_k = |x_k - x_{k+1}| + l \geq |x_k - x_{k+1}| + dist(x_{k+1}, \partial\Omega) > |x_k - x_{k+1}| + \frac{1}{2}\delta_k$$

where l is the length of the rest of the segment L (not the part from x_k to x_{k+1}). Therefore, $\frac{1}{2}\delta_k > |x_k - x_{k+1}| \geq dist(x_k, \partial\Omega_{k+1})$.

The next step is to apply Jerison's Lemma 7.2 (compare with Theorem 2.4.1) to the function $u(x) - \lambda_{k+1}$ on the set Ω_{k+1}. For completeness, we include the proof of this lemma at the end of the argument.

Lemma 2.4.3 *(Lemma 7.2 in [5]) Let E be an open convex set and suppose $u \in C(\bar{E})$ is convex and is 0 on ∂E. Then there exists a dimensional constant C such that*

$$|u(x)|^n \leq C\delta(x, E)|E|Mu(E).$$

This gives

$$|u(x_k) - \lambda_{k+1}|^n \leq C\delta(x_k, \Omega_{k+1})|\Omega_{k+1}|Mu(\Omega_{k+1})$$

since $u(x_k) = \lambda_k$, $u(x_k) - \lambda_{k+1} = -s_{k+1}$. Then we see that

$$s_{k+1}^n \leq C\delta(x_k, \Omega_{k+1})|\Omega_{k+1}|Mu(\Omega_{k+1}). \tag{2.2}$$

Note that Ω_{k+1} is convex. It is a section of u of height $|\min u| + \lambda_{k+1}$ with base point at the minimum of u with slope 0. Let L now be a shortest segment from x_k to $\partial\Omega_{k+1}$ and let z be the endpoint on $\partial\Omega_{k+1}$. Since Ω_{k+1} is convex, the hyperplane Π through z and normal to L is a support plane for Ω_{k+1}. Let $\rho = |L| = |x_k - z|$. Let Π' be the support plane parallel to Π on the opposite side of Ω_{k+1} (so that Ω_{k+1} is contained in the slab between Π and Π') and let $r = dist(\Pi, \Pi')$. Then since $\Omega_{k+1} \subset B_1(0)$, there exists a dimensional constant such that $|\Omega_{k+1}| \leq Cr$.

Let T be an affine transformation normalizing Ω_{k+1}. Then $dist(T(\Pi), T(\Pi'))$ ≈ 1 and, by a similar triangle argument, $dist(T(x_k), T(\Pi)) \approx \frac{\rho}{r}$.

On the other hand, by Lemma 2.3.3,

$$\delta(x_k, \Omega_{k+1}) = \delta(T(x_k), T(\Omega_{k+1})) \leq C dist(T(x_k), T(\Omega_{k+1}))$$

$$\leq C dist(T(x_k), T(\Pi)) \leq C\frac{\rho}{r}.$$

Inserting this into (2.2), we get

$$s_{k+1}^n \leq C\frac{\rho}{r}|\Omega_{k+1}|Mu(\Omega_{k+1}) \leq C\rho Mu(\Omega_{k+1}) < C\delta_{k+1}Mu(\Omega_{k+1}).$$

By the choice of k, $\delta_{k+1} \leq \delta_k \leq 2^{-k}\delta_0 \leq C2^{-k}\delta(x_0, \Omega)$, where $\delta_0 = dist(\Omega_0, \partial\Omega) \leq dist(x_0, \partial\Omega)$. Therefore,

$$\delta_{k+1}Mu(\Omega_{k+1}) = \delta_{k+1}^{\epsilon}\delta_{k+1}^{1-\epsilon}\int_{\Omega_{k+1}} dMu(x) = \delta_{k+1}^{\epsilon}\int_{\Omega_{k+1}} \delta_{k+1}^{1-\epsilon}dMu(x). \quad (2.3)$$

For every $x \in \Omega_{k+1}$, we have that $\delta(x, \Omega) \geq C dist(x, \partial\Omega) \geq C dist(\Omega_{k+1}, \partial\Omega) = C\delta_{k+1}$, so that

$$(2.3) \leq \delta_{k+1}^{\epsilon}\int_{\Omega_{k+1}} C\delta(x, \Omega)^{1-\epsilon}dMu(x)$$

$$\leq C2^{-k\epsilon}\delta(x_0, \Omega)^{\epsilon}\int_{\Omega_{k+1}} \delta(x, \Omega)^{1-\epsilon}dMu(x) = C2^{-k\epsilon}A.$$

Hence $s_{k+1}^n = s^n 2^{-n(k+1)\beta} \leq C2^{-k\epsilon}A$. Since $\beta n \leq \epsilon$, we get that $s^n 2^{-\epsilon(k+1)} \leq C2^{-k\epsilon}A$, implying that $s^n \leq CA$, where C depends on ϵ. This completes the proof of the theorem. □

Proof of Lemma 2.4.3 Let T normalize E, and let $v(y) = u(T^{-1}y)$. By Theorem 2.4.1 applied to v in $T(E)$ and using Lemma 2.3.3 and (2.1),

$$\begin{aligned} |u(x)|^n &= |v(Tx)|^n \leq C_n(diam T(E))^{n-1}dist(Tx, \partial T(E))Mv(T(E)) \\ &\leq C_n dist(Tx, \partial T(E))Mv(T(E)) \leq C_n\delta(Tx, T(E))Mv(T(E)) \\ &= C_n\delta(x, E)Mv(T(E)) = C_n\delta(x, E)|\det T^{-1}|Mu(E) \\ &\leq C_n\delta(x, E)|E|Mu(E), \end{aligned}$$

where the last inequality holds because $|\det T^{-1}| = \frac{|E|}{|T(E)|} \leq C|E|$. □

There is an example (see [5], pp. 44-45) showing that if $\epsilon = 0$, there is no way to estimate the size of u at a point in terms of its distance to the boundary. This is the reason for requiring that the exponent appearing in the definition of D_ϵ is strictly less than 1.

To conclude this section, we prove a useful relationship between the minimum of a function u and the integral of the normalized distance over the domain that holds under certain conditions. This is the analog of Proposition 3.2.3 in [3].

Proposition 2.4.4 *Let Ω be open convex and normalized, $u \in C(\bar{\Omega})$, convex and $u|_{\partial\Omega} = 0$. Suppose $\int_\Omega \delta(x,\Omega)^{1-\epsilon} dMu \le CMu(\frac{1}{2}\Omega)$. Then there exist two constants $C_1 = C_1(n,\epsilon)$ and $C_2 = C_2(n,\epsilon,C)$ such that*

$$C_1 |\min_\Omega u|^n \le \int_\Omega \delta(x,\Omega)^{1-\epsilon} dMu \le C_2 |\min_\Omega u|^n.$$

Proof The first inequality follows directly from Theorem 2.4.2 and the fact that $\delta(x,\Omega) \le 1$. In fact, for this inequality the hypothesis concerning the integral is not needed. For the second inequality,

$$\int_\Omega \delta(x,\Omega)^{1-\epsilon} dMu \le CMu(\frac{1}{2}\Omega) \le \bar{C} |\min_\Omega u|^n$$

where the second inequality is a consequence of the following lemma and the fact that since Ω is normalized, $dist(\frac{1}{2}\Omega, \partial\Omega) \ge C(n)$. □

Lemma 2.4.5 *(Lemma 3.2.1 in [3]) Let $\Omega \subset \mathbb{R}^n$ be a bounded convex and open, and ϕ a convex function in Ω such that $\phi \le 0$ on $\partial\Omega$. If $x \in \Omega$ and $l(y) = \phi(x) + p \cdot (y - x)$ is a supporting hyperplane to ϕ at the point $(x, \phi(x))$ then*

$$|p| \le \frac{-\phi(x)}{dist(x,\partial\Omega)}.$$

More generally, if $\bar{\Omega}_0 \subset \Omega$, then

$$\partial\phi(\Omega_0) \subset B\left(0, \frac{\max_{\Omega_0}(-\phi)}{dist(\Omega_0,\partial\Omega)}\right).$$

Proof. The claim is clearly true if $p = 0$, so assume $p \neq 0$. We have $\phi(y) \geq \phi(x) + p\cdot(y-x)$ for every $y \in \Omega$. If $0 < r < dist(x, \partial\Omega)$ then $y_0 = x + r\frac{p}{|p|} \in \Omega$ and $0 \geq \phi(y_0) \geq \phi(x) + r|p|$. Hence $|p| \leq \frac{-\phi(x)}{r}$ for any $r \in (0, dist(x, \partial\Omega))$. This proves the first statement. For the second, notice that for any $x \in \Omega_0$, $-\phi(x) \leq \max_{\Omega_0}(-\phi)$ and $dist(x, \partial\Omega) \geq dist(\Omega_0, \partial\Omega)$. □

2.5 Hausdorff Metric

The Hausdorff metric introduces a topology on the set of nonempty compact subsets of $\mathbb{R}^n$. This metric is the proper framework to analyze the convergence of sections of solutions of the Monge-Ampère equation, and plays an important role in the proofs of the regularity results appearing below. The basic definitions and results come from the book [7]. Let K^n denote the set of nonempty compact subsets of $\mathbb{R}^n$.

Definition 2.5.1 *For $K, L \in K^n$, the Hausdorff metric is defined by*

$$d_H(K, L) = \max\{\max_{x\in K}\min_{y\in L}|x-y|, \max_{x\in L}\min_{y\in K}|x-y|\},$$

or equivalently by

$$d_H(K, L) = \min\{\lambda \geq 0 : K \subset L + \lambda B_1(0),\ L \subset K + \lambda B_1(0)\}.$$

We also define the Minkowski support function of a closed convex set K.

Definition 2.5.2 *Let $K \subset \mathbb{R}^n$ be closed, nonempty and convex. The Minkowski support function is the map $h(K, \cdot) : \mathbb{R}^n \to \bar{\mathbb{R}}$ given by*

$$h(K, u) = \sup_{x\in K}\{< x, u >\}.$$

We quote the following result establishing the connection between the Hausdorff metric and the support function.

Theorem 2.5.3 *(Theorem 1.8.11 in [7]) For any convex bodies K and L,*

$$d_H(K, L) = \sup_{u\in S^{n-1}}|h(K, u) - h(L, u)|.$$

We now collect some properties of the Hausdorff metric that will be used later. First we present the following theorem due to Blaschke.

Theorem 2.5.4 ***(Blaschke Selection Theorem)*** *From each bounded sequence of convex bodies, one can select a subsequence converging (in the Hausdorff metric) to a convex body.*

For our purposes, we will not need the full strength of this result, but will need only the following special case of this theorem. We include the proof for this case; the argument is contained in the proof of the Selection Lemma (Lemma 5.3.1) in [3].

Theorem 2.5.5 *Let $\{K_n\}$ be a bounded sequence of convex bodies. Suppose additionally that there is point x and a number $\epsilon > 0$, such that $B_\epsilon(x) \subset K_n$ for all n. Then there is a subsequence K_{n_j} converging to a convex body K.*

Proof of Theorem 2.5.5 Without loss of generality, take $x = 0$. Let $F_j(x)$ be the function whose graph is the cone in $\mathbb{R}^{n+1}$ with vertex $(0, -1)$ and passing through the set $\partial K_j \times \{0\}$. Then $K_j = \{x \in \mathbb{R}^n : F_j(x) \leq 0\}$. Note that $F_j(x)$ is convex.

If $p \in \partial F_j(K_j) = \partial F_j(0)$, then $|p| \leq \frac{1}{\epsilon}$. Let $R > 0$ be such that $K_j \subset B_R(0)$ for all j. Then for any j, $-1 \leq F_j(x) \leq C$ in $B_{2R}(0)$, where $C = C(n, \epsilon)$. This holds because the slope of any ray in a cone emanating from the common vertex is uniformly bounded. Therefore, the functions $\{F_j(x)\}$ are uniformly bounded and equicontinuous in $B_{2R}(0)$. So by Arzelà-Ascoli, there exists a uniformly convergent subsequence, which we also denote F_j, converging to say $F(x)$. Since F is the limit of convex functions, it is convex. Define the set K to be $\{x \in \mathbb{R}^n : F(x) \leq 0\}$. K is convex because of the convexity of F.

We now show that $K_j \to K$ in the Hausdorff metric. We first demonstrate that given ϵ, $K_j \subset K + \epsilon B_1(0)$, for all j sufficiently large. If this is not true, there exists an $\epsilon > 0$ and a subsequence K_{j_k} such that $K_{j_k} \not\subset K + \epsilon B_1(0)$. In other words, there is a point $x_{j_k} \in K_{j_k}$ such that $x_{j_k} \notin K + \epsilon B_1(0)$. This will mean that $B_{\frac{\epsilon}{2}}(x_{j_k}) \cap K = \emptyset$. Since $\{x_{j_k}\}$ is bounded we may assume that $x_{j_k} \to \bar{x}$. Then:

$$|F_{j_k}(x_{j_k}) - F(\bar{x})| \leq |F_{j_k}(x_{j_k}) - F(x_{j_k})| + |F(x_{j_k}) - F(\bar{x})|.$$

Both of the terms on the right go to 0 (by uniform convergence and by the continuity of F respectively). Hence $F_{j_k}(x_{j_k}) \to F(\bar{x})$, but since $x_{j_k} \in K_{j_k}$, $F_{j_k}(x_{j_k}) \leq 0$, implying that $F(\bar{x}) \leq 0$, so $\bar{x} \in K$, but $B_{\frac{\epsilon}{2}}(x_{j_k}) \cap K = \emptyset$. This is a contradiction.

The next step is to show that $K \subset K_j + \epsilon B_1(0)$ for all $j \geq J(\epsilon)$. Let $x \in K$. Then $F(x) \leq 0$. We need to show that if $j \geq J(\epsilon)$, there exists $x_j \in K_j$ for which $|x_j - x| < \epsilon$. If not, there exists $\epsilon > 0$ and a subsequence for which $K_{j_k} \cap B_\epsilon(x) = \emptyset$. This implies that $F_{j_k}|_{B_{\frac{\epsilon}{2}}(x)} > 0$. In particular, $F_{j_k}(x) > 0$, but $F_{j_k}(x) \to F(x)$. The only way this is possible (since $x \in K$) is if $F(x) = 0$, i.e. $x \in \partial K$. Now choose a sequence $\{y_k\}$ converging to x, such that $y_k \in B_{\frac{\epsilon}{2}}(x) \cap int(K)$. Then $F(y_k) < 0$ so we must have $F_{j_k}(y_k) < 0$ for k sufficiently large, but $F_{j_k}|_{B_{\frac{\epsilon}{2}}(x)} > 0$. This is a contradiction, and therefore, for all $\epsilon > 0$, there is $J = J(\epsilon)$ such that $K \subset K_j + \epsilon B_1(0)$ for all $j \geq J$. □

The next lemma connects convergence in the Hausdorff metric with pointwise convergence of characteristic functions.

Lemma 2.5.6 *Suppose $\{K_n\}$, a sequence of compact convex sets converge to K in the Hausdorff metric. Then $\chi_{K_n}(x) \to \chi_K(x)$ pointwise for every $x \notin \partial K$.*

Proof of Lemma 2.5.6 By Theorem 2.5.4, K is compact and convex. First suppose $x \notin K$. Then $dist(x, K) = \rho > 0$. Hence $x \notin K + \frac{\rho}{2}B_1(0)$. By the convergence of K_n to K, this means that $x \notin K_n$ for all n sufficiently large. Therefore, $\chi_{K_n}(x) = 0 = \chi_K(x)$ for all $n \geq N$.

Now suppose $x \in int(K)$. Let $0 < \rho < dist(x, \partial K)$. Then $B_\rho(x) \subset K$. Change the coordinates by a translation so that in the new coordinates $x = 0$. From now on K_n and K represent the translations of the original sets in question. For any $v \in S^{n-1}$, K is contained between the parallel planes

$$\{z \in \mathbb{R}^n :< z, v >= h(K, v)\} \text{ and } \{z \in \mathbb{R}^n :< z, v >= h(K, -v)\}.$$

Since $B_\rho(0) \subset K$, $h(K, v) \geq \rho$ for all $v \in S^{n-1}$. For $n \geq N(\rho)$, we have that $|h(u, K_n) - h(u, K)| \leq \frac{\rho}{2}$ for all $u \in S^{n-1}$. Therefore, $h(u, K_n) \geq \frac{\rho}{2}$ for any unit vector u, so that $B_{\frac{\rho}{2}}(0) \subset K_n$ for any n large enough. This shows that $0 \in K_n$ for such n, and translating back to the original coordinates, we get that $x \in K_n$ for all $n \geq N$ for some N, proving the claim. □

We now prove if a sequence of convex bodies $\{K_n\}$ converges to K, then the sequence $\{\frac{1}{2}K_n\}$ converges to $\frac{1}{2}K$.

Lemma 2.5.7 *If $K_n \to K$ in the Hausdorff metric, then $\frac{1}{2}K_n \to \frac{1}{2}K$, where $\frac{1}{2}K_n$ is the dilation of K_n (respectively $\frac{1}{2}K$) with respect to its center of mass $c(K_n)$.*

Proof We have the following formulas:

$$\frac{1}{2}K_n = \{\frac{1}{2}(c(K_n) + y) : y \in K_n\} \text{ and}$$

$$[c(K_n)]_i = \frac{1}{|K_n|}\int_{K_n} x_i \, dx.$$

By Theorem 1.8.16 in [7], the volume map is continuous in the Hausdorff metric. In other words, if $S_n \to S$, then $|S_n| \to |S|$. We now show that the center of mass map is also continuous with respect to the Hausdorff metric on the class of convex bodies. To prove this we need to demonstrate that $\int_{K_n} x_i \, dx \to \int_K x_i \, dx$. Since $\{K_n\}$ converges, there exists an $R > 0$ such that $K, K_n \subset B_R(0)$ for all n. Then:

$$\left|\int_{K_n} x_i dx - \int_K x_i dx\right| \leq \int_{K_n \cap K^c} |x_i| dx + \int_{K \cap K_n^c} |x_i| dx$$

$$\leq R(|K_n \cap K^c| + |K \cap K_n^c|) \to 0$$

as $n \to \infty$, since $K_n \to K$, so by Lemma 2.5.6, $\chi_{K_n} \to \chi_K$ almost everywhere.

Now let $\epsilon > 0$. Then there exists N such that $n \geq N$ implies that $K \subset K_n + \epsilon B_1$, $K_n \subset K + \epsilon B_1$, and $|c(K_n) - c(K)| \leq \epsilon$. We need to show that there exists N' such that if $n \geq N'$, the following two inclusions hold: $\frac{1}{2}K \subset \frac{1}{2}K_n + \epsilon B_1$ and $\frac{1}{2}K_n \subset \frac{1}{2}K + \epsilon B_1$.

Let $x \in \frac{1}{2}K$. Then $x = \frac{1}{2}(c(K) + y)$ for some $y \in K$. We want to show that for n sufficiently large, $x \in \frac{1}{2}K_n + \epsilon B_1$. In other words, we need to find a point $z_n \in \frac{1}{2}K_n$ such that $|x - z_n| \leq \epsilon$.

For each $n \geq N$, there is a point $\bar{z}_n \in K_n$ such that $|\bar{z}_n - y| \leq \epsilon$. Let $z_n = \frac{1}{2}(c(K_n) + \bar{z}_n) \in \frac{1}{2}K_n$. Then:

$$|x - z_n| = |\frac{1}{2}(c(K) + y) - \frac{1}{2}(c(K_n) + \bar{z}_n)| \leq \frac{1}{2}(|c(K) - c(K_n)| + |y - \bar{z}_n|) \leq \epsilon.$$

The other inclusion is proved by contradiction. If the claim is not true, then there exists $\epsilon > 0$ and a subsequence K_{n_j} such that $\frac{1}{2}K_{n_j} \not\subset \frac{1}{2}K + \epsilon B_1(0)$. This means that we can find a sequence of points $\{x_{n_j}\}$ such that $x_{n_j} \in \frac{1}{2}K_{n_j}$ and $dist(x_{n_j}, \frac{1}{2}K) > \epsilon$ for all points in the subsequence. Write $x_{n_j} = \frac{1}{2}(c(K_{n_j}) + \bar{x}_{n_j})$ where $\bar{x}_{n_j} \in K_{n_j}$. By passing to another subsequence, we can assume that $\bar{x}_{n_j} \to \bar{x}$. Then by letting $j \to \infty$, we see that x_{n_j} also approaches a limit, namely $\frac{1}{2}(c(K) + \bar{x})$. Theorem 1.8.7 (b) in [7] states that if $K_n \to K$ and $\{x_{n_j}\}$ converges to x, where $x_{n_j} \in K_{n_j}$, then $x \in K$. This theorem implies that $\bar{x} \in K$ and therefore $\{x_{n_j}\}$ converges to a point in $\frac{1}{2}K$. However, this is impossible if $dist(x_{n_j}, \frac{1}{2}K) > \epsilon$. □

The last two results in this section concern the convergence of the normalized distances to the boundaries of convex sets converging in the Hausdorff sense, and the continuity of sections in the parameter t respectively.

Lemma 2.5.8 *Let $\{S_j\}$ be a sequence of convex bodies converging in the Hausdorff metric to the convex body S. Then for every $x \in S$, $\delta(x, S_j) \to \delta(x, S)$. In fact, the functions $f_j(x) = \delta(x, S_j)$ converge uniformly to $f(x) = \delta(x, S)$ on compact subsets of S.*

Proof Let $x \in S$. Then $dist(x, \partial S) = \rho > 0$. Since $S_j \to S$, Lemma 2.5.6 implies that $x \in S_j$ for all j sufficiently large (depending on ρ). Then for these j, $\delta(x, S_j)$ is defined. Let l be any line through x. Let x_1 and x_2 be the endpoints of the segment $l \cap \bar{S}$. Let x_1^j and x_2^j be the endpoints of the segment $l \cap \bar{S}_j$, with x_1^j being in the same ray (emanating from x) as x_1. We make the claim that $x_1^j \to x_1$ and $x_2^j \to x_2$ "uniformly" in the sense that this convergence does not depend on l.

Then:

$$\frac{|x_1^j - x|}{|x_2^j - x|} \to \frac{|x_1 - x|}{|x_2 - x|}$$

and this implies that $\delta(x, S_j) \to \delta(x, S)$.

We prove the claim by considering two cases. First, consider the case that $|x_1^j - x| > |x_1 - x|$. Let u be the unit vector from x along l, pointing in the direction of x_1. Let Π be a support plane to S at x_1; let v be its unit normal (away from S). Let Π_j be the plane parallel to Π that supports S_j at some point. Let $R_j = dist(\Pi, \Pi_j)$. Let θ be the angle between u and v. Then $|x_1^j - x_1| \leq R_j \sec\theta$. Construct a right triangle with one vertex at x. The angle at x is θ, and the sides intersecting at x are given by the rays emanating from x with directions u and v. The second vertex A is the point in ∂S where the ray starting at x in the direction v hits ∂S. The third vertex is the point B lying in the ray from x with direction u that lies in a plane parallel to Π through A. Then

$$\cos\theta = \frac{|x - A|}{|x - B|} \geq \frac{\rho}{diam(S)} > 0.$$

Hence, there is a number M for which $\sec\theta < M$. Therefore, $|x_1^j - x_1| \leq MR_j$. By Theorem 2.5.3, $R_j \leq d_H(S_j, S) \to 0$. This is because the number R_j is $h(v, S_j) - h(v, S)$, where $h(\cdot, \cdot)$ is the Minkowski support function. This shows that $x_1^j \to x_1$ at least for those j such that $|x_1^j - x| > |x_1 - x|$.

We now consider the other possibility, that $|x_1 - x| > |x_1^j - x|$. As before, let u be a unit vector from x along l pointing in the direction of x_1. Let Π_j be a support plane to S_j at x_1^j and let v_j be its unit normal (pointing away from S_j). Let Π_j' be parallel to Π_j and support S at some point. Denote by R_j the distance between the parallel planes Π_j and Π_j'. Let θ_j be the angle between

u and v_j. Then $|x_1^j - x_1| \leq R_j \sec\theta_j$. For each j we construct a right triangle with vertex x and sides intersecting at x given by the rays starting at x with directions u and v_j. The second vertex of the triangle, A_j lies at the end of the side with direction v_j and is in the boundary of S_j. The third vertex, B_j, is found by intersecting the side with direction u with the plane parallel to Π_j that passes through A_j. Then

$$\cos\theta_j = \frac{|x - A_j|}{|x - B_j|} \geq \frac{\rho/2}{diam(S)} > 0.$$

Then, as in the first case, $x_1^j \to x_1$ for those j for which $|x_1 - x| > |x_1^j - x|$.

For both of the cases considered, the same arguments show the corresponding result for x_2. Combining the two cases, we get that $x_1^j \to x_1$ and $x_2^j \to x_2$. Notice that the convergence does not depend on the particular line l. This allows us to conclude that $\delta(x, S_j) \to \delta(x, S)$ pointwise for every $x \in int(S)$. The claim about uniform convergence on compact subsets follows since the only property of the point x needed in the above argument was its distance from the boundary of S. Therefore, if $K \Subset S$, $dist(x, \partial S) \geq \rho > 0$ for all $x \in K$ for some ρ. Then we can apply the above argument to deduce the uniform convergence. □

Lemma 2.5.9 *Let u be a convex function defined on a domain Ω. Let $S = S(x_0, p, t)$ be a section of u, such that $S \Subset \Omega$. Let $\rho > 0$. Denote by S^ρ the section $S(x_0, p, t - \rho)$, and the section $S(x_0, p, t + \rho)$ by S_ρ. Then*

$$\lim_{\rho \to 0} S^\rho = \lim_{\rho \to 0} S_\rho = S,$$

where the limits are in the Hausdorff metric.

Proof

Let $l(x) = u(x_0) + p \cdot (x - x_0) + t$ be the affine function defining S, i.e. $S = \{x \in \Omega : u(x) < l(x)\}$. First we show that $\lim_{\rho \to 0} S_\rho = S$. For every ρ, $S \subset S_\rho$, so we only need to prove that for every $\rho_0 > 0$,

$$S_\rho \subset S + \rho_0 B_1(0)$$

for all ρ sufficiently small. If this is not true, then there exists $\rho_0 > 0$ such that for each $n \in \mathbb{N}$, there exists $\epsilon_n < \frac{1}{n}$ and a point $x_n \in \partial S_{\epsilon_n} \cap (S + \rho_0 B_1(0))^c$. This implies that $dist(x_n, \partial S) > \rho_0$. But this leads to a contradiction. Choose a subsequence $x_{n_j} \to \bar{x}$. Then:

$$\begin{array}{ccc} u(x_{n_j}) & = & u(x_0) + p \cdot (x_{n_j} - x_0) + t + \epsilon_{n_j} \\ \downarrow & & \downarrow \\ u(\bar{x}) & = & u(x_0) + p \cdot (\bar{x} - x_0) + t. \end{array}$$

This means that $\bar{x} \in \partial S$, but $dist(\bar{x}, \partial S) > \rho_0$. This is a contradiction.

Now we prove that $S^\rho \to S$, as $\rho \to 0$. Since $S^\rho \subset S$ for all ρ, we only need to show that for all $\rho_0 > 0$,

$$S \subset S^\rho + \rho_0 B_1(0)$$

for all ρ sufficiently small. Again, the proof of this inclusion is by contradiction. If this does not hold, there exists a $\rho_0 > 0$ such that for all n, there exists $\epsilon_n < \frac{1}{n}$ such that $S \not\subset S^{\epsilon_n} + \rho_0 B_1(0)$. In other words, there is a point $x_n \in S$ such that $x_n \notin S^{\epsilon_n} + \rho_0 B_1(0)$, meaning that $dist(x_n, \partial S^{\epsilon_n}) > \rho_0$. Then we can choose a subsequence $x_{n_j} \to \bar{x} \in \bar{S}$. Each $x_{n_j} \notin S^{\epsilon_{n_j}}$, so

$$l(x_{n_j}) - \epsilon_{n_j} \leq u_0(x_{n_j}) < l(x_{n_j}).$$

Let $j \to \infty$ to conclude that $l(\bar{x}) \leq u_0(\bar{x}) \leq l(\bar{x})$. This means that $\bar{x} \in \partial S$. We have the inequality:

$$dist(\bar{x}, \partial S^{\epsilon_{n_j}}) \leq |\bar{x} - x_{n_j}| + dist(x_{n_j}, \partial S^{\epsilon_{n_j}}) \text{ and}$$

This implies that $|dist(\bar{x}, \partial S^{\epsilon_{n_j}}) - dist(x_{n_j}, \partial S^{\epsilon_{n_j}})| \leq |\bar{x} - x_{n_j}|$. We also have that $|\bar{x} - x_{n_j}| \to 0$ and $dist(x_{n_j}, \partial S^{\epsilon_{n_j}}) \geq \rho_0$. This implies that $dist(\bar{x}, \partial S^{\epsilon_{n_j}}) \geq \frac{\rho_0}{2}$ for all j large enough. Therefore, $B_{\frac{\rho_0}{2}}(\bar{x}) \cap S^{\epsilon_{n_j}} = \emptyset$ for large j. However, since $\bar{x} \in \partial S$, there is a point $z \in B_{\frac{\rho_0}{2}}(\bar{x})$ such that $z \in S$. This means that $u_0(z) < l(z)$, so $u_0(z) < l(z) - \epsilon_{n_j}$ for j large enough, implying that $z \in S^{\epsilon_{n_j}}$. This is a contradiction. □

CHAPTER 3

REGULARITY PROPERTIES

3.1 Comparison of D_1 and D_ϵ

The following theorem shows that the two conditions, D_ϵ and doubling, are equivalent for convex functions defined on the whole of $\mathbb{R}^n$. The converse of this theorem is a trivial consequence of the fact that $\delta(\cdot,\cdot) \leq 1$.

Theorem 3.1.1 *If $u : \mathbb{R}^n \to \mathbb{R}$ is convex and $Mu \in D_\epsilon$ for some $\epsilon \in (0,1)$, then Mu is doubling.*

The proof of this result uses the characterization of doubling Monge-Ampère measures on $\mathbb{R}^n$ due to Gutiérrez and Huang:

Theorem 3.1.2 *(Theorem 3.3.5 in [3]) $Mu = \mu$ is doubling on $\mathbb{R}^n$ if and only if there exist constants $0 < \tau, \lambda < 1$, such that for all x_0 and $t > 0$, $S(x_0, p, \tau t) \subset \lambda S(x_0, p, t)$.*

Proof of Theorem 3.1.1 : Let $S = S_u(x_0, p, t)$ be any section of u. Let T be an affine transformation that normalizes S, $Tx = Ax + b$ for an invertible matrix A, and denote $T(S)$ by S^*. Define $v(x) = u(T^{-1}x)$. Then $T(S_u(x_0, p, \lambda t)) = S_v(Tx_0, q, \lambda t)$ for any $\lambda > 0$, where $q = (A^{-1})^t p$. We also have $T(\lambda S) = \lambda T(S) = \lambda S^*$.

Let $v^*(x) = v(x) - v(Tx_0) - q\cdot(x - Tx_0) - t$. Then $\partial v^* = \partial v - q$, and by the translation invariance of Lebesgue measure $Mv^* = Mv$. Also, $v^*|_{\partial S^*} = 0$. Let $y \in S^*\backslash\lambda S^*$, where $\lambda < 1$ is to be chosen, close to 1.

Since S^* is normalized, $dist(y, \partial S^*) \leq (1-\lambda)$ implies that $\delta(y, S^*) \leq C_n(1-\lambda)$. Then by Theorem 2.4.2 and Proposition 2.4.4:

$$\begin{aligned} |v^*(y)|^n &\leq C\delta(y,S^*)^\epsilon \int_{S^*} \delta(x,S^*)^{1-\epsilon} dMv^* \\ &\leq C(1-\lambda)^\epsilon |\min_{S^*} v^*|^n \\ &= C(1-\lambda)^\epsilon t^n. \end{aligned}$$

Therefore $v^*(y) \geq -C(1-\lambda)^{\frac{\epsilon}{n}} t$, meaning that

$$v(y) - v(Tx_0) - q\cdot(y - Tx_0) \geq (1 - C(1-\lambda)^{\frac{\epsilon}{n}})t.$$

Choose λ so that the term on the right hand side is positive. Then choose $0 < \tau < 1 - C(1-\lambda)^{\frac{\epsilon}{n}}$. Therefore,

$$v(y) \geq v(Tx_0) + q\cdot(y - Tx_0) + \tau t,$$

so $y \notin S_v(Tx_0, q, \tau t)$. This implies that $S_v(Tx_0, q, \tau t) \subset \lambda S^*$. By applying T^{-1}, we get the inclusion $S_u(x_0, p, \tau t) \subset \lambda S_u(x_0, p, t)$. So by Theorem 3.1.2, Mu is doubling. □

For functions defined only on bounded domains in $\mathbb{R}^n$, the two conditions are not equivalent. A simple example of a function whose Monge-Ampère measure is $D_{\frac{1}{2}}$ but not doubling on sections is presented next.

Example Define the function $u : [0,1] \to \mathbb{R}$ by

$$u(x) = \begin{cases} x\log x & 0 < x \leq 1, \\ 0 & x = 0. \end{cases}$$

This function is continuous on $[0,1]$, convex and satisfies $Mu = \frac{1}{x}\,dx$.

Since u is zero on the boundary (i.e. $u(0) = u(1) = 0$), the interval $(0,1)$ is a section. By considering this section, we see that the measure cannot be doubling, because

$$\begin{aligned} Mu((0,1)) &= \int_0^1 \frac{dx}{x} = \infty, \text{ but} \\ Mu(\frac{1}{2}(0,1)) &= \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{dx}{x} = \ln 3 < \infty. \end{aligned}$$

This means that there is no constant C for which $Mu((0,1)) \le CMu(\frac{1}{2}(0,1))$. Therefore, the measure is not doubling on sections.

$D_{\frac{1}{2}}$ means that for any section $(a,b) \subset (0,1)$,

$$\int_a^b \delta(x,[a,b])^{\frac{1}{2}} \frac{dx}{x} \le CMu(\frac{1}{2}(a,b)) = C \int_{\frac{3a+b}{4}}^{\frac{3b+a}{4}} \frac{dx}{x} = C \ln\left(\frac{3b+a}{3a+b}\right).$$

Here

$$\delta(x,[a,b]) = \begin{cases} \frac{x-a}{b-x} & a < x \le \frac{a+b}{2}, \\ \\ \frac{b-x}{x-a} & \frac{a+b}{2} < x < b. \end{cases}$$

To show that $Mu \in D_{\frac{1}{2}}$, four cases will be considered:

1. $a, b > C_1 > 0$,
2. $0 < a < C_1$, $\quad b > C_2 a$, where $C_2 > \frac{3}{2}$,
3. $0 < a < C_1$, $\quad b \le C_2 a$,
4. $a = 0$.

Case 1. When $a, b > C_1$, $1 \le \frac{1}{x} < \frac{1}{C_1}$. On sets of this form Mu is comparable to Lebesgue measure, so Mu is doubling and hence satisfies $D_{\frac{1}{2}}$.

Case 2. We write

$$\int_a^b \delta(x,[a,b])^{\frac{1}{2}} \frac{dx}{x} = \int_a^{\frac{a+b}{2}} \frac{\sqrt{x-a}}{x\sqrt{b-x}}\, dx + \int_{\frac{a+b}{2}}^b \frac{\sqrt{b-x}}{x\sqrt{x-a}}\, dx = I + II.$$

We consider integral I first:

$$I \le \int_a^{\frac{a+b}{2}} \frac{dx}{\sqrt{x}\sqrt{b-x}} \le \frac{\sqrt{2}}{\sqrt{b-a}} \int_a^{\frac{a+b}{2}} \frac{dx}{\sqrt{x}} = \frac{2\sqrt{2}}{\sqrt{b-a}}\left(\sqrt{\frac{a+b}{2}} - \sqrt{a}\right)$$

$$\leq \frac{2\sqrt{2}}{\sqrt{b-a}}\left(\sqrt{\frac{a+b}{2}}\right) = 2\sqrt{\frac{a+b}{b-a}} \leq 2\sqrt{\frac{2b}{b(1-C_2^{-1})}} = 2\sqrt{\frac{2}{1-C_2^{-1}}}.$$

We now turn to II:

$$II \leq \sqrt{\frac{2}{b-a}}\int_{\frac{a+b}{2}}^{b} \frac{\sqrt{b-x}}{x}\,dx \leq \frac{2\sqrt{2}}{(a+b)\sqrt{b-a}}\int_{\frac{a+b}{2}}^{b} \sqrt{b-x}\,dx$$

$$\leq \frac{2\sqrt{2}}{(a+b)\sqrt{b-a}}\sqrt{\frac{b-a}{2}}\left(\frac{b-a}{2}\right) = \frac{b-a}{a+b} \leq 1.$$

Therefore, $I + II \leq 2\sqrt{\frac{2}{(1-C_2^{-1})}} + 1$.

The right-hand side for the $D_{\frac{1}{2}}$ condition is

$$\ln\left(\frac{3b+a}{3a+b}\right) \geq \ln\left(\frac{3b}{b(\frac{3}{C_2}+1)}\right) = \ln\left(\frac{3}{\frac{3}{C_2}+1}\right).$$

Choose C such that $1+2\sqrt{\frac{2}{(1-C_2^{-1})}} \leq C\ln\left(\frac{3}{\frac{3}{C_2}+1}\right)$. For the term on the right to be positive (so that the choice of C is possible), C_2 must be $> \frac{3}{2}$.

Case 3. Let $b = a+h$, where $h \leq (C_2-1)a$. We want to show that the following inequality holds:

$$\int_a^{a+\frac{h}{2}} \frac{\sqrt{x-a}}{x\sqrt{b-x}}\,dx + \int_{a+\frac{h}{2}}^{b} \frac{\sqrt{b-x}}{x\sqrt{x-a}}\,dx = I + II$$

$$\leq C\ln\left(\frac{3b+a}{3a+b}\right) = C\ln\left(\frac{4a+3h}{4a+h}\right). \tag{3.1}$$

Since $II \leq I$ (see below), it is enough to prove the estimate for I. We have that:

$$I \leq \int_a^{a+\frac{h}{2}} \frac{1}{x}\,dx = \ln\left(\frac{a+\frac{h}{2}}{a}\right) = \ln\left(1+\frac{h}{2a}\right) \leq \frac{h}{2a}.$$

For the right-hand side of (3.1),

$$\ln\left(\frac{4a+3h}{4a+h}\right) = \ln\left(1+\frac{2h}{4a+h}\right) \geq \tilde{C}\left(\frac{2h}{4a+h}\right), \tag{3.2}$$

where $\tilde{C} < 1$ is chosen so that $\ln(1+x) \geq \tilde{C}x$, for all $x \in [0, \frac{C_2-1}{2}]$. Then the last inequality in (3.2) holds since $\frac{2h}{4a+h} \leq \frac{2(C_2-1)a}{4a} = \frac{C_2-1}{2}$. The claim will hold if $\tilde{C}$ can be chosen such that:

$$\frac{h}{2a} \leq \tilde{C}\frac{2h}{4a+h} \text{ or equivalently } \tilde{C}^{-1} \leq \frac{4a}{4a+h}.$$

Now

$$\frac{4a}{4a+h} \geq \frac{4a}{(3+C_2)a} = \frac{4}{3+C_2}.$$

Hence, take $\bar{C} = \frac{4}{3+C_2}$. Then

$$I \leq \frac{h}{2a} \leq \bar{C}\frac{2h}{4a+h} \leq \frac{C}{\bar{C}} \ln\left(\frac{4a+3h}{4a+h}\right) = CMu\left(\frac{1}{2}(a,b)\right).$$

Case 4. When $a = 0$, the right-hand side is $\ln 3$. Therefore we just need to dominate the left-hand side by a constant. The left-hand side is

$$\int_0^{\frac{b}{2}} \frac{\sqrt{x}}{x\sqrt{b-x}}\, dx + \int_{\frac{b}{2}}^{b} \frac{\sqrt{b-x}}{x\sqrt{x}}\, dx = I + II.$$

Note: Again $II \leq I$, so it is enough to estimate I. This is done as follows:

$$I \leq \sqrt{\frac{2}{b}} \int_0^{\frac{b}{2}} \frac{1}{\sqrt{x}}\, dx = 2.$$

Therefore, the desired inequality also holds in Case 4.

We now prove that $II \leq I$. For any interval, $(a,b) \subset (0,1)$,

$$\int_a^b \delta(x,[a,b])^{\frac{1}{2}} \frac{dx}{x} = \int_a^{\frac{a+b}{2}} \delta(x,[a,b])^{\frac{1}{2}} \frac{dx}{x} + \int_{\frac{a+b}{2}}^{b} \delta(x,[a,b])^{\frac{1}{2}} \frac{dx}{x} = I + II.$$

Then

$$I \geq \frac{2}{a+b} \int_a^{\frac{a+b}{2}} \delta(x,[a,b])^{\frac{1}{2}}\, dx = I', \text{ and}$$

$$II \leq \frac{2}{a+b} \int_{\frac{a+b}{2}}^{b} \delta(x,[a,b])^{\frac{1}{2}}\, dx = II'.$$

Now notice that $I' = II'$ since in this setting δ has the symmetry property: $\delta(x,[a,b]) = \delta(\bar{x},[a,b])$, where if $x = a+\epsilon$, then $\bar{x} = b - \epsilon$, for $0 < \epsilon < \frac{b-a}{2}$.

Since the four cases considered above exhaust all possibilities for sections, this completes the proof that $Mu \in D_{\frac{1}{2}}$, but is not doubling. □

3.2 Extremal Points and Strict Convexity

This section contains the analog for D_ϵ of Caffarelli's theorem concerning extremal points in the case $\lambda \leq Mu \leq \Lambda$. The proof in the case of D_ϵ follows the same outline as in the book of Gutiérrez (Theorem 5.2.1 in [3]). Before stating and proving this theorem, we introduce some terminology and basic facts of convex geometry. For a more detailed description of these topics see [7].

Definition The point $x_0 \in \partial\Gamma$ is an extremal point of $\Gamma \subset \mathbb{R}^n$ if it is not a convex combination of other points in $\bar{\Gamma}$.

Remark Let E be the set of extremal points of Γ, a bounded convex subset of $\mathbb{R}^n$. Then the convex hull of E equals $\bar{\Gamma}$.

Lemma Let $\Gamma \neq \emptyset$ be a closed, convex and bounded subset of $\mathbb{R}^n$. Then the set E of extremal points of Γ is not empty.

Theorem 3.2.1 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex, and let $u \in C(\Omega)$ be convex. Suppose $Mu \in D_\epsilon$. Assume $u \geq 0$ and let $\Gamma = \{x \in \Omega : u(x) = 0\}$. If $\Gamma \neq \emptyset$ and contains more than one point, then Γ has no extremal points in Ω.*

Proof: The proof is by contradiction. Suppose $x_0 \in \Omega$ is an extremal point of Γ. Since $u \geq 0$, Γ is convex. We apply the following result.

Lemma 3.2.2 *(Lemma 5.1.4 in [3])Let x_0 be an extremal point of Ω. Then given $\delta > 0$ there exist a supporting hyperplane $l(x)$ at some point of $\partial\Omega$ (not necessarily x_0), and $\epsilon_0 > 0$ such that*

(a) $\Omega \subset \{x : l(x) \geq 0\}$.

(b) $diam\{x \in \bar{\Omega} : 0 \leq l(x) \leq \epsilon_0\} < \delta$, *and*

(c) $0 \leq l(x_0) < \epsilon_0$.

This result is applied to the the set Γ with $\delta = \rho < \frac{1}{2}dist(x_0, \partial\Omega)$. Let x_1 be the point at which $l(x)$ is a supporting hyperplane.

Define

$$S = \{x \in \Gamma : 0 \leq l(x) \leq \epsilon_0\}.$$
$$\Pi_1 = \{x : l(x) = \epsilon_0\}.$$
$$\Pi_2 = \{x : l(x) = 0\}.$$

For $\epsilon_1 > 0$, define the convex set

$$\Gamma_{\epsilon_1} = \{x \in \Omega : u(x) \leq \epsilon_1(\epsilon_0 - l(x))\}.$$

Notice that $S \subset \Gamma_{\epsilon_1}$ for all $\epsilon_1 > 0$. Therefore $S \subset \bigcap_{\epsilon_1>0} \Gamma_{\epsilon_1}$. We claim that we also have that $\bigcap_{\epsilon_1>0} \Gamma_{\epsilon_1} \subset S$. If $x \in \Gamma_{\epsilon_1}$, the non-negativity of u implies that $l(x) \leq \epsilon_0$, and if $x \in \Gamma_{\epsilon_1}$ for all $\epsilon_1 > 0$, then we must have that $u(x) = 0$, and hence $x \in \Gamma$ and $l(x) \geq 0$. Since $diam(S) < \rho < \frac{1}{2}dist(x_0, \partial\Omega)$ and $x_0 \in S$, $S \subset int(\Omega)$. Because the Γ_{ϵ_1} decrease as $\epsilon_1 \to 0$, and $S = \bigcap_{\epsilon_1>0} \Gamma_{\epsilon_1}$, $\Gamma_{\epsilon_1} \subset int(\Omega)$ for all ϵ_1 sufficiently small.

We point out that Γ_{ϵ_1} is the closure of a section of u in Ω. The set Γ_{ϵ_1} is the set of points in Ω for which the graph of u lies below the graph of the hyperplane $\bar{l}(x) = \epsilon_1(\epsilon_0 - l(x))$. Let η_0 be the largest value of η for which $\{x \in \Omega : u(x) \leq \bar{l}(x) - \eta\}$ is not empty. By continuity there is a point $\xi \in \Omega$ where $u(\xi) = \bar{l}(\xi) - \eta_0$. Then Γ_{ϵ_1} is a section of u with height η_0 with base point ξ. What this amounts to geometrically is lowering the plane $\bar{l}$ until the graph of u lies above it.

Now slide Π_2 in a parallel fashion away from Π_1 until it touches $\partial\Gamma_{\epsilon_1}$ at a point x_{ϵ_1} and let Π_3 denote the resulting plane, i.e.,

$$\Pi_3 = \{x : l(x) = -\rho_{\epsilon_1}\}, \qquad x_{\epsilon_1} \in \Pi_3, \qquad \Gamma_{\epsilon_1} \subset \{x : -\rho_{\epsilon_1} \leq l(x) \leq \epsilon_0\},$$

with $\rho_{\epsilon_1} > 0$.

Let $u_{\epsilon_1}(x) = u(x) - \epsilon_1(\epsilon_0 - l(x))$. This function is convex, and $\inf_{\Gamma_{\epsilon_1}} u_{\epsilon_1} < 0$. This second assertion can be seen from considering the point x_0. We have that $0 \leq l(x_0) < \epsilon_0$, and that $u(x_0) = 0 \leq \epsilon_1(\epsilon_0 - l(x_0))$ for all ϵ_1 so $x_0 \in \Gamma_{\epsilon_1}$ for every ϵ_1. Then $u_{\epsilon_1}(x_0) = 0 - \epsilon_1(\epsilon_0 - l(x_0)) < 0$. The set Γ_{ϵ_1} is a section of u_{ϵ_1} in the set Ω. Note also that $u_{\epsilon_1}|_{\partial\Gamma_{\epsilon_1}} = 0$.

We have that:

$$dist(\Pi_2, \Pi_3) = \rho_{\epsilon_1}, \qquad dist(\Pi_1, \Pi_2) = \epsilon_0, \qquad \text{so } \frac{dist(\Pi_2, \Pi_3)}{dist(\Pi_1, \Pi_2)} = \frac{\rho_{\epsilon_1}}{\epsilon_0}.$$

Since the Γ_{ϵ_1} shrink to S as $\epsilon_1 \to 0$, $\rho_{\epsilon_1} \to 0$ as $\epsilon_1 \to 0$.

Now consider the quantity

$$\frac{|u_{\epsilon_1}(x_1)|}{|\inf_{\Gamma_{\epsilon_1}} u_{\epsilon_1}|}.$$

We have that

$$u_{\epsilon_1}(x_1) = u(x_1) - \epsilon_1(\epsilon_0 - l(x_1)) = 0 - \epsilon_1(\epsilon_0 - 0) = -\epsilon_1\epsilon_0 < 0, \text{ and}$$

$$u_{\epsilon_1}(x) = u(x) - \epsilon_1(\epsilon_0 - l(x)) \geq \inf_{\Gamma_{\epsilon_1}} u - \epsilon_1(\epsilon_0 - l(x))$$

for all $x \in \Gamma_{\epsilon_1}$. Then since $\inf_{\Gamma_{\epsilon_1}} u = 0$, we get that

$$u_{\epsilon_1}(x) \geq -\epsilon_1(\epsilon_0 - l(x)).$$

This implies that

$$\inf_{\Gamma_{\epsilon_1}} u_{\epsilon_1}(x) \geq \inf_{\Gamma_{\epsilon_1}}[-\epsilon_1(\epsilon_0 - l(x))] = -\sup_{\Gamma_{\epsilon_1}} \epsilon_1(\epsilon_0 - l(x)) = -\epsilon_1(\epsilon_0 + \rho_{\epsilon_1}).$$

Therefore,

$$\frac{|u_{\epsilon_1}(x_1)|}{|\inf_{\Gamma_{\epsilon_1}} u_{\epsilon_1}|} \geq \frac{\epsilon_1\epsilon_0}{\epsilon_1(\epsilon_0 + \rho_{\epsilon_1})} = \frac{\epsilon_0}{\epsilon_0 + \rho_{\epsilon_1}} \to 1$$

as $\epsilon_1 \to 0$ since $\rho_{\epsilon_1} \to 0$.

We conclude that $\liminf_{\epsilon_1 \to 0} \frac{|u_{\epsilon_1}(x_1)|}{|\inf_{\Gamma_{\epsilon_1}} u_{\epsilon_1}|} = 1$.

Let T_{ϵ_1} normalize Γ_{ϵ_1} and $u^*_{\epsilon_1}(x) = u_{\epsilon_1}(T_{\epsilon_1}^{-1}x)$. Then $Mu^*_{\epsilon_1} \in D_\epsilon$ and $u^*_{\epsilon_1}$ is zero on $\partial\Gamma^*_{\epsilon_1}$ where $\Gamma^*_{\epsilon_1} = T_{\epsilon_1}(\Gamma_{\epsilon_1})$.

Then, as above,

$$\frac{|u^*_{\epsilon_1}(T_{\epsilon_1}x_1)|}{|\inf_{\Gamma^*_{\epsilon_1}} u^*_{\epsilon_1}|} \geq C_1 > 0$$

for ϵ_1 sufficiently small. This implies (by Proposition 2.4.4) that

$$|u^*_{\epsilon_1}(T_{\epsilon_1}x_1)|^n \geq C_1^n |\inf_{\Gamma^*_{\epsilon_1}} u^*_{\epsilon_1}|^n \geq C \int_{\Gamma^*_{\epsilon_1}} \delta(x, \Gamma^*_{\epsilon_1})^{1-\epsilon} dMu^*_{\epsilon_1}. \tag{3.3}$$

We now explain why Proposition 2.4.4 holds in this case. Using (2.1) we obtain that:

$$Mu^*_{\epsilon_1}(\frac{1}{2}\Gamma^*_{\epsilon_1}) = Mu^*_{\epsilon_1}(\frac{1}{2}T(\Gamma_{\epsilon_1})) = Mu^*_{\epsilon_1}(T(\frac{1}{2}\Gamma_{\epsilon_1}))$$
$$= |\det T^{-1}| Mu_{\epsilon_1}(\frac{1}{2}\Gamma_{\epsilon_1}) = |\det T^{-1}| Mu(\frac{1}{2}\Gamma_{\epsilon_1}).$$

The following is also true (again by (2.1) and Lemma 2.3.3):

$$\int_{\Gamma^*_{\epsilon_1}} \delta(x, \Gamma^*_{\epsilon_1})^{1-\epsilon}\, dMu^*_{\epsilon_1} = \int_{T(\Gamma_{\epsilon_1})} \delta(x, T(\Gamma_{\epsilon_1}))^{1-\epsilon}\, dMu^*_{\epsilon_1}$$

$$= \int_{\Gamma_{\epsilon_1}} \delta(Ty, T(\Gamma_{\epsilon_1}))^{1-\epsilon} |\det T^{-1}|\, dMu_{\epsilon_1} = \int_{\Gamma_{\epsilon_1}} \delta(y, \Gamma_{\epsilon_1})^{1-\epsilon} |\det T^{-1}|\, dMu_{\epsilon_1}$$

$$= \int_{\Gamma_{\epsilon_1}} \delta(y, \Gamma_{\epsilon_1})^{1-\epsilon} |\det T^{-1}|\, dMu.$$

Since Γ_{ϵ_1} is a section of u and $Mu \in D_\epsilon$,

$$\int_{\Gamma_{\epsilon_1}} \delta(y, \Gamma_{\epsilon_1})^{1-\epsilon} dMu \leq CMu(\frac{1}{2}\Gamma_{\epsilon_1}).$$

Therefore, by canceling the factor $|\det T^{-1}|$ we see that we have the inequality

$$\int_{\Gamma^*_{\epsilon_1}} \delta(x, \Gamma^*_{\epsilon_1})^{1-\epsilon}\, dMu^*_{\epsilon_1} \leq CMu^*_{\epsilon_1}(\frac{1}{2}\Gamma^*_{\epsilon_1})$$

which is what we need to apply Proposition 2.4.4, so (3.3) holds.

The next step is to show that $dist(T_{\epsilon_1}x_1, \partial\Gamma^*_{\epsilon_1}) \to 0$ as $\epsilon_1 \to 0$. Let Π^*_i denote $T_{\epsilon_1}\Pi_i$ for $i = 1, 2$ and 3. We first prove that as $\epsilon_1 \to 0$,

$$\frac{dist(\Pi^*_2, \Pi^*_3)}{dist(\Pi^*_1, \Pi^*_2)} \to 0.$$

We have that $dist(\Pi^*_1, \Pi^*_2) \leq dist(\Pi^*_1 \cap \Gamma^*_{\epsilon_1}, \Pi^*_2 \cap \Gamma^*_{\epsilon_1}) \leq 2$ since $\Gamma^*_{\epsilon_1}$ is normalized. Then

$$\frac{dist(\Pi^*_2, \Pi^*_3)}{dist(\Pi^*_1, \Pi^*_2)} = \frac{dist(\Pi_2, \Pi_3)}{dist(\Pi_1, \Pi_2)} = \frac{\rho_{\epsilon_1}}{\epsilon_0} \to 0$$

as $\epsilon_1 \to 0$, and hence $dist(\Pi^*_2, \Pi^*_3) \to 0$.

Now let $\partial\Gamma_2^* \subset \partial\Gamma_{\epsilon_1}^*$ be that portion of the boundary lying between the planes Π_2^* and Π_3^*. Let $P_{\epsilon_1} \in \partial\Gamma_2^*$ be the point such that the line through
$|T_{\epsilon_1}x_1 - P_{\epsilon_1}| \leq dist(\Pi_2^*, \Pi_3^*) \to 0.$
$ist(T_{\epsilon_1}x_1, \partial\Gamma_{\epsilon_1}^*) \leq dist(T_{\epsilon_1}x_1, \partial\Gamma_2^*) \leq$
On the other hand, by Theorem 2.4.2.
$|T_{\epsilon_1}x_1 - P_{\epsilon_1}| \leq dist(\Pi_2^*, \Pi_3^*) \to 0.$

On the other hand, by Theorem 2.4.2.

$$|u_{\epsilon_1}^*(T_{\epsilon_1}x_1)|^n \leq C\delta(T_{\epsilon_1}x_1, \Gamma_{\epsilon_1}^*)^\epsilon \int_{\Gamma_{\epsilon_1}^*} \delta(x, \Gamma_{\epsilon_1}^*)^{1-\epsilon} dMu_{\epsilon_1}^*.$$

From (3.3) above,

$$|u_{\epsilon_1}^*(T_{\epsilon_1}x_1)|^n \geq C \int_{\Gamma_{\epsilon_1}^*} \delta(x, \Gamma_{\epsilon_1}^*)^{1-\epsilon} dMu_{\epsilon_1}^*.$$

This implies that $\delta(T_{\epsilon_1}x_1, \Gamma_{\epsilon_1}^*)^\epsilon \geq C$, but $dist(T_{\epsilon_1}x_1, \partial\Gamma_{\epsilon_1}^*) \to 0$. However, since $\Gamma_{\epsilon_1}^*$ is normalized, $\delta(T_{\epsilon_1}x_1, \Gamma_{\epsilon_1}^*) \approx dist(T_{\epsilon_1}x_1, \partial\Gamma_{\epsilon_1}^*)$. This is a contradiction. □

Corollary 3.2.3 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex. Suppose $u \in C(\bar{\Omega})$ is convex and zero on $\partial\Omega$. Then if $Mu \in D_\epsilon$, either u is strictly convex or identically zero.*

Proof Suppose u is not strictly convex. Then the graph of u contains a line segment, say L. If $(x_0, u(x_0)) \in int(L)$, then any supporting hyperplane $l(x)$ to u at x_0 must contain L.

Apply Theorem 3.2.1 to the function $u(x) - l(x)$. This function is non-negative on Ω, and the set $\Gamma = \{x \in \Omega : u(x) = l(x)\}$ contains more than one point. Therefore, Γ has no extremal points inside Ω, so all of its extremal points are in $\partial\Omega$.

Write $x_0 = \sum_{i=1}^m \lambda_i x_i$, where the $x_i \in \partial\Omega$ are extremal points of Γ, $\lambda_i > 0$, and $\sum \lambda_i = 1$. Then $u(x_0) = l(x_0) = \sum \lambda_i l(x_i) = \sum \lambda_i u(x_i) = 0$, since $u(x_i) = 0$. Then because x_0 is an interior point of Ω and $u(x_0) = 0$, $u \equiv 0$ by convexity. □

Theorem 3.2.4 *(Compare with Theorem 5.4.7 in [3]) Let Ω be bounded, open and convex, and let $u \in C(\bar{\Omega})$ be convex. Suppose $Mu \in D_\epsilon$ for some ϵ, and $Mu > \lambda > 0$. Then if $u = f$ on $\partial\Omega$, where $f \in C^{1,\beta}(\partial\Omega)$ for $\beta > 1 - \frac{2}{n}$ $(n \geq 3)$, then u is strictly convex.*

Proof Suppose u is not strictly convex. Then the graph of u contains a line segment, say from $(x_0, u(x_0))$ to $(x_1, u(x_1))$ for some x_0 and x_1 in Ω. Let $l(x)$ be a supporting hyperplane to u that contains this segment. Let $E = \{x \in \Omega : u(x) - l(x) = 0\}$. The segment from x_0 to x_1 is contained in E. Therefore, by Theorem 3.2.1, all of the extremal points of E (denote this set by E^*) are in $\partial\Omega$. Since E^* generates E, there exist two points z_0 and z_1 in E^* such that $\overline{z_0 z_1} \cap E \neq \emptyset$. By a translation and rotation, we can assume that the segment from z_0 to z_1 lies on the x_1 coordinate axis, and its midpoint is the origin. Let u^* represent the values of $u - l$ after this change of coordinates, and let f^* denote the new boundary values. We then have that $z_0 = t_0 e_1$ and $z_1 = -t_0 e_1$ for some $t_0 > 0$. Then $u^* \geq 0$ for all $x \in \Omega$ and is 0 along the x_1 axis, $Mu^* \in D_\epsilon$, and $Mu^* > \lambda$.

Write $x = (t, x')$ for $t \in \mathbb{R}$ and $x' = (x_2, \ldots, x_n)$. Now construct a thin cylindrical tube T_ρ with axis $x' = 0$, and radius 2ρ, for ρ small. We will introduce a barrier function $B = B(t, x')$ such that $B > u^*$ on ∂T_ρ, $MB = \lambda < Mu^*$, and $B(0, 0) = 0$. Therefore by Theorem 2.1.6, $B > u^*$ in T_ρ, so that $u^*(0) < 0$, but this is a contradiction, and hence u is strictly convex. Define

$$B(t, x') = K(a^{n-1} t^2 + \frac{1}{a}|x'|^2)$$

for constants K and a to be determined.

Let $(t, x') \in T_\rho$. We first claim that $0 \leq u^*(t, x') \leq C_1 |x'|^{1+\beta}$. We can write $(t, x') = \theta z_0' + (1 - \theta) z_1'$ for some z_0' and z_1' in $T_\rho \cap \partial\Omega$, and we can choose z_0' and z_1' such that they lie on the same straight line parallel to $x' = 0$. Then by convexity,

$$0 \leq u^*(t, x') \leq \theta u^*(z_0') + (1 - \theta) u^*(z_1') = \theta f^*(z_0') + (1 - \theta) f^*(z_1'). \quad (3.4)$$

We have that $f^*(z_i) = u^*(z_i) = 0$ for $i = 0, 1$. Since $u^* \geq 0$ on $\bar{\Omega}$, this implies that f^* has a minimum at z_i, so that $Df^*(z_i) = 0$. By the assumed regularity of f^*,

$$f^*(z_0') - f^*(z_0) = Df^*(z_0) \cdot (z_0' - z_0) + o(|z_0' - z_0|^{1+\beta}) = o(|z_0' - z_0|^{1+\beta}).$$

Since $\partial\Omega$ is Lipschitz, $|z_0' - z_0| \leq C|x'|$ (We can parameterize $\partial\Omega$ near z_0 by a Lipschitz function of x', say p. Then $|z_0' - z_0| = |p(x') - p(0)| \leq C|x'|$). Similarly, $f^*(z_1') \leq C|x'|^{1+\beta}$. Inserting these estimates into (3.4), we obtain the claim.

The next step is to compare the values of u^* and B on ∂T_ρ. We first consider the lateral side (i.e. $\partial T_\rho \cap \Omega$). Here we have that $|x'| = \rho$. Therefore, $B(t, x') \geq K\frac{\rho^2}{a}$. From the last claim, $u^*(t, x') \leq C_1\rho^{1+\beta}$. In order to guarantee that $B > u^*$, we need that

$$K\frac{\rho^2}{a} > C_1\rho^{1+\beta}. \tag{3.5}$$

We now deal with the rest of the boundary, $T_\rho \cap \partial\Omega$. Since $\partial\Omega$ is Lipschitz, $|t| \approx t_0$, so that there is a constant C_2 such that $|t| \geq C_2 t_0$. Then $B(t, x') \geq Ka^{n-1}t^2 \geq KC_2^2 a^{n-1}t_0^2$, and again we employ the estimate $u(t, x') \leq C_1\rho^{1+\beta}$. Therefore we require that

$$KC_2^2 a^{n-1} t_0^2 > C_1\rho^{1+\beta}. \tag{3.6}$$

We now compute MB. $MB = (2K)^n$, so if K is chosen smaller that $(\frac{\lambda}{2})^{\frac{1}{n}}$, we get $MB < Mu^*$.

To determine an appropriate value for a, we let $a = \frac{\rho^{1-\beta}}{\gamma}$ where γ is a large constant to be chosen shortly. From (3.5) we get that $\gamma K\rho^{1+\beta} > C_1\rho^{1+\beta}$; so we take $\gamma K > C_1$.

From (3.6), we see that

$$\frac{KC_2^2 t_0^2 \rho^{(1-\beta)(n-1)}}{\gamma^{n-1}} > C_1\rho^{1+\beta},$$

or equivalently,

$$\frac{KC_2^2 t_0^2}{C_1\gamma^{n-1}} > \rho^{1+\beta+(\beta-1)(n-1)}.$$

The left hand side of the last inequality is now a fixed quantity, so in order to make such an estimate possible (for small values of ρ) we need the exponent to be positive. This will happen precisely when $\beta > 1 - \frac{2}{n}$. □

We remark that Theorem 3.2.4 is sharp. There are examples (due to Pogorelov, see pp. 81-84 in [6]) of functions whose Monge-Ampère measures are bounded between two positive constants (and hence doubling and D_ϵ) and are $C^{1,1-\frac{2}{n}}$ on the boundary, but are not strictly convex.

3.3 Selection Lemma

The following compactness result is crucial for proving further regularity results and has independent interest. Compare with Lemma 5.3.1 in [3].

Lemma 3.3.1 ***(Selection Lemma)*** *Let $\{\Omega_j\}_1^\infty$ be a sequence of normalized convex domains, and let $u_j \in C(\bar{\Omega}_j)$ be convex, $u_j|_{\partial\Omega_j} = 0$, with $Mu_j \in D_\epsilon(C)$ for all j. Assume also that Mu_j is absolutely continuous with respect to Lebesgue measure for each j. Then if $\lambda \leq |\inf_{\Omega_j} u_j| \leq \Lambda$ for all j, there exist:*

(a) a normalized convex domain Ω_0,

(b) $u_0 \in C(\bar{\Omega}_0)$, convex, with $Mu_0 \in D_\epsilon(C)$, Mu_0 absolutely continuous with respect to Lebesgue measure, $u|_{\partial\Omega_0} = 0$ and $\lambda \leq |\inf u_0| \leq \Lambda$; and a subsequence of the u_j that converges uniformly on compact subsets to u_0.

If, in addition, for each j, there exists $x_j \in \Omega_j$ with $dist(x_j, \partial\Omega_j) \geq \epsilon'$, and $l_j(x)$ a support plane to u_j at x_j such that $S_j = \{x \in \Omega_j : u_j(x) < l_j(x) + \frac{1}{j}\} \not\subset \{x \in \Omega_j : u_j(x) < -\bar{C}\epsilon'\}$, then there exist:

(c) a point $x_0 \in \Omega_0$ such that $dist(x_0, \partial\Omega_0) \geq \epsilon'$, and

(d) a support plane l_0 to u_0 at x_0 such that $S_0 = \{x \in \Omega_0 : u_0(x) = l_0(x)\} \not\subset \{x \in \Omega_0 : u_0(x) < -\bar{C}\epsilon'\} = T_0$.

Remark A The boundedness condition on the minima of the u_j is necessary (and restricting just the D_ϵ constant is not enough) to guarantee the existence of a uniformly convergent subsequence as the following example demonstrates. For each $N \in \mathbb{N}$ we can uniquely solve the problem (provided Ω is strictly convex)

$$\begin{cases} \det D^2 u &= N dx \\ u|_{\partial\Omega} &= 0. \end{cases}$$

Then for any section S_N of u_N, $Mu_N(S_N) = N|S_N|$ and $Mu_N(\frac{1}{2}S_N) = N|\frac{1}{2}S_N| = \frac{1}{2^n}N|S_N|$. Therefore, Mu_N is doubling on the sections of u_N for all N with the same doubling constant. If Ω is normalized, Proposition 2.4.4 tells us that $|\min_\Omega u_N|^n \approx N$. From this, one sees that the sequence $\{min_\Omega u_N\}$ is unbounded, and therefore, $\{u_N\}$ cannot have a uniformly convergent subsequence.

Remark B If a measure $\mu \in D_\epsilon$, it is possible for μ to have singular part with respect to Lebesgue measure. There is an example of a measure that is doubling on intervals in $\mathbb{R}$, which is totally singular with respect to dx. See [8] (p.40) for details.

Proof of Lemma 3.3.1 The domain Ω_0 can be produced by the special case of the Blaschke Selection Theorem, Theorem 2.5.5. $\{\bar{\Omega}_j\}$ is a sequence of compact, convex sets. Denote also by $\{\bar{\Omega}_j\}$ the subsequence guaranteed by this result, and let $\bar{\Omega}_0$ be the limiting set. Take $\Omega_0 = int(\bar{\Omega}_0)$. Then given $\rho > 0$, for all j sufficiently large, we have that:

$$\Omega_j \subset \Omega_0 + \rho B_1(0) \qquad \text{and} \qquad \Omega_0 \subset \Omega_j + \rho B_1(0).$$

Since each Ω_j is normalized, these inclusions imply that Ω_0 is as well. This demonstrates (a).

The proof of the rest of the theorem will be done in stages.

Step 1 The first step in proving (b) is to show that for every compact $K \Subset \Omega_0$, there are positive constants $j_0(K)$ and $c(K)$ such that

$$K \subset \{x \in \Omega_j : dist(x, \partial\Omega_j) > c(K)\} \tag{3.7}$$

for all $j \geq j_0(K)$.

Let $dist(K, \partial\Omega_0) = \rho > 0$, and let $x \in K$. Translate the coordinates so that x is now 0. Let $\tilde{\Omega}_j = \Omega_j - x$, and $\tilde{\Omega}_0 = \Omega_0 - x$. Since $\Omega_j \to \Omega_0$, there exists J such that if $j \geq J$, then $d_H(\Omega_j, \Omega_0) < \frac{\rho}{2}$. Since d_H is invariant under translations, $d_H(\Omega_j, \Omega_0) = d_H(\tilde{\Omega}_j, \tilde{\Omega}_0)$. By Theorem 2.5.3, for any $u \in S^{n-1}$, $d_H(\tilde{\Omega}_j, \tilde{\Omega}_0) \geq |h(\tilde{\Omega}_j, u) - h(\tilde{\Omega}_0, u)|$. We have that $h(\tilde{\Omega}_0, u) \geq \rho$ for all unit vectors u, since $B_\rho(0) \subset \tilde{\Omega}_0$. Therefore,

$$\frac{\rho}{2} \geq |h(\tilde{\Omega}_j, u) - h(\tilde{\Omega}_0, u)| \geq \rho - |h(\tilde{\Omega}_j, u)|.$$

This implies that $h(\tilde{\Omega}_j, u) \geq \frac{\rho}{2}$. Therefore, $B_{\frac{\rho}{2}}(0) \subset \tilde{\Omega}_j$; translating back we obtain that $B_{\frac{\rho}{2}}(x) \subset \Omega_j$ for all $j \geq J$. Since $x \in K$ was arbitrary, we have that $dist(K, \partial\Omega_j) \geq \frac{\rho}{2}$ for j sufficiently large.

Step 2 Now we show that for every compact $K \Subset \Omega_0$, there is a constant $C(K)$ such that for every $x \in K$ and every $p \in \partial u_j(x)$,

$$|u_j(x)| + |p| \leq C(K). \tag{3.8}$$

By assumption, we have that the $\{u_j\}$ are uniformly bounded. Also by virtue of the last step, $\rho > 0$ can be chosen so that

$$K_\rho - \{x : dist(x, K) \leq \rho\} \Subset \Omega_j$$

for $j \geq j_0(K)$. Let $0 \neq p \in \partial u_j(K)$ for any $j \geq j_0(K)$, say $p \in \partial u_j(\bar{x})$, where $\bar{x} \in K$. Then

$$u_j(x) \geq u_j(\bar{x}) + p \cdot (x - \bar{x})$$

for all $x \in \Omega_j$. In particular, this is true for $x = \bar{x} + \rho\omega$, where $|\omega| = 1$. Then

$$u_j(\bar{x} + \rho\omega) \geq u_j(\bar{x}) + \rho|p| \text{ implying } \max_{K_\rho} u_j \geq \min_K u_j + \rho|p|.$$

and therefore,

$$|p| \leq \frac{\max_{K_\rho} u_j - \min_K u_j}{\rho} < \infty$$

since by hypothesis the u_j are uniformly bounded.

Step 3 We now produce the function u_0. To begin, we demonstrate that given $K \Subset \Omega_0$, the $\{u_j\}$ are uniformly Lipschitz on K, for j large enough, i.e. we show that $|u_j(x) - u_j(z)| \leq C(K)|x - z|$ for all $x, z \in K$ and for all $j \geq j_0(K)$. The proof of this claim is as in the proof of Lemma 1.1.6 in [3]. Let $x \in K$ and $p \in \partial u_j(x)$. Then by Step 2, $|p| \leq C(K)$. For any $z \in K$,

$$u_j(z) \geq u_j(x) + p \cdot (z - x) \text{ or } u_j(z) - u_j(x) \geq -|p|\,|z - x| \geq -C(K)|z - x|.$$

Therefore, $|u_j(x) - u_j(z)| \le C(K)|x - z|$ for all $j \ge j_0(K)$. In particular the $\{u_j\}$ are equicontinuous on K.

Therefore, by Arzelà-Ascoli, there exists a uniformly convergent subsequence in K. Write $\Omega_0 = K_1 \cup K_2 \cup K_3 \cup \cdots$, with $K_1 \Subset K_2 \Subset \cdots$. By a diagonal process, we can extract a subsequence of the u_j that converges uniformly on compact subsets of Ω_0. Define $u_0(x)$ to be the limit of this subsequence.

Because u_0 is the limit of convex functions, it is convex. If $x \in \Omega_0$, then $x \in \Omega_j$ for j large. Then since $u_j|_{\partial\Omega_j} = 0$, $u_j(x) \le 0$. By letting $j \to \infty$, we get that $u_0(x) \le 0$. Also, since $\lambda \le |\min u_j| \le \Lambda$ for every j, we have that $\lambda \le |\min u_0| \le \Lambda$.

Step 4 The next step is to show that $u_0 \in C(\bar{\Omega}_0)$ and $u_0|_{\partial\Omega_0} = 0$. To this end, we first show that for every $\eta > 0$, there is a number $j_0(\eta)$ such that

$$\{x \in \Omega_j : dist(x, \partial\Omega_j) \ge \eta\} \subset \{x \in \Omega_0 : dist(x, \partial\Omega_0) \ge \frac{\eta}{2}\}, \qquad (3.9)$$

for all $j \ge j_0(\eta)$. This can be shown by an argument similar to that in Step 1. For $j \ge J_1$, we have that $d_H(\Omega_j, \Omega) < \frac{\eta}{2}$. Let $j \ge J_1$ and let $x \in \Omega_j$ satisfy $dist(x, \partial\Omega_j) \ge \eta$. Define $\tilde{\Omega}_j = \Omega_j - x$, and $\tilde{\Omega}_0 = \Omega_0 - x$. This change of coordinates takes x to the origin, and, as before, we have that $d_H(\tilde{\Omega}_j, \tilde{\Omega}_0) = d_H(\Omega_j, \Omega_0)$. Then for any $u \in S^{n-1}$ (by Theorem 2.5.3):

$$\frac{\eta}{2} > d_H(\tilde{\Omega}_j, \tilde{\Omega}_0) \ge |h(\tilde{\Omega}_j, u) - h(\tilde{\Omega}_0, u)|.$$

Since $B_\eta(0) \subset \tilde{\Omega}_j$, $h(\tilde{\Omega}_j, u) \ge \eta$. Hence, $h(\tilde{\Omega}_0, u) \ge \frac{\eta}{2}$, so $B_{\frac{\eta}{2}}(0) \subset \tilde{\Omega}_0$. This last inclusion is equivalent to $B_{\frac{\eta}{2}}(x) \subset \Omega_0$ and (3.9) holds.

We now show that

$$\{x \in \Omega_0 : u_0(x) < -\rho\} \Subset \Omega_0$$

for each $\rho > 0$. By Theorem 2.4.2 and Proposition 2.4.4,

$$|u_j(x)|^n \le C\delta(x, \Omega_j)^\epsilon \int_{\Omega_j} \delta(y, \Omega_j)^{1-\epsilon} dMu_j \le C\Lambda\delta(x, \Omega_j)^\epsilon \le \tilde{C} dist(x, \Omega_j)^\epsilon.$$

This implies that $|u_j(x)|^{\frac{n}{\epsilon}} \leq \tilde{C}dist(x,\Omega_j)$, so if $u_j(x) < \frac{-\rho}{2}$, then $(\frac{\rho}{2})^{\frac{n}{\epsilon}} \leq Cdist(x,\partial\Omega_j)$. In other words the following inclusion holds:

$$\{x \in \Omega_j : u_j(x) < \frac{-\rho}{2}\} \subset \{x \in \Omega_j : dist(x,\partial\Omega_j) \geq C\rho^{\frac{n}{\epsilon}}\}.$$

Then by (3.9), for j large enough (depending on ρ)

$$\{x \in \Omega_j : dist(x,\partial\Omega_j) \geq C\rho^{\frac{n}{\epsilon}}\} \subset \{x \in \Omega_0 : dist(x,\partial\Omega_0) \geq \frac{C}{2}\rho^{\frac{n}{\epsilon}}\} \equiv K(\rho).$$

Therefore,

$$\bigcup_{j \geq j_0(\rho)} \{x \in \Omega_j : u_j(x) < \frac{-\rho}{2}\} \subset K(\rho).$$

This implies that $\{x \in \Omega_0 : u_0(x) < -\rho\} \subset K(\rho)$. This is because if $x \in \Omega_0$, then $x \in \Omega_j$ for all j large enough (by Lemma 2.5.6), and if $u(x) < -\rho$, then $u_j(x) < -\frac{\rho}{2}$ for all large j. This proves that

$$\lim_{x \to \partial\Omega_0} u_0(x) = 0.$$

Step 5 Now we will produce the point x_0 and the supporting hyperplane with the desired properties. By hypothesis,

$$S_j = \{x \in \Omega_j : u_j(x) < l_j(x) + \frac{1}{j}\} \not\subseteq \{x \in \Omega_j : u_j(x) < -C\epsilon'\} = T_j.$$

Then there is a point $y_j \in S_j \cap T_j^c$. Then $u_j(y_j) < l_j(y_j) + \frac{1}{j}$ and we can assume $u_j(y_j) = -C\epsilon'$. By (3.9), $x_j \in \Omega_0$ and

$$dist(x_j,\partial\Omega_0) \geq \frac{\epsilon'}{2} \qquad \text{for all } j > j_0(\epsilon'). \tag{3.10}$$

By Theorem 2.4.2 and Proposition 2.4.4,

$$C\epsilon'^n = |u_j(y_j)|^n \leq C_n\delta(y_j,\Omega_j)^\epsilon \int_{\Omega_j} \delta(y,\Omega_j)^{1-\epsilon} dMu_j \leq C(n,\Lambda)\delta(y_j,\Omega_j)^\epsilon.$$

This implies that $dist(y_j,\partial\Omega_j) \geq C\epsilon'^{\frac{n}{\epsilon}}$, so by (3.9), $dist(y_j,\Omega_0) \geq C(n,\epsilon',\Lambda)$ for j large enough.

Therefore, for all j sufficiently large, $\{x_j\}$ and $\{y_j\}$ are contained in a compact subset of Ω_0. By passing to subsequences, $x_j \to x_0 \in \Omega_0$ and $y_j \to y_0 \in$

Ω_0. Furthermore, $dist(x_0, \partial\Omega_0), dist(y_0, \partial\Omega_0) \geq C(\epsilon')$. This shows statement ($c$) of the theorem. Let $p_j \in \mathbb{R}^n$ define $l_j(x)$, i.e. $l_j(x) = u_j(x_j) + p_j \cdot (x - x_j)$. Then since the x_j are away from $\partial\Omega_0$, the p_j are bounded (Step 2). Choose a subsequence so that $p_j \to p_0$. Now $u_j(x) \geq u_j(x_j) + p_j \cdot (x - x_j) = l_j(x)$ for all $x \in \Omega_j$. Let $j \to \infty$ to get $u_0(x) \geq u_0(x_0) + p_0 \cdot (x - x_0)$ for all $x \in \Omega_0$. This means that $l_0(x) = u_0(x_0) + p_0 \cdot (x - x_0)$ is a support plane to u_0 at x_0. Now $u_j(y_j) = -C\epsilon'$ for all j implying that $u_0(y_0) = -C\epsilon'$. Also, $u_j(y_j) \leq l_j(y_j) + \frac{1}{j}$. Let $j \to \infty$ and get $u_0(y_0) \leq l_0(y_0)$ so that $u_0(y_0) = l_0(y_0)$. Therefore, $y_0 \in S_0 \cap T_0^c$. This proves (d).

Step 6 It remains to show that $Mu_0 \in D_\epsilon(C)$. To prove this, it must be demonstrated that for any section S of u_0,

$$\int_S \delta(x, S)^{1-\epsilon} dMu_0 \leq CMu_0\left(\frac{1}{2}S\right).$$

To prove this inequality, two cases will be considered: the case where $\bar{S} \Subset \Omega_0$, and the case where $\bar{S} \cap \partial\Omega_0 \neq \emptyset$.

Case 1 $\bar{S} \Subset \Omega_0$. The idea is to approximate S by $\{S_j\}$, a sequence of sections of u_j, with the property that $S_j \to S$ in the Hausdorff metric. Once the possibility of this approximation is demonstrated, we will show that this implies that

$$Mu_j\left(\frac{1}{2}S_j\right) \to Mu_0\left(\frac{1}{2}S\right) \text{ and } \tag{3.11}$$

$$\int_{S_j} \delta(x, S_j)^{1-\epsilon} dMu_j \to \int_S \delta(x, S)^{1-\epsilon} dMu_0. \tag{3.12}$$

This will establish the claim, since we have for all j ($Mu_j \in D_\epsilon(C)$)

$$\int_{S_j} \delta(x, S_j)^{1-\epsilon} dMu_j \leq CMu_j\left(\frac{1}{2}S_j\right).$$

Let S be such a section for u_0, $S = S(x_0, p, t) \Subset \Omega_0$. Let $l(x) = u_0(x_0) + p \cdot (x - x_0) + t$. Since $l(x) < 0$ for all $x \in \Omega_0$ and $\Omega_j \to \Omega_0$, $l(x) < 0$ for all $x \in \Omega_j$, for $j \geq J_0$.

Then $l(x)$ determines a section S_j of u_j in Ω_j at some point (slide $l(x)$ down until it touches the graph of u_j at one point, this will be the desired base point), with some parameter (equal to the distance that $l(x)$ must be lowered). In other words, $S_j = \{x \in \Omega_j : u_j(x) < l(x)\}$.

By the uniform convergence, for any $\rho > 0$, there exists J_1 such that for all $j \geq J_1$,

$$u_0(x) - \rho \leq u_j(x) \leq u_0(x) + \rho,$$

for every $x \in U$, where $\bar{S} \subset \bar{U} \Subset \Omega_0$. Take ρ small enough that $S_\rho \equiv S(x_0, p, t + \rho) \Subset U$, and $t - \rho > 0$. Define $S^\rho \equiv S(x_0, p, t - \rho)$. Then we have

$$S^\rho \subset S_j \subset S_\rho \text{ for } j \geq \max(J_0, J_1).$$

Now by Lemma 2.5.9 in Section 2.5, we have that

$$\lim_{\rho \to 0} S^\rho = \lim_{\rho \to 0} S_\rho = S,$$

implying that $\lim_{j \to \infty} S_j = S$, and we are able to approximate the section S as claimed.

The next step is to prove (3.11). Since $S_j \to S$ in the Hausdorff metric, $\frac{1}{2}S_j \to \frac{1}{2}S$ by Lemma 2.5.7. Also, since $u_j \to u_0$ uniformly on compact subsets, the measures Mu_j converge to Mu_0 weakly (by Lemma 2.1.4). Since Mu_j is absolutely continuous with respect to Lebesgue measure, and $Mu_j \to Mu_0$ weakly, Mu_0 is absolutely continuous with respect to Lebesgue measure at least away from $\partial\Omega_0$. Let $\rho > 0$. We need to show that $|Mu_j(\frac{1}{2}S_j) - Mu_0(\frac{1}{2}S)| < \rho$ for all $j \geq J(\rho)$.

Let $f \in C_0^0(S)$, $0 \leq f \leq 1$, $f \equiv 1$ on $\frac{1}{2}S$ be such that

$$\left| \int_S f(x)\, dMu_0 - Mu_0(\frac{1}{2}S) \right| < \rho.$$

Then by the weak convergence, there exists $J_1 = J_1(\rho)$ such that if $j \geq J_1$, then $|\int_S f(x)\, dMu_j - \int_S f(x)\, dMu_0| < \rho$. Therefore,

$$\left| \int_S f(x)\, dMu_j - Mu_0\left(\frac{1}{2}S\right) \right| \leq$$

$$\left| \int_S f(x) dMu_j - \int_S f(x)\, dMu_0 \right| + \left| \int_S f(x)\, dMu_0 - Mu_0\left(\frac{1}{2}S\right) \right| < 2\rho$$

and the claim will follow if $|Mu_j(\frac{1}{2}S_j) - \int_S f(x)\, dMu_j| < \rho$ for all j sufficiently large. We estimate this quantity in the following way:

$$|Mu_j(\frac{1}{2}S_j) - \int_S f(x)\, dMu_j|$$

$$\leq |Mu_j(\frac{1}{2}S_j) - Mu_j(\frac{1}{2}S)| + |Mu_j(\frac{1}{2}S) - \int_S f(x)\, dMu_j| = I + II.$$

We begin by examining integral I:

$$I \leq \int |\chi_{\frac{1}{2}S_j}(x) - \chi_{\frac{1}{2}S}(x)|\, dMu_j \equiv \int f_j\, dMu_j.$$

We want to show that $\lim_{j\to\infty} \int f_j\, dMu_j = 0$. We have that $f_j \to 0$ pointwise a.e. (with respect to Mu_0; here is where the absolute continuity is needed) and $|f_j| \leq 1$ for all j. Since $\frac{1}{2}S_j \to \frac{1}{2}S$ and $\frac{1}{2}S \Subset \Omega_0$, there exists a set A such that $\frac{1}{2}S_j, \frac{1}{2}S \subset A \Subset \Omega_0$, for all $j \geq J_2$. By Egorov's theorem, there is a set E, with $Mu_0(E) < \rho$, such that $f_j \to 0$ uniformly on $A \cap E^c$. Since f_j can take on the values 0 and 1 only, this means that $f_j(x) = 0$ for all $j \geq J_3$ and all $x \in A \cap E^c$. Then

$$\int f_j\, dMu_j = \int_A f_j\, dMu_j = \int_E f_j\, dMu_j + \int_{A\cap E^c} f_j\, dMu_j \leq Mu_j(E) + 0.$$

By the weak convergence, if $j \geq J_4$, $|Mu_j(E) - Mu_0(E)| \leq \rho$, implying that $Mu_j(E) \leq 2\rho$. Therefore, $I \leq 2\rho$ for $j \geq \max(J_1, J_2, J_3, J_4)$.

We now consider the second integral:

$$II = \int_S f(x)\, dMu_j - Mu_j\left(\frac{1}{2}S\right) = \int_{S\cap\frac{1}{2}S^c} f(x)\, dMu_j.$$

Let $j \to \infty$. By the weak convergence, the left-hand side of the equation becomes

$$\int_S f(x)\, dMu_0 - Mu_0\left(\frac{1}{2}S\right), \text{ which is equal to } \int_{S\cap\frac{1}{2}S^c} f(x)\, dMu_0.$$

This means that

$$\lim_{j\to\infty} \int_{S\cap\frac{1}{2}S^c} f(x)\, dMu_j = \int_{S\cap\frac{1}{2}S^c} f(x)\, dMu_0 < \rho.$$

Therefore, if $j \geq J_5$, $II < \rho$. Combining all of these inequalities, we get that $Mu_j(\frac{1}{2}S_j) \to Mu_0(\frac{1}{2}S)$.

Now we prove (3.12). For $j \geq J_0$, we have $S_j \Subset \Omega_0$ and $S \Subset \Omega_j$. We estimate the difference as follows:

$$\left|\int_{S_j} \delta(x, S_j)^{1-\epsilon}\, dMu_j - \int_S \delta(x, S)^{1-\epsilon}\, dMu_0\right|$$

$$\leq \left|\int_{S_j} \delta(x, S_j)^{1-\epsilon}\, dMu_j - \int_S \delta(x, S)^{1-\epsilon}\, dMu_j\right|$$

$$+\left|\int_S \delta(x, S)^{1-\epsilon}\, dMu_j - \int_S \delta(x, S)^{1-\epsilon}\, dMu_0\right| = I + II.$$

The integral II goes to 0 by the weak convergence. The support of $\chi_S\delta(x, S)$ is $\bar{S} \Subset \Omega_0$. Since $\delta(x, S)$ is continuous in x (Lemma 2.3.4), and $\lim_{x\to\partial S} \delta(x, S) = 0$, we have that $\chi_S\delta(x, S)$ is continuous and has compact support.

To estimate I, we write

$$I = \left|\int \chi_{S_j}(x)\delta(x, S_j)^{1-\epsilon}\, dMu_j - \int \chi_S(x)\delta(x, S)^{1-\epsilon}\, dMu_j\right|$$

$$\equiv \left|\int f_j(x)\, dMu_j - \int f(x)\, dMu_j\right| \leq \int |f_j(x) - f(x)|\, dMu_j.$$

We claim that $|f_j - f| \leq 1$ and $(f_j - f) \to 0$ pointwise. The first assertion is trivial. If $x \in S$, $\chi_{S_j}(x) \to \chi_S(x) = 1$, and by the continuity of δ in the second argument with respect to the Hausdorff metric(Lemma 2.5.8), $\delta(x, S_j) \to \delta(x, S)$. If $x \in \partial S$, then $f(x) = 0$: for each j with $x \in S_j$, $\delta(x, S_j) \to 0$, and if $x \notin S_j$, $f_j(x) = 0$, so in any case, $f_j(x) \to f(x)$ for $x \in \partial S$. Finally, if $x \notin \bar{S}$, $x \notin S_j$ for large j, so $f_j(x) = f(x) = 0$. This establishes that $(f_j - f) \to 0$ pointwise.

Then by an argument similar to the one above (using Egorov, etc.), $I \to 0$. Therefore,

$$\int_{S_j} \delta(x, S_j)^{1-\epsilon}\, dMu_j \to \int_S \delta(x, S)^{1-\epsilon}\, dMu_0$$

for $S \Subset \Omega_0$ and Mu_0 satisfies the D_ϵ condition for these sections.

Case 2 $\partial S \cap \partial\Omega_0 \neq \emptyset$. The idea here is to approximate $S = S(x_0, p, t)$ by $S_n = S(x_0, p, t - \frac{1}{n}) \Subset \Omega_0$ and use Case 1.

From Case 1, $\int_{S_n} \delta(x, S_n)^{1-\epsilon} dMu_0 \leq CMu_0(\frac{1}{2}S_n)$. We have that $\chi_{\frac{1}{2}S_n}(x) \to \chi_{\frac{1}{2}S}(x)$ pointwise except possibly on $\partial(\frac{1}{2}S)$ (Lemma 2.5.6) and $\frac{1}{2}S_n, \frac{1}{2}S \subset A \Subset \Omega_0$ for some set A. Then $\chi_{\frac{1}{2}S_n}(x) \leq \chi_A \in L^1(Mu_0)$. Therefore, by the Dominated Convergence Theorem, $Mu_0(\frac{1}{2}S_n) \to Mu_0(\frac{1}{2}S)$.

Since $S_n \to S$ in the Hausdorff sense, $\delta(x, S_n) \to \delta(x, S)$ and $\chi_{S_n}(x) \to \chi_S(x)$. Hence, using Fatou,

$$\int_S \delta(x, S)^{1-\epsilon}\, dMu_0 = \int_S \lim_{n\to\infty} (\chi_{S_n}(x)\delta(x, S_n)^{1-\epsilon})\, dMu_0$$

$$\leq \liminf_{n\to\infty} \int_S \chi_{S_n}(x)\delta(x, S_n)^{1-\epsilon}\, dMu_0 \leq \liminf_{n\to\infty} CMu_0\left(\frac{1}{2}S_n\right) = CMu_0\left(\frac{1}{2}S\right).$$

Therefore, $Mu_0 \in D_\epsilon$ on the sections of u_0 in Ω_0. This completes the proof of the Selection Lemma. □

3.4 Hölder Continuity of the Gradient

Theorem 3.4.1 *(Compare with Theorem 5.3.3 in [3]) Given Ω a convex, bounded and normalized domain in $\mathbb{R}^n$, consider $u \in C(\bar{\Omega})$, convex, with $u|_{\partial\Omega} = 0$ and such that $Mu \in D_{\epsilon_1}(C)$ and is absolutely continuous with respect to Lebesgue measure. Suppose also that $0 < \lambda \leq |\inf_\Omega u| \leq \Lambda$. Then for each $\epsilon > 0$, there exists $\rho = \rho(\epsilon)$ such that for all Ω normalized, for all x_0 with $dist(x_0, \partial\Omega) \geq \epsilon$, for all functions u satisfying the above conditions, and for all supporting hyperplanes $l(x)$ to u at x_0, we have that $\{x \in \Omega : u(x) < l(x) + \rho\} \Subset \Omega$.*

Moreover, $\{x \in \Omega : u(x) < l(x) + \rho\} \subset \{x \in \Omega : u(x) < -\bar{C}\epsilon\}$, where $\bar{C} = \bar{C}(C, \epsilon_1, n, \lambda, \Lambda)$ and ρ depends only on $\epsilon, \epsilon_1, n, \lambda$ and Λ.

Proof By contradiction. Suppose there exists $\epsilon > 0$ such that for each $\rho = \frac{1}{j}$, there exists a normalized convex domain Ω_j, a point $x_j \in \Omega_j$ with

$dist(x_j, \partial\Omega_j) \geq \epsilon$, a function u_j satisfying the hypotheses in Ω_j and l_j, a supporting hyperplane to u_j at x_j such that

$$S_j = \{x \in \Omega_j : u_j(x) < l_j(x) + \frac{1}{j}\} \not\subset \{x \in \Omega_j : u_j(x) < -\bar{C}\epsilon\}.$$

Then by Lemma 3.3.1, there exists:
(a) Ω_0, a normalized convex domain, (b) u_0, a convex function in Ω_0 satisfying $Mu_0 \in D_{\epsilon_1}(C)$, $u_0|_{\partial\Omega_0} = 0$, and
$\lambda \leq |\inf_{\Omega_0} u_0| \leq \Lambda$.
(c)a point $x_0 \in \Omega_0$ such that $dist(x_0, \partial\Omega_0) \geq \epsilon$, and
(d) a supporting hyperplane l_0 to u_0 at x_0
such that $S_0 = \{x \in \Omega_0 : u_0(x) = l_0(x)\} \not\subset \{x \in \Omega_0 : u_0(x) < -\bar{C}\epsilon\} = T_0$.
So there is a point $z \in S_0$ with $u_0(z) \geq -\bar{C}\epsilon$. Since l_0 is a supporting hyperplane at x_0, it follows that $u_0 = l_0$ on the segment from x_0 to z.

We now apply the following lemma (Lemma 5.1.6 in [3]) to u_0 in the normalized domain Ω_0.

Lemma 3.4.2 *Let Γ be a convex and bounded domain in $\mathbb{R}^n$, and let $u \in C(\bar{\Gamma})$ be convex and zero on $\partial\Gamma$. If T is an affine transformation that normalizes Γ then*

$$\{x \in \Gamma : dist(Tx, \partial T(\Gamma)) > \eta\} \subset \{x \in \Gamma : u(x) \leq \eta\theta_n \min_{\Gamma} u\},$$

for all $0 < \eta < 1$, where θ_n is a dimensional constant.

We obtain:

$$\{x \in \Omega_0 : dist(x, \partial\Omega_0) \geq \epsilon\} \subset \{x \in \Omega_0 : u_0(x) < -\tilde{C}\epsilon\}$$

where $\tilde{C} = \theta_n |\min_{\Omega_0} u_0|$. In our case, $|\min_{\Omega_0} u_0|$ can be replaced by a constant depending on the structure since $|\min_{\Omega_0} u_0| \geq \lambda$, this gives us the constant $\bar{C}$.

The point x_0 is in T_0 since $dist(x_0, \partial\Omega_0) \geq \epsilon$, implying that $u_0(x_0) < -\bar{C}\epsilon$. Therefore $x_0 \neq z$. Then $\overline{x_0 z} \subset S_0$, so that S_0 contains more than one point. By applying Theorem 3.2.1 to $u_0 - l_0$ in Ω_0, we conclude that S_0 has no extremal points inside Ω_0. The contradiction arises by showing that S_0 must have an extremal point in Ω_0.

For every $x \in \Omega_0$, $u_0(x) < 0$, because otherwise by convexity $u_0 \equiv 0$, contradicting $|\inf_{\Omega_0} u_0| \geq \lambda > 0$.

Then by the following result (Lemma 5.3.2 in [3]), we see that S_0 must have an extremal point in Ω_0.

Lemma 3.4.3 *Let u be convex in Γ, such that for some $x_0 \in \Gamma^0$, we have $u(x) > u(x_0)$ for all $x \in \partial\Gamma$. Let l_{x_0} be any supporting hyperplane to u at x_0. If the set*

$$E = \{x \in \Gamma : u(x) = l_{x_0}(x)\}$$

contains more than one point, then E has an extremal point inside Γ.

This completes the proof of Theorem 3.4.1. □

We next prove two technical lemmas concerning the relationship of the dilation of a normalized domain and the dilation of a sublevel set. These results will be used in the first part of the argument for the $C^{1,\alpha}$ of functions whose Monge-Ampère measures are D_ϵ.

Lemma 3.4.4 *(Compare with Lemma 5.4.1 in [3]) Suppose Ω is a normalized convex domain, $u \in C(\bar{\Omega})$ is convex and $u|_{\partial\Omega} = 0$, $Mu \in D_\epsilon(C)$ and is absolutely continuous with respect to Lebesgue measure and $u(x_0) = \min_\Omega u$. Suppose also that $0 < \lambda \leq |\min_\Omega u| \leq \Lambda$. Given $0 < \eta \leq 1$, define*

$$\Omega_\eta = \{x \in \Omega : u(x) < (1-\eta)\min_\Omega u\}.$$

Then there exists a constant ν, $0 < \nu < 1$, depending on n, ϵ, the D_ϵ constant C, λ and Λ, such that $\frac{1}{2}\Omega \subset \nu\Omega_{\frac{1}{2}}$, where the dilations are with respect to x_0.

Proof The proof is by contradiction. By the extremal points theorem (Theorem 3.2.1), there is a unique point where the minimum is attained. If the lemma is not true, then for each $j = 2, 3, 4, \ldots$ and $\nu_j = 1 - \frac{1}{j}$, there exists a convex normalized domain Ω_j and a convex function u_j satisfying $u_j|_{\partial\Omega_j} = 0$,

$Mu_j \in D_\epsilon(C)$, Mu_j absolutely continuous, and $\lambda \le |\min_{\Omega_j} u_j| \le \Lambda$, but $\frac{1}{2}\Omega_j \not\subset \nu_j(\Omega_j)_{\frac{1}{2}}$, where the dilations are with respect to x_j, the point satisfying $u_j(x_j) = \min_{\Omega_j} u_j$.

In other words,

$$\frac{1}{2}\Omega_j = \{x_j + \frac{1}{2}(x - x_j) : x \in \Omega_j\} \not\subset \nu_j(\Omega_j)_{\frac{1}{2}} = \{x_j + \nu_j(x - x_j) : x \in (\Omega_j)_{\frac{1}{2}}\}.$$

Note $x_j \in \frac{1}{2}\Omega_j \cap \nu_j(\Omega_j)_{\frac{1}{2}}$ for all j, so by convexity $\frac{1}{2}\Omega_j \cap \partial(\nu_j(\Omega_j)_{\frac{1}{2}}) \neq \emptyset$. Let $y_j \in \frac{1}{2}\Omega_j \cap \partial(\nu_j(\Omega_j)_{\frac{1}{2}})$. Then $\{y_j\} \subset B_1(0)$, so $\{y_j\}$ has a convergent subsequence say, $y_j \to y_0$.

Therefore, by Lemma 3.3.1, there exists a normalized convex domain Ω_0, a convex function $u_0 \in C(\bar{\Omega}_0)$ satisfying $Mu_0 \in D_\epsilon(C)$ with $u_0|_{\partial\Omega_0} = 0$, and $u_j \to u_0$ uniformly on compact subsets. We also have the following inclusion for any $\epsilon_0 > 0$ (see (3.9)),

$$\{x \in \Omega_j : dist(x, \partial\Omega_j) \ge \epsilon_0\} \subset \{x \in \Omega_0 : dist(x, \partial\Omega_0) \ge \frac{\epsilon_0}{2}\}$$

for all j sufficiently large depending on ϵ_0.

Claim: $y_0 \in \overline{\frac{1}{2}\Omega_0} \cap \partial(\Omega_0)_{\frac{1}{2}}$, where the dilation is with respect to the (unique) point x_0 for which $u_0(x_0) = \min_{\Omega_0} u_0$. We assume this claim for now.

Construct the line through x_0 and y_0. This crosses $\partial\Omega_0$ at some point y_0^*. Since $y_0 \in \partial(\Omega_0)_{\frac{1}{2}}$, $u_0(y_0) = \frac{1}{2}\min_{\Omega_0} u_0 = \frac{1}{2}u_0(x_0)$. Now write $y_0 = \theta x_0 + (1-\theta)y_0^*$ for some $0 < \theta < 1$. Then

$$u_0(y_0) = \frac{1}{2}\min_{\Omega_0} u_0 \le \theta u_0(x_0) + (1-\theta)u_0(y_0^*) = \theta \min_{\Omega_0} u_0.$$

Since $u_0 \le 0$, this means that $\theta \le \frac{1}{2}$. Now suppose that $\theta < \frac{1}{2}$. Then $y_0 = x_0 + (1-\theta)(y_0^* - x_0) \in (1-\theta)\partial\Omega_0$. But $1 - \theta > \frac{1}{2}$, contradicting the fact that $y_0 \in \overline{\frac{1}{2}(\Omega_0)}$, and therefore $\theta = \frac{1}{2}$.

This means that u_0 is linear on the segment L from x_0 to y_0^*. This implies that $L \subset \{u_0 = l_{x_0}\}$, where l_{x_0} is any supporting hyperplane to u_0. Hence the set where u_0 and l_{x_0} agree has more than one point. Therefore, this set has no extremal points in Ω_0, but $0 = u_0(x) > u_0(x_0)$ for all $x \in \partial\Omega_0$. Hence by Lemma 3.4.3, this set has an interior extremal point. This provides the contradiction.

It remains to prove the claim. Since $y_j \in \frac{1}{2}\Omega_j$, $y_j = x_j + \frac{1}{2}(z_j - x_j)$ for some $z_j \in \Omega_j$. Since $\{|u_j(x_j)|\}$ is bounded between two positive constants and Ω_j is normalized, we have that $dist(x_j, \partial\Omega_j) \geq \epsilon_0$ for all j by Theorem 2.4.2 and Proposition 2.4.4.

From (3.9), $dist(x_j, \partial\Omega_0) \geq \frac{\epsilon_0}{2}$ for all j large enough. So by passing to a subsequence, we may assume that $x_j \to \bar{x}$. We will show that $\bar{x} = x_0$. For j sufficiently large, x_j and $\bar{x}$ are contained in a compact subset of Ω_0 (since $\Omega_j \to \Omega_0$); on this compact set $u_j \to u_0$ uniformly. Therefore for any $x \in \Omega_0$, we have $u_j(x_j) \leq u_j(x)$; letting $j \to \infty$, we obtain $u_0(\bar{x}) \leq u_0(x)$. Hence, $\bar{x}$ is a minimum for u_0, so $\bar{x} = x_0$.

Select a subsequence of the $\{z_j\}$, such that $z_j \to z_0$. Since $z_j \in \Omega_j$ and $\Omega_j \to \Omega_0$, we get that $z_0 \in \bar{\Omega}_0$. Let $j \to \infty$ in the equation

$$y_j = x_j + \frac{1}{2}(z_j - x_j)$$

to get

$$y_0 = x_0 + \frac{1}{2}(z_0 - x_0)$$

so that $y_0 \in \overline{\frac{1}{2}\Omega_0}$. It remains to show that $y_0 \in \partial(\Omega_0)_{\frac{1}{2}}$. We have that $y_j \in \partial(\nu_j(\Omega_j)_{\frac{1}{2}}) = \nu_j(\partial(\Omega_j)_{\frac{1}{2}})$, so that $y_j = x_j + \nu_j(w_j - x_j)$ for some $w_j \in \partial(\Omega_j)_{\frac{1}{2}}$. Then we have that $u_j(w_j) = \frac{1}{2}u_j(x_j) = \frac{1}{2}\min_{\Omega_j} u_j$. As before, Theorem 2.4.2 and Proposition 2.4.4 guarantee that the w_j lie at least a uniform distance from $\partial\Omega_j$, implying that $dist(w_j, \partial\Omega_0) \geq \bar{\epsilon} > 0$ for j large enough. Select a subsequence $w_j \to w_0$. Then by the uniform convergence of the u_j on compact subsets, we obtain $u_0(w_0) = \lim_{j\to\infty} u_j(w_j)$.

Now let $j \to \infty$ in the equation $y_j = x_j + \nu_j(w_j - x_j)$ to get $y_0 = x_0 + (w_0 - x_0) = w_0$, since $\nu_j \to 1$. So we have that $u_j(w_j) \to u_0(w_0)$, but on the other hand, $u_j(w_j) = \frac{1}{2}u_j(x_j) \to \frac{1}{2}u_0(x_0)$. This means that $u_0(w_0) = \frac{1}{2}u_0(x_0)$, i.e. $w_0 \in \partial(\Omega_0)_{\frac{1}{2}}$. This proves the claim and hence the lemma. □

Lemma 3.4.5 *(See Corollary 5.4.4 in [3]) Suppose that Ω is a normalized convex domain, $u \in C(\bar{\Omega})$ is convex and $u|_{\partial\Omega} = 0$, $Mu \in D_\epsilon(C)$ and is absolutely continuous with respect to Lebesgue measure, and $u(x_0) = \min_\Omega u = -1$. Then there exists a constant $\nu \in (0, 1)$ such that*

$$\Omega \subset (2\nu)^k \Omega_{\frac{1}{2^k}}$$

for $k = 1, 2, 3, \ldots$, $\nu = \nu(n, \epsilon, D_\epsilon$ constant), *where the dilations are with respect to the point x_0.*

Proof The case $k = 1$ is covered by the previous lemma. The ν in the preceding result depends on n, ϵ, the D_ϵ constant, and $\min_\Omega u$. Since this minimum is -1, in this case we can remove the dependence of ν on $\min_\Omega u$.

Case k = 2: Let T_1 normalize $\Omega_{\frac{1}{2}}$, and denote $T_1(\Omega_{\frac{1}{2}})$ by Ω_1^*. Let $v_1(x) = 2[u(T_1^{-1}x) + \frac{1}{2}]$. Then $v_1|_{\partial\Omega_1^*} = 0$, $Mv_1 \in D_\epsilon(C)$, and $\min_{\Omega_1^*} v_1 = v_1(T_1 x_0) = -1$. Apply the previous lemma to v_1 in Ω_1^*, to get $\frac{1}{2}\Omega_1^* \subset \nu(\Omega_1^*)_{\frac{1}{2}}$. Now

$$(\Omega_1^*)_{\frac{1}{2}} = \{x \in \Omega_1^* : v_1(x) < \frac{1}{2}\min_{\Omega_1^*} v_1\} = \{x \in \Omega_1^* : 2[u(T_1^{-1}x) + \frac{1}{2}] < -\frac{1}{2}\}$$

$$= \{x \in \Omega_1^* : u(T_1^{-1}x) < -\frac{3}{4}\} = T_1(\{x \in \Omega_{\frac{1}{2}} : u(x) < -\frac{3}{4}\}) = T_1(\Omega_{\frac{1}{4}}).$$

Therefore, $\nu(T_1(\Omega_{\frac{1}{4}})) = T_1(\nu\Omega_{\frac{1}{4}})$ contains $\frac{1}{2}(T_1(\Omega_{\frac{1}{2}})) = T_1(\frac{1}{2}\Omega_{\frac{1}{2}})$. Hence, by applying T_1^{-1}, we get $\frac{1}{2}\Omega_{\frac{1}{2}} \subset \nu\Omega_{\frac{1}{4}}$, or $\Omega_{\frac{1}{2}} \subset (2\nu)\Omega_{\frac{1}{4}}$. Combining this with the preceding step, we get

$$\Omega \subset (2\nu)\Omega_{\frac{1}{2}} \subset (2\nu)^2\Omega_{\frac{1}{4}}.$$

Case k = 3: Let T_2 normalize $\Omega_{\frac{1}{4}}$, and denote $T_2(\Omega_{\frac{1}{4}})$ by Ω_2^*. Let $v_2(x) = 4[u(T_2^{-1}x) + \frac{3}{4}]$. Then $v_2|_{\partial\Omega_2^*} = 0$, $Mv_2 \in D_\epsilon(C)$, and $\min_{\Omega_2^*} v_2 = v_2(T_2 x_0) = -1$. By Lemma 3.4.4 applied to v_2 in Ω_2^*, $\nu(\Omega_2^*)_{\frac{1}{2}}$ contains $\frac{1}{2}\Omega_2^*$. Then:

$$(\Omega_2^*)_{\frac{1}{2}} = \{x \in \Omega_2^* : v_2(x) < -\frac{1}{2}\} = \{x \in \Omega_2^* : u(T_2^{-1}x) < -\frac{7}{8}\}$$

$$= T_2(\{x \in \Omega_{\frac{1}{4}} : u(x) < -\frac{7}{8}\}) = T_2(\Omega_{\frac{1}{8}}).$$

Therefore, $\frac{1}{2}\Omega_{\frac{1}{4}} \subset \nu\Omega_{\frac{1}{8}}$, or $\Omega_{\frac{1}{4}} \subset (2\nu)\Omega_{\frac{1}{8}}$. Hence, by the previous steps:

$$\Omega \subset (2\nu)\Omega_{\frac{1}{2}} \subset (2\nu)^2\Omega_{\frac{1}{4}} \subset (2\nu)^3\Omega_{\frac{1}{8}}.$$

In general, let T_k normalize $\Omega_{\frac{1}{2^k}}$ and $v_k(x) = 2^k[u(T_k^{-1}x) + (1 - \frac{1}{2^k})]$. As above, we conclude that $\Omega_{\frac{1}{2^{k-1}}} \subset (2\nu)\Omega_{\frac{1}{2^k}}$. □

Theorem 3.4.6 *(Compare with Theorem 5.4.5 in [3]) Let Ω be bounded, open and convex, and let $u \in C(\bar{\Omega})$ be convex, with $u|_{\partial\Omega} = 0$. Then if $Mu \in D_\epsilon$ for some $\epsilon \in (0,1]$ and is absolutely continuous with respect to Lebesgue measure, then u is $C^{1,\alpha}$ in the interior of Ω for some $0 < \alpha < 1$.*

Before proving this theorem, we make a few remarks. First we explain what it means to say that a function is $C^{1,\alpha}$ at a point. By definition, a function u is $C^{1,\alpha}$ on an open set if it is differentiable there and its gradient is Hölder continuous. We want to expand this definition to include functions that may not be differentiable everywhere.

We say that a function u is $C^{1,\alpha}$ at the point x_0, where x_0 is a minimum for u if $0 \le u(x) - u(x_0) \le C|x - x_0|^{1+\alpha}$ holds for all x. This is a sensible generalization of Hölder continuity of the gradient, since if u was $C^{1,\alpha}$ on a domain containing x_0, and x_0 is a minimum for u, this estimate would be satisfied. Indeed, in this case we would have (by Taylor)

$$0 \le u(x) - u(x_0) = \nabla u(\xi)\cdot(x - x_0) = \nabla u(\xi)\cdot(x - x_0) - \nabla u(x_0)\cdot(x - x_0)$$

for some ξ lying between x and x_0. Taking absolute value, we get

$$|u(x) - u(x_0)| \le |\nabla u(\xi) - \nabla u(x_0)||x - x_0| \le C|\xi - x_0|^\alpha|x - x_0| \le C|x - x_0|^{1+\alpha}.$$

Suppose now that u is convex and satisfies the inequality $|u(x) - l_{x_0}(x)| \le C|x - x_0|^{\alpha+1}$ for any supporting hyperplane $l_{x_0}(x)$ to u at x_0, at every point of the domain x_0. Then we claim that u is $C^{1,\alpha}$. We first show that this inequality implies that at each point x_0, u has a unique supporting hyperplane. Let $l^1_{x_0}$ and $l^2_{x_0}$ be two support planes to u at x_0. Then for any x:

$$|l^1_{x_0}(x) - l^2_{x_0}(x)| \le |u(x) - l^1_{x_0}(x)| + |u(x) - l^2_{x_0}(x)| \le 2C|x - x_0|^{1+\alpha}.$$

Write $l^1_{x_0}(x) = u(x_0)+p_1\cdot(x-x_0)$ and $l^2_{x_0}(x) = u(x_0)+p_2\cdot(x-x_0)$. Substituting these expressions into the previous inequality we obtain $|(p_1-p_2)\cdot(x-x_0)| \leq 2C|x-x_0|^{1+\alpha}$. By choosing x close to x_0 and such that $(p_1-p_2)\cdot(x-x_0) = |p_1-p_2||x-x_0|$, we see that this is possible only if $p_1 = p_2$. This means that there is exactly one supporting hyperplane to u at each point x_0. Let this hyperplane be defined by the vector $p(x_0) = \nabla u(x_0)$. Then from the hypothesis we have

$$|u(x) - u(x_0) - \nabla u(x_0)\cdot(x-x_0)| \leq C|x-x_0|^{1+\alpha}.$$

By dividing both sides by $|x-x_0|$, we get

$$\frac{|u(x) - u(x_0) - \nabla u(x_0)\cdot(x-x_0)|}{|x-x_0|} \leq C|x-x_0|^{\alpha}.$$

In other words, u is differentiable at x_0. We can repeat the same argument for any point in the domain, and conclude that u is differentiable everywhere. The next step is to show that u is $C^{1,\alpha}$. Fix any two points x and x_0. Then $|(\nabla u(x) - \nabla u(x_0))\cdot(x-x_0)|$

$$= |u(x) - \nabla u(x_0)\cdot(x-x_0) - u(x_0) + u(x_0) - \nabla u(x)\cdot(x_0-x) - u(x)|$$

$$\leq |u(x) - \nabla u(x_0)\cdot(x-x_0) - u(x_0)| + |u(x_0) - \nabla u(x)\cdot(x_0-x) - u(x)|$$

$$\leq C|x-x_0|^{1+\alpha} + C|x-x_0|^{1+\alpha}.$$

Therefore $|\nabla u(x) - \nabla u(x_0)||x-x_0| \leq 2C|x-x_0|^{1+\alpha}$, or $|\nabla u(x) - \nabla u(x_0)| \leq 2C|x-x_0|^{\alpha}$. This completes the proof of the claim.

Proof of Theorem 3.4.6 The proof proceeds in a sequence of steps.

Step 1 If Ω is normalized and $\min_\Omega u = u(x_0) = -1$, then u is $C^{1,\alpha}$ at x_0.

The point x_0 where the minimum is attained is unique by the result on extremal points. By Theorem 2.4.2 and Proposition 2.4.4, $dist(x_0, \partial\Omega) > \rho$, where ρ depends on n, ϵ, and the D_ϵ constant. From this we see that $B_\rho(x_0) \subset \Omega$. Let $x \in \Omega$, $x \neq x_0$. Then there exists a $k \geq 1$ such that $2^{-k} \leq u(x) - u(x_0) \leq 2^{-k+1}$. Then since $u(x_0) = -1$, we get that $u(x) \geq -(1-2^{-k})$.

so $x \notin \Omega_{\frac{1}{2^k}}$. So by Lemma 3.4.5. $x \notin (2\nu)^{-k}B_\rho(x_0) = B_{\frac{\rho}{(2\nu)^k}}(x_0)$. and hence $|x - x_0| \geq \rho(2\nu)^{-k}$.

We can take $\nu > \frac{1}{2}$. so $\nu = 2^{-\theta}$ for some $\theta \in (0, 1)$. Hence. $|x - x_0| \geq \rho(2^{1-\theta})^{-k} = \rho(2^{-k})^{1-\theta}$. Since $u(x) - u(x_0) < 2^{-k+1}$. we see that $2^{-k} > \frac{u(x)-u(x_0)}{2}$. By raising both sides of the last inequality to the power $1 - \theta$. and then multiplying by ρ. we get the inequality

$$|x - x_0| \geq \rho(2^{-k})^{1-\theta} > \rho\left(\frac{u(x) - u(x_0)}{2}\right)^{1-\theta}.$$

From this. we obtain $0 \leq u(x) - u(x_0) \leq 2\left(\frac{1}{\rho}\right)^{\frac{1}{1-\theta}} |x - x_0|^{\frac{1}{1-\theta}}$. proving the claim for this step. since $\frac{1}{1-\theta} > 1$.

Step 2 If Ω is not necessarily normalized. and $\min_\Omega u$ is not necessarily -1. then u is $C^{1,\alpha}$ at its minimum x_0.

Let T normalize Ω ($Tx = Ax + b$ for an invertible matrix A and some $b \in \mathbb{R}^n$). and define $u^*(y) = |\min_\Omega u|^{-1}u(T^{-1}y)$. Then $Mu^* \in D_\epsilon(C)$ in $\Omega^* = T(\Omega)$ and $\min_{\Omega^*} u^* = -1$ and this minimum is attained at Tx_0. Then by Step 1 we have that.

$$0 \leq u^*(y) - u^*(Tx_0) \leq C(\epsilon, n, D_\epsilon\text{constant})|y - Tx_0|^{\alpha+1}.$$

Let $y = Tx$. Then:

$$0 \leq u(x) - u(x_0) \leq C(\epsilon, n, D_\epsilon\text{constant})|\min_\Omega u||Tx - Tx_0|^{\alpha+1} \text{ or}$$

$$0 \leq u(x) - u(x_0) \leq C||A||^{\alpha+1}|\min_\Omega u||x - x_0|^{\alpha+1}.$$

Step 3 If Ω is normalized. then u is $C^{1,\alpha}$ in the interior of Ω.

We prove that if $dist(\bar{x}, \partial\Omega) \geq \rho$. then

$$|u(x) - l_{\bar{x}}(x)| \leq C(n, \epsilon, \rho, D_\epsilon const, |\min_\Omega u|)|x - \bar{x}|^{\alpha+1}.$$

where $l_{\bar{x}}(x)$ is any support plane to u at $\bar{x}$.

By Theorem 3.4.1. there exists $\rho_0 = \rho_0(n, \epsilon, \rho, D_\epsilon const, |\min_\Omega u|)$ such that

$$\Omega_{\bar{x},\rho_0} \equiv \{x \in \Omega : u(x) < l_{\bar{x}}(x) + \rho_0\} \subset \{x \in \Omega : u(x) < -\bar{C}\rho\} \Subset \Omega.$$

where $\bar{C} = \bar{C}(n, \epsilon, D_\epsilon const, |\min_\Omega u|)$. Let T normalize $\Omega_{\bar{x},\rho_0}$ and let $v(x) = u(x) - l_{\bar{x}}(x) - \rho_0$. Then $v|_{\partial\Omega_{\bar{x},\rho_0}} = 0$ and $v(\bar{x}) = \min_{\Omega_{\bar{x},\rho_0}} v = -\rho_0$. Then by Step 2,

$$0 \leq v(x) - v(\bar{x}) \leq C(\epsilon, n, D_\epsilon const)|\min_{\Omega_{\bar{x},\rho_0}} v|||A||^{1+\alpha}|x - \bar{x}|^{1+\alpha}.$$

Then since $|\min_{\Omega_{\bar{x},\rho_0}} v| = \rho_0$, the claim holds if $||A||$ can be dominated as claimed. As on p.98 of [3], $||A|| = \max \lambda_i^{-1}$, where the λ_i are the lengths of the axes of the minimum ellipsoid of $\Omega_{\bar{x},\rho_0}$, and $\det A = (\lambda_1)^{-1} \cdots (\lambda_n)^{-1}$.

Define $u^*(x) = |\det A|^{\frac{2}{n}} v(T^{-1}x)$. We claim that $|\min_{T(\Omega_{\bar{x},\rho_0})} u^*|^n$ is comparable to $Mu^*(\frac{1}{2}T(\Omega_{\bar{x},\rho_0}))$. Indeed by Proposition 2.4.4 and the D_ϵ condition, we have that $|\min_{T(\Omega_{\bar{x},\rho_0})} u^*|^n \leq C \int_{T(\Omega_{\bar{x},\rho_0})} \delta(x, T(\Omega_{\bar{x},\rho_0}))^{1-\epsilon} dMu^* \leq Mu^*(\frac{1}{2}T(\Omega_{\bar{x},\rho_0}))$. For the other inequality, we again use Proposition 2.4.4 and also use the fact that if $x \in \frac{1}{2}T(\Omega_{\bar{x},\rho_0})$, then $\delta(x, T(\Omega_{\bar{x},\rho_0})) \geq C_n$. More precisely,

$$|\min_{T(\Omega_{\bar{x},\rho_0})} u^*|^n \geq C \int_{T(\Omega_{\bar{x},\rho_0})} \delta(x, T(\Omega_{\bar{x},\rho_0}))^{1-\epsilon} dMu^*$$

$$\geq C \int_{\frac{1}{2}T(\Omega_{\bar{x},\rho_0})} \delta(x, T(\Omega_{\bar{x},\rho_0}))^{1-\epsilon} dMu^* \geq CMu^*(\frac{1}{2}T(\Omega_{\bar{x},\rho_0})).$$

We have that $|\min_{T(\Omega_{\bar{x},\rho_0})} u^*|^n = |\det A|^2 \rho_0^n$ and (by (2.1))

$$Mu^*(\frac{1}{2}T(\Omega_{\bar{x},\rho_0})) = |\det A| Mu(T^{-1}(\frac{1}{2}T(\Omega_{\bar{x},\rho_0})).$$

This implies that

$$|\det A|^2 \rho_0^n \approx |\det A| Mu \left(T^{-1}\left(\frac{1}{2}T(\Omega_{\bar{x},\rho_0})\right)\right)$$

or

$$|\det A| \leq C\rho_0^{-n} Mu(T^{-1}(\frac{1}{2}T(\Omega_{\bar{x},\rho_0})) \leq C\rho_0^{-n} Mu(\{x \in \Omega : u(x) < -C\rho\}).$$

Since $\lambda_i \leq 1$, $||A|| = \max\{\lambda_i^{-1}\} = \lambda_j^{-1} \leq (\lambda_1)^{-1} \cdots (\lambda_n)^{-1} = \det A$. Therefore $||A||$ can be estimated by n, ϵ, the D_ϵ constant, ρ, $|\min u|$ and $Mu(\{x \in \Omega : u(x) < -C\rho\})$. By the remark preceding Step 1, we see that this establishes the claim for this step.

Step 4 If Ω is not normalized, then u is $C^{1,\alpha}$ in the interior of Ω.

Let T be an affine transformation that normalizes Ω, and define $u^*(y) = u(T^{-1}y)$ for $y \in T(\Omega)$. Now apply Step 3 to u^* in the normalized domain $T(\Omega)$. The constant appearing in the inequality will also depend on $||T||$, which depends on the eccentricity and volume of Ω. □

CHAPTER 4

ESTIMATES FOR THE PARABOLIC MONGE-AMPÈRE EQUATION

In this chapter we show that the estimates of Jerison also hold for parabolically convex solutions of the parabolic Monge-Ampère equation $-u_t \det D_x^2 u = f$ on bowl-shaped domains. First we will need to introduce some notation and define some terminology.

Let $D \subset \mathbb{R}^{n+1}$ and let $t \in \mathbb{R}$. Then denote

$$D(t) = \{x \in \mathbb{R}^n : (x, t) \in D\}.$$

The set D is said to be a bowl-shaped domain if $D(t)$ is convex for every t and $D(t_1) \subset D(t_2)$ whenever $t_1 \leq t_2$. Now suppose that D is bounded and let $t_0 = \inf\{t : D(t) \neq \emptyset\}$. The parabolic boundary of D is defined to be

$$\partial_p D = (\bar{D}(t_0) \times \{t_0\}) \cup \left(\bigcup_{t \in \mathbb{R}} (\partial D(t) \times \{t\}) \right).$$

For a bowl-shaped domain D we define the set D_{t_0} to be $D_{t_0} = D \cap \{(x, t) : t \leq t_0\}$.

A function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $u = u(x,t)$, is called parabolically convex if it is continuous, convex in x and non-increasing in t.

We next define the parabolic normal map and parabolic Monge-Ampère measure. As in the elliptic case, this will lead to the notion of weak solution for this operator. Let $D \subset \mathbb{R}^{n+1}$ be an open, bounded bowl-shaped domain, and u be a continuous real-valued function on D. The parabolic normal mapping of u at a point (x_0, t_0) is the set-valued function

$$P_u(x_0, t_0) = \{(p, h) : u(x,t) \geq u(x_0, t_0) + p \cdot (x - x_0) \text{for all } x \in D(t) \text{ with } t \leq t_0,$$
$$h = p \cdot x_0 - u(x_0, t_0)\}.$$

As before, the parabolic normal mapping of a set $E \subset D$ is defined to be the union of the parabolic normal maps of each point in the set. The family of subsets E of D for which $P_u(E)$ is Lebesgue measurable is a Borel σ-algebra and the map taking such a set E to its Lebesgue measure is a measure, called the parabolic Monge-Ampère measure associated to the function u. In what follows, the notation $|E|_k$ denotes the Lebesgue measure of the set E in $\mathbb{R}^k$.

There is a parabolic analog of Aleksandrov's estimate due to Gutiérrez and Huang [G-H p. 1463].

Theorem 4.0.1 *Let $D \subset \mathbb{R}^{n+1}$ be an open bounded bowl-shaped domain, and let $u \in C(\bar{D})$ be a parabolically convex function with $u = 0$ on $\partial_p D$. If $(x_0, t_0) \in D$, then*

$$|u(x_0, t_0)|^{n+1} \leq C_n dist(x_0, \partial D(t_0)) diam(D(t_0))^{n-1} |P_u(D_{t_0})|_{n+1}$$

where C_n is a dimensional constant.

We now prove the following parabolic versions of Jerison's estimates. The proofs follow the same outline as those given in Chapter 2.

Lemma 4.0.2 *(Compare with Lemma 2.4.3)Let D be a bounded, open bowl-shaped domain in $\mathbb{R}^{n+1}$. Suppose $u \in C(\bar{D})$ is parabolically convex, $u|_{\partial_p(D)} = 0$. Then there exists a dimensional constant C_n such that*

$$|u(x_0, t_0)|^{n+1} \leq C_n \delta(x_0, D(t_0)) |D(t_0)|_n |P_u(D_{t_0})|_{n+1}$$

for any $(x_0, t_0) \in D$, *where* $\delta(x_0, D(t_0))$ *is the normalized distance from* (x_0, t_0) *to the boundary of the n-dimensional convex set* $D(t_0)$.

Proof $D(t_0)$ is a bounded convex subset of a copy of $\mathbb{R}^n$. Let T be an affine transformation of $\mathbb{R}^n$ that normalizes $D(t_0)$. Define $\tilde{T} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by $\tilde{T}(x, t) = (Tx, t)$. Then $\tilde{T}(D_{t_0}) \subset B_1(0) \times [K, t_0]$ for some $K < t_0$. Let $v(z) = u(\tilde{T}^{-1}z)$ for $z \in \tilde{T}(D)$.

Then $\tilde{T}(D)$ is a bowl-shaped domain, v is continuous on the closure of $\tilde{T}(D)$, is parabolically convex, and zero on $\partial_p\tilde{T}(D)$.

Now apply the parabolic Aleksandrov estimate (Theorem 4.0.1) to v in $\tilde{T}(D)$ to obtain

$$|u(x_0, t_0)|^{n+1} = |v(\tilde{T}(x_0, t_0))|^{n+1}$$

$$\leq C_n dist(Tx_0, \partial\tilde{T}(D(t_0)))[diam(\tilde{T}(D(t_0)))]^{n-1}|P_v(\tilde{T}(D_{t_0}))|_{n+1}. \qquad (4.1)$$

We next claim that $|P_v(\tilde{T}(D_{t_0}))|_{n+1} = |\det T^{-1}|\ |P_u(D_{t_0})|_{n+1}$. Let $p \in \partial u(x_0)$. Then

$$u(x, t_0) \geq u(x_0, t_0) + p \cdot (x - x_0)$$

for all $x \in D(t_0)$. Since u is non-increasing in t,

$$u(x, t) \geq u(x, t_0) \geq u(x_0, t_0) + p \cdot (x - x_0)$$

for all $x \in D(t)$ and $t \leq t_0$, so $(p, h) \in P_u(x_0, t_0)$ where $h = p \cdot x_0 - u(x_0, t_0)$. If $p \notin \partial u(x_0)$, $(p, h) \notin P_u(x_0, t_0)$, so therefore, $p \in \partial u(x_0)$ if and only if $(p, h) \in P_u(x_0, t_0)$.

We also know that $p \in \partial u(x_0)$ if and only if $(T^{-1})^t p \in \partial v(Tx_0)$. Let $y = Tx$. Then as above, for $t \leq t_0$,

$$v(y, t) \geq v(y, t_0) + (T^{-1})^t p \cdot (y - Tx_0).$$

Hence, $(T^{-1})^t p \in \partial v(Tx_0)$ if and only if $((T^{-1})^t p, \tilde{h}) \in P_v(Tx_0, t_0)$, where $\tilde{h} = (T^{-1})^t p \cdot Tx_0 - v(Tx_0, t) = p \cdot x_0 - u(x_0) = h$. In other words, $(p, h) \in P_u(x_0, t_0)$ if and only if $((T^{-1})^t p, h) \in P_v(Tx_0, t_0)$.

We also have $(\tilde{T}^{-1})^t(p,h) = ((T^{-1})^t p, h)$ which implies that $(\tilde{T}^{-1})^t P_u(E) = P_v(\tilde{T}(E))$ for any Borel set $E \subset D$. In particular, $(\tilde{T}^{-1})^t P_u(D_{t_0}) = P_v(\tilde{T}(D_{t_0}))$. This implies that

$$|\det \tilde{T}^{-1}|\, |P_u(D_{t_0})| = |P_v(\tilde{T}(D_{t_0}))|,$$

but $\det \tilde{T}^{-1} = \det T^{-1}$.

Then using this last claim, Lemma 2.3.3, and the fact that $|\det T^{-1}| \leq C(n)|D(t_0)|_n$, we continue from (4.1) and prove the claimed estimate:

$$\begin{aligned} &\leq C_n\delta(Tx_0, T(D(t_0)))|P_v(\tilde{T}(D_{t_0}))|_{n+1} \\ &= C_n\delta(x_0, D(t_0))|P_v(\tilde{T}(D_{t_0}))|_{n+1} \\ &= C_n\delta(x_0, D(t_0))|\det T^{-1}||P_u(D_{t_0})|_{n+1} \\ &\leq C_n\delta(x_0, D(t_0))|D(t_0)|_n|P_u(D_{t_0})|_{n+1}. \end{aligned}$$

□

Lemma 4.0.3 *Let $0 < \epsilon < 1$. Let E be a bounded open bowl-shaped domain in $\mathbb{R}^{n+1}$, such that $E \subset B_1(0) \times (-\infty, \infty)$. Suppose $u \in C(\bar{E})$ is parabolically convex and zero on $\partial_p E$. Let M be the parabolic Monge-Ampère measure associated to u. Then there exists $C = C(\epsilon, n)$ such that*

$$|u(x_0, t_0)|^{n+1} \leq C\delta((x_0, t_0), E(t_0))^\epsilon \int_{E_{t_0}} \delta((x, t_0), E(t_0))^{1-\epsilon} dM(x, t).$$

Proof Without loss of generality, multiply u by a positive constant so that $u(x_0, t_0) = -1$. Let $s_k = s2^{-k\beta}$ where s and β are positive and chosen to satisfy $\beta(n+1) \leq \epsilon$ and $\sum_{k=1}^\infty s_k \leq \frac{1}{2}$. Let A denote the quantity

$$\delta((x_0, t_0), E(t_0))^\epsilon \int_{E_{t_0}} \delta((x, t_0), E(t_0))^{1-\epsilon} dM(x, t).$$

We want to show that $A \geq C(s)$, since s depends on ϵ and $|u(x_0, t_0)| = 1$. Let $E_k = \{(x, t) \in E : u(x, t) \leq \lambda_k = -1 + s_1 + \cdots + s_k\}$. Define $E_0 = \{(x, t) \in E : u(x, t) \leq -1\}$. Notice that $E_k \subset E_{k+1}$ for every k. The set E_k is bowl-shaped and $u|_{\partial_p E_k} = \lambda_k$. Fix t and let $\delta_k(t) = dist(\partial E_k(t), \partial E(t))$.

Since $\delta_k(t) \not\to 0$ as $k \to \infty$ (if $\delta_k(t) \to 0$, then u would be smaller than $-\frac{1}{2}$ somewhere on the parabolic boundary of E), choose k to be the smallest nonnegative integer for which $\delta_{k+1}(t) > \frac{1}{2}\delta_k(t)$.

Let $(x_k, t) \in \partial E_k(t)$ be a point closest to $\partial E(t)$. Then as in the proof of Theorem 2.4.2, we have that

$$dist((x_k, t), \partial E_{k+1}(t)) < \frac{1}{2}\delta_k(t) < \delta_{k+1}(t). \tag{4.2}$$

Now apply Lemma 4.0.2 to the function $u(x,t) - \lambda_{k+1}$ on the set E_{k+1} to get

$$|u(x_k, t) - \lambda_{k+1}|^{n+1} \leq C_n \delta((x_k, t), E_{k+1}(t))|E_{k+1}(t_0)|_n M((E_{k+1})_t).$$

The point $(x_k, t) \in \partial E_k(t)$, so $u(x_k, t) = \lambda_k$ and $|u(x_k, t) - \lambda_{k+1}| = |\lambda_k - \lambda_{k+1}| = s_{k+1}$. Thus,

$$s_{k+1}^{n+1} \leq C_n \delta((x_k, t), E_{k+1}(t))|E_{k+1}(t)|_n M((E_{k+1})_t). \tag{4.3}$$

Let L_t be a shortest segment from (x_k, t) to $\partial E_{k+1}(t)$ and let $(z, t) \in \partial E_{k+1}(t)$ be the other endpoint. Let ρ denote $|L_t| = |x_k - z|$.

The set $E_{k+1}(t)$ is convex, so we can proceed as in the proof Theorem 2.4.2. The hyperplane Π (of dimension $n-1$) normal to L_t through (z, t) is a support plane for $E_{k+1}(t)$. Let Π' be the support plane parallel to Π on the opposite side of $E_{k+1}(t)$, so that $E_{k+1}(t)$ is contained between the two planes, and let $r = dist(\Pi, \Pi')$. Then since $E_{k+1}(t) \subset B_1(0) \times \{t\}$, there exists a constant $C = C(n)$ such that $|E_{k+1}(t)|_n \leq Cr$ (In fact C is the volume of the unit ball in $\mathbb{R}^{n-1}$).

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation normalizing $E_{k+1}(t)$. Then $dist(T(\Pi), T(\Pi')) \approx 1$ and $dist((Tx_k, t), T(\Pi)) \approx \frac{\rho}{r}$. Define $\tilde{T} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by $\tilde{T}(x, t) = (Tx, t)$.

By Lemma 2.3.3, we have:

$$\begin{aligned}\delta((x_k,t),E_{k+1}(t)) &= \delta((Tx_k,t),T(E_{k+1}(t)))\\ &\le Cdist((Tx_k,t),\partial T(E_{k+1}(t)))\\ &\le Cdist((Tx_k,t),T(\Pi))\\ &\le C\frac{\rho}{r}.\end{aligned}$$

Insert this into (4.3) to get

$$s_{k+1}^{n+1}\le C\frac{\rho}{r}|E_{k+1}(t)|_n M((E_{k+1})_t)\le C\rho M((E_{k+1})_t)<C\delta_{k+1}(t)M((E_{k+1})_t),$$

where the last inequality holds since $\rho = dist((x_k,t),\partial E_{k+1}(t)) < \delta_{k+1}(t)$ from (4.2).

Therefore we have:

$$s_{k+1}^{n+1}<C\delta_{k+1}(t)M((E_{k+1})_t). \tag{4.4}$$

By the choice of k, $\delta_{k+1}(t) < \delta_k(t) \le 2^{-k}\delta_0(t)$. For some values of t, $\delta_0(t)$ might not be defined. This is the case when $u > -1$ on $E(t)$. For $t = t_0$, $\delta_0(t)$ is defined. Since u is non-increasing in t, $\delta_0(t)$ is defined for any $t \ge t_0$. Take $t = t_0$.

Then $2^{-k}\delta_0(t_0) \le C2^{-k}\delta((x_0,t_0),E(t_0))$ for a dimensional constant C. This is true since $\delta_0(t_0) \le dist((x_0,t_0),\partial E(t_0))$ and the general fact that $dist(x,\partial E)\le (diam(E))\delta(x,E)$. In this case $diam(E(t_0)) \le 2$.

Therefore,

$$\begin{aligned}\delta_{k+1}(t_0)M((E_{k+1})_{t_0}) &= \delta_{k+1}(t_0)^{\epsilon}\delta_{k+1}(t_0)^{1-\epsilon}\int_{(E_{k+1})_{t_0}} dM(y,s)\\ &= \delta_{k+1}(t_0)^{\epsilon}\int_{(E_{k+1})_{t_0}}\delta_{k+1}(t_0)^{1-\epsilon}dM(y,s)\\ &\le 2^{-k\epsilon}\delta((x_0,t_0),E(t_0))^{\epsilon}\int_{(E_{k+1})_{t_0}}\delta_{k+1}(t_0)^{1-\epsilon}dM(y,s)\end{aligned}$$

$$\le 2^{-k\epsilon}\delta((x_0,t_0),E(t_0))^{\epsilon}\int_{(E_{k+1})_{t_0}}\delta((y,t_0),E(t_0))^{1-\epsilon}dM(y,s).$$

The last inequality holds since

$$\delta_{k+1}(t_0) = dist(\partial E_{k+1}(t_0), \partial E(t_0)) \leq dist((y, t_0), \partial E(t_0)) \leq \delta((y, t_0), E(t_0))$$

for all $y \in E_{k+1}(t_0)$. Then from (4.4) we obtain that

$$s_{k+1}^{n+1} \leq 2^{-k\epsilon}\delta((x_0, t_0), E(t_0))^{\epsilon} \int_{(E_{k+1})_{t_0}} \delta((y, t_0), E(t_0))^{1-\epsilon} dM(y, s) \leq 2^{-k\epsilon}A.$$

Now recall that

$$s_{k+1}^{n+1} = s^{n+1}2^{-(n+1)(k+1)\beta} \geq s^{n+1}2^{-\epsilon(k+1)}$$

since $\beta(n+1) \leq \epsilon$. Hence

$$s^{n+1}2^{-\epsilon(k+1)} \leq 2^{-k\epsilon}A \Rightarrow s^{n+1} \leq CA,$$

where C depends on ϵ, so $A \geq C(s)$ as desired. □

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