Comprehensive Examination in Geometry & Topology
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Part I. Solve three of the following problems.

I.1 Let M be a closed (compact without boundary) smooth manifold and $f: M \to M$ a smooth map. Say a fixed point x of f is *nondegenerate* if $df_x: T_xM \to T_xM$ does not have 1 as an eigenvalue. Say that f itself is *nondegenerate* if each of its fixed points is.

a) Let $\Delta = \{(x, x) \in M \times M\}$ and define $F: M \to M \times M$ by $F(x) = (x, f(x))$. Show that the fixed point x of f is nondegenerate if and only if F is transverse to Δ at $(x, x) \in M \times M$.

b) Prove that if $f: M \to M$ is nondegenerate, then it has finitely many fixed points.

I.2 Let M be a smooth n–manifold with boundary. Construct a smooth vector field X on M that is nowhere tangent to ∂M and points into M.

I.3 Let X and Y be path connected, locally path connected topological spaces whose universal covers are $\widetilde{X}$ and $\widetilde{Y}$, respectively. Prove or disprove the following statements:

a) If X is homotopy equivalent to Y, then $\widetilde{X}$ is homeomorphic to $\widetilde{Y}$.

b) If X is homeomorphic to Y, then $\widetilde{X}$ is homeomorphic to $\widetilde{Y}$.

c) If X is homotopy equivalent to Y, then $\widetilde{X}$ is homotopy equivalent to $\widetilde{Y}$.

I.4 Construct a cell complex X whose singular homology groups are as follows:

$$H_i(X) = \begin{cases} \mathbb{Z}^2 & i = 0, \\ \mathbb{Z} \oplus \mathbb{Z}_2 & i = 1, \\ \mathbb{Z}_7 & i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Part II. Solve two of the following problems.

II.1 Let $\omega = P dx + Q dy$ be a closed 1–form on $\mathbb{R}^2$. Given a point $(x_0, y_0) \in \mathbb{R}^2$, let γ_+ be the piecewise linear path that follows a vertical segment from $(0,0)$ to $(0, y_0)$ and then a horizontal segment from $(0, y_0)$ to (x_0, y_0). Let γ_- be the piecewise linear path that follows a horizontal segment from $(0,0)$ to $(x_0, 0)$ and then a vertical segment from $(x_0, 0)$ to (x_0, y_0).

a) Prove that $\int_{\gamma_+} \omega = \int_{\gamma_-} \omega$.

b) Define a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ by setting $f(x_0, y_0) = \int_\gamma \omega$, where γ is one of the above paths from $(0,0)$ to (x_0, y_0). Prove that $df = \omega$, hence ω is exact.

II.2 Let R be the wedge of two circles, which are oriented and labeled a and b. Let $f\colon R \to R$ be a homeomorphism such that $f(a) = b$ and $f(b) = a$, and let X_f be the mapping torus of f:

$$X_f = \frac{R \times [0,1]}{(x,1) \sim (f(x), 0)}$$

a) Describe a cell complex structure on X_f, and use it to compute $\pi_1(X_f)$.

b) Prove that $\pi_1(X_f)$ is not abelian.

II.3 Let T be a torus with an open disk removed. Let X be the topological space obtained by taking 3 copies of T and identifying all of their boundaries by a homeomorphism.

a) Compute the singular homology groups $H_i(X)$.

b) Prove that X is not homotopy equivalent to any orientable closed manifold.