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Characterizations of Matrices Enjoying the Perron-Frobenius Property and Generalizations of M-Matrices Which May Not Have Nonnegative Inverses

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A Dissertation
Submitted to
the Temple University Graduate Board

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by
Abed Elhashash
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ABSTRACT

Characterizations of Matrices Enjoying the Perron-Frobenius Property and Generalizations of M-Matrices Which May Not Have Nonnegative Inverses

Abed Elhashash
DOCTOR OF PHILOSOPHY

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Professor Daniel B. Szyld, Chair

General matrices with a positive dominant eigenvalue and a corresponding nonnegative eigenvector are studied. Such matrices are said to possess the Perron-Frobenius property. The latter property is naturally enjoyed by nonnegative matrices and has a wide variety of applications. In this dissertation, general matrices, which are not necessarily nonnegative, that possess the Perron-Frobenius property are analyzed. Several characterizations of matrices having the Perron-Frobenius property are presented: spectral, combinatorial, and geometric characterizations. In some cases, a full characterization is obtained, while in others only certain aspects are studied. In addition, some combinatorial, topological and spectral properties of matrices enjoying the Perron-Frobenius property are presented and the similarity transformations preserving the Perron-Frobenius property are completely described. Furthermore, generalizations of M-matrices are studied, including the new class of GM-matrices. Matrices in the latter class are of the form $sI - B$ where B and its transpose possess the Perron-Frobenius property and the spectral radius of B is less than s. Results analogous to those known for M-matrices are demonstarted. Also, various splittings of GM-matrices are studied along with conditions for their convergence.

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To my parents,

for their love and encouragement.

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LIST OF SYMBOLS

A^T	The transpose of the matrix or the vector A.
$\rho(A)$	The spectral radius of the matrix A.
PF_n	The collection of matrices A for which both A and A^T possess the strong Perron-Frobenius property.
WPF_n	The collection of matrices A for which both A and A^T possess the Perron-Frobenius property.
$\sigma(A)$	The spectrum of the matrix A.
$mult_\lambda(A)$	The algebraic multiplicity of an eigenvalue λ of A.
$index_\lambda(A)$	The index of an eigenvalue λ of A.
$\mathcal{N}(A)$	The nullspace of the matrix A.
$E_\lambda(A)$	The ordinary eigenspace of an eigenvalue λ of A.
$G_\lambda(A)$	The generalized eigenspace of an eigenvalue λ of A.
$J_s(\lambda)$	The $s \times s$ Jordan block corresponding to λ.
I_s	The $s \times s$ identity matrix.
$A \oplus \tilde{A}$	The direct sum of the matrix A with the matrix $\tilde{A}$.
$Box(\lambda)$	The Jordan box of an eigenvalue λ.
$\mathbb{C}^{n\times n}$	The collection of $n \times n$ complex matrices.
$\mathbb{R}^{n\times n}$	The collection of $n \times n$ real matrices.
$GL(n,\mathbb{R})$	The collection of $n \times n$ invertible real matrices.
$GL(n,\mathbb{C})$	The collection of $n \times n$ invertible complex matrices.
$\bigoplus_{j=1}^k V_j$	The direct sum of the vector spaces $V_1, \ldots, V_k$.
e_i	The ith standard unit vector of $\mathbb{R}^n$.
$G(A)$	The graph of the matrix A.
$A[\alpha,\beta]$	The submatrix of A whose rows are indexed by α and whose columns are indexed by β.
$A[\alpha]$	The principal submatrix of A whose rows and columns are indexed by α.
A^+	The positive part of the matrix A.
A^-	The negative part of the matrix A.
$Even_k$	The collection of maps $\tau : \{1,2,\ldots,k\} \to \{+,-\}$ having the the cardinality of $\tau^{-1}\{-\}$ even.

Odd_k	The collection of maps $\tau : \{1, 2, \ldots, k\} \rightarrow \{+, -\}$ having the cardinality of $\tau^{-1}\{-\}$ odd.
$\succ$	The symbol of weight dominance of one graph over the other.
$\prod_{i=1}^{k} G(A_i)$	The product of the graphs $G(A_i)$ in the order $i = 1, 2, \ldots, k$.
A_κ	The matrix obtained by rearranging the columns and rows of A according to the ordered partition κ.
$Hull(A)$	The convex hull of the transposed rows of A.
$H(y)$	The half-space consisting of vectors making an acute or right angle with vector y.
A_{i*}	The ith row of the matrix A.
A_{*j}	The jth column of the matrix A.
$R(A)$	The reduced graph of the matrix A.
$\overline{R(A)}$	The transitive closure of the reduced graph of the matrix A.
D_A	The denominator set of the matrix A.
P_A	The set of problematic powers of the matrix A.
N_A	The set of nice powers of the matrix A.
$\lceil \delta \rceil$	The smallest integer greater than or equal to δ.
$\lfloor \delta \rfloor$	The largest integer less than or equal to δ.
$\|\|A\|\|_2$	The spectral norm of the matrix A.
$\|\|v\|\|_2$	The l_2-norm of the vector v.
$diag(v)$	The diagonal matrix whose (i, i)-entry is the ith entry of vector v.
$Re(\lambda)$	The real part of the complex number λ.
$Im(\lambda)$	The imaginary part of the complex number λ.

CHAPTER 1

INTRODUCTION

1.1 Background

We say that a real matrix A is nonnegative (positive, nonpositive, negative, respectively) if it is entry-wise nonnegative (positive, nonpositive, negative, respectively) and we write $A \geq 0$ ($A > 0$, $A \leq 0$, $A < 0$, respectively). This notation and nomenclature is also used for vectors. In 1907, Perron [33] proved that a positive matrix has the following properties:

1. Its spectral radius is a simple positive eigenvalue.
2. The eigenvector corresponding to the spectral radius can be chosen to be positive (called a Perron vector).
3. No other eigenvalue has a positive eigenvector.
4. The spectral radius is a strictly increasing function of the matrix entries.

Later in 1912, this result was extended by Frobenius [15] to nonnegative irreducible matrices and consequently to nonnegative matrices, using a perturbation argument. In the latter case, there exists a nonnegative dominant eigenvalue with a corresponding nonnegative eigenvector. These results, known now as the Perron-Frobenius theory, have been widely applied to problems with nonnegative matrices, and also with M-matrices and H-matrices;

see, e.g., the monographs [2], [22], [44], [49]. Applications include stochastic processes [44], Markov chains [47], population models [28], solutions of partial differential equations [1], and asynchronous parallel iterative methods [16], among others.

A natural question is: which matrices other than the nonnegative ones have some of the properties 1–4? It turns out that eventually nonnegative and eventually positive matrices do satisfy some of the properties 1-4. A matrix A is said to be eventually nonnegative (positive) if $A^k \geq 0$ ($A^k > 0$, respectively) for all $k \geq k_0$ for some positive integer k_0.

Friedland [14] introduced eventually nonnegative matrices and showed that for such matrices the spectral radius is an eigenvalue. Other authors, [19], [20], [31], [40], [51], [52], studied some of the properties in the Perron-Frobenius theory exhibited by eventually positive and eventually nonnegative matrices, while others studied the combinatorial properties of nonnegative and eventually nonnegative matrices, [5], [21], [42]. In particular, Carnochan Naqvi and McDonald [5] studied the combinatorial properties of eventually nonnegative matrices whose index is 0 or 1 by considering their Frobenius normal forms, whereas Eschenbach and Johnson [9] gave combinatorial characterization of matrices that have their spectral radius as an eigenvalue.

In a series of papers, [26], [27], [48], Tarazaga and his co-authors extended the Perron-Frobenius theory to matrices with some negative entries by studying closed cones of matrices whose central ray is the matrix having all entries equal to one and by giving the maximal angles in which eigenvalue dominance and eigenvector positivity are retained. In [27], limitations of extending the Perron-Frobenius theory outside the cone of positive matrices are discussed. In [38], theorems of the Perron-Frobenius type are proved for quasi-compact and quasi-positive operators on cones in Banach spaces, while, in [39], [43], only cones of positive semidefinite matrices are considered.

We also mention the work of Rump [36], [37], who generalized the concept of a positive dominant eigenvalue and defined a new quantity for real matrices known as the *sign-real spectral radius* for which he derived various properties similar to those in the Perron-Frobenius theory.

We call a column or a row vector v semipositive if v is nonzero and nonnegative. Likewise, if v is nonzero and nonpositive, then we call v seminegative. We denote the spectral radius of a matrix A by $\rho(A)$. Following [31], we say that a real matrix A possesses the *Perron-Frobenius property* if A has a positive dominant eigenvalue with a corresponding nonnegative eigenvector. We say that A possesses the *strong Perron-Frobenius property* if A has a simple, positive, and strictly dominant eigenvalue with a positive eigenvector. If a matrix A satisfies $Av = \rho(A)v$ for some semipositive vector v, then we say that A has a *Perron-Frobenius Eigenpair* $(\rho(A), v)$. In the latter case, if $\rho(A) > 0$, we call v a *right Perron-Frobenius eigenvector* for A. Similarly, if $\rho(A) > 0$, and $w^T A = \rho(A)w^T$ for some semipositive vector w, then we call w a *left Perron-Frobenius eigenvector* for A.

Following [26], we let PF_n denote the collection of $n \times n$ real matrices whose spectral radius is a simple, positive, and strictly dominant eigenvalue having positive right and left eigenvectors, or equivalently, the collection of matrices A for which both A and its transpose possess the strong Perron-Frobenius property; see, e.g., [26], [31], [52]. Similarly, WPF_n denotes the collection of $n \times n$ real matrices whose spectral radius is a positive eigenvalue having nonnegative left and right eigenvectors. Equivalently, WPF_n is the collection of matrices A for which both A and its transpose possess the Perron-Frobenius property

One of the main goals of this dissertation is to characterize as much as possible the collection of eventually nonnegative matrices, PF_n and WPF_n. As we shall see, in some cases, a full characterization is obtained, while in others only certain aspects are studied. New characterizations of PF_n and WPF_n are given in terms of the spectral projector. Combinatorial characterizations of eventually nonnegative and eventually positive matrices are given in terms of walks in the graph and in terms of products and unions of graphs. Also, convex sets determined by the rows and columns of a matrix are used to characterize eventually nonnegative and eventually positive matrices. One of the questions answered in this dissertation is: which similarity transformations leave invariant the sets WPF_n, PF_n, and the collection of matrices with the Perron-Frobenius property? Another result pertaining to the recent work of Tarazaga and his coauthors [26], [27], [48], is showing that the set of eventually positive symmetric matrices extends beyond a known cone centered at the matrix of all ones. Moreover, topological aspects of WPF_n, theorems that are counterparts to those known for nonnegative matrices, and some applications are presented.

Another aspect of this dissertation is the presentation of various generalizations of the class of M-matrices and the proof of results that are counterparts to those known for M-matrices. The class of GM-matrices, which generalizes the class of M-matrices using the Perron-Frobenius property, is introduced. Also other classes, such as the class of EM- and pseudo-M-matrices, which generalize the class of M-matrices using eventual nonnegativity and eventual positivity, respectively, are studied. Complete characterizations of nonsingular GM- and pseudo-M-matrices are given. As a result, a spectral characterization of inverse GM-matrices is established. The latter partially answers the question: which nonnegative matrices are inverse M-matrices? Moreover, a characterization of M-matrices using positive stability on the class of Z-matrices is presented for GM-matrices using the generalized Z-matrices, the GZ-matrices. The latter are introduced in this dissertation. Other generalizations of this type for GM- and pseudo-M-matrices are proved. Furthermore, some combinatorial properties of EM-matrices are studied.

New splittings for an arbitrary nonsingular matrix and for a GM-matrix are introduced in this dissertation. One of them, the splitting having the Perron singular property, is a splitting for an arbitrary nonsingular matrix. The other splittings are the G-regular splitting, the GM-splitting, the overlapping splitting, and the commuting bounded splitting. The latter splittings are for a GM-matrix. The G-regular splitting and the GM-splitting generalize the known regular splitting and M-splitting, respectively. Conditions for convergence of all of these new splittings are explored and an example on each splitting is given.

1.2 Notation and Preliminary Definitions

The spectrum of matrix A is denoted by $\sigma(A)$. We call an eigenvalue of A a *simple eigenvalue* if its algebraic multiplicity in the characteristic polynomial is 1. We call an eigenvalue $\lambda \in \sigma(A)$ *dominant* if $|\lambda| = \rho(A)$. We call an eigenvalue $\lambda \in \sigma(A)$ *strictly dominant* if $|\lambda| > |\mu|$ for all $\mu \in \sigma(A)$, $\mu \neq \lambda$. The algebraic multiplicity of an eigenvalue $\lambda \in \sigma(A)$ is its multiplicity as a root of the characteristic polynomial of A and is denoted by $mult_\lambda(A)$, while the index of an eigenvalue $\lambda \in \sigma(A)$ is its multiplicity as a root of the minimal polynomial of A and is denoted by $index_\lambda(A)$. Sometimes, as a shorthand, we write *index of A* for $index_0(A)$.

The ordinary eigenspace of A for the eigenvalue λ is denoted by $E_\lambda(A)$. By definition, $E_\lambda(A) = \mathcal{N}(A - \lambda I)$, the null space of $A - \lambda I$. The nonzero vectors in $E_\lambda(A)$ are called ordinary eigenvectors of A corresponding to λ. The generalized eigenspace of A for the eigenvalue λ is denoted by $G_\lambda(A)$. Note that $G_\lambda(A) = \{v \mid (A - \lambda I)^k v = 0 \ \text{ where } k = index_\lambda(A)\} = \mathcal{N}(A - \lambda I)^k$. The generalized eigenspace $G_\lambda(A)$ is also known as the algebraic eigenspace of A for the eigenvalue λ. The nonzero vectors of $G_\lambda(A)$ are called generalized eigenvectors for A corresponding to λ. We call the projection operator onto $G_\lambda(A)$ a *spectral projector* if $|\lambda| = \rho(A)$ and the projection is along the direct sum of the other generalized eigenspaces. For any $\lambda \in \mathbb{C}$, $J_s(\lambda)$ denotes the $s \times s$ Jordan block corresponding to λ, i.e., $J_s(\lambda) = \lambda I_s + N_s$ where I_s is the $s \times s$ identity matrix and N_s is the matrix whose first superdiagonal consists of 1's while all other entries are zeroes. Note that $N_s = 0$ if $s = 1$. The $s \times s$ zero matrix is denoted by O_s. When the dimension of the zero matrix is clear we just write O. If $\tilde{A}$ is another real or complex $r \times r$ matrix, then $A \oplus \tilde{A}$ is the direct sum of A with $\tilde{A}$. The Jordan canonical form of matrix A is denoted by $J(A)$. By $Box(\lambda)$ we denote the Jordan box corresponding to an eigenvalue λ in $J(A)$, i.e., $Box(\lambda)$ is the direct sum of all of the Jordan blocks corresponding to λ in $J(A)$.

We say that $A \in \mathbb{C}^{1\times 1}$ is *reducible* if $A = [0]$. We say that $A \in \mathbb{C}^{n\times n}$ $(n \geq 2)$ is *reducible* if A is permutationally similar to $\begin{bmatrix} B & O \\ C & D \end{bmatrix}$ where B and D are square matrices. We say that a matrix $A \in \mathbb{C}^{n\times n}$ $(n \geq 1)$ is *irreducible* if A is not reducible. We call a matrix $A \in \mathbb{R}^{n\times n}$ *normal* if A commutes with its transpose.

1.3 Relations among Sets

We present in this short section the inclusion relations among the different sets mentioned in the previous two sections. We begin by mentioning that

$$PF_n = \{\text{Eventually Positive Matrices}\}. \tag{1.1}$$

This equality follows from [26, Theorem 1], [31, Theorem 2.2], and [52, Theorem 4.1 and Remark 4.2]. Obviously, every eventually positive matrix is eventually nonnegative. However, the converse is not true, e.g., one could take the identity matrix. Moreover, part of the collection of eventually nonnegative matrices is in WPF_n.

We begin by Lemma 1.1 whose proof can be found in [31]. Here, we have added the necessary hypothesis of having at least one nonzero eigenvalue or equivalently being nonnilpotent.

Lemma 1.1 *If $A \in \mathbb{R}^{n\times n}$ is eventually nonnegative and has at least one nonzero eigenvalue, then, both matrices A and A^T possess the Perron-Frobenius property, i.e., $A \in WPF_n$.*

We illustrate with the following example the need of at least one nonzero eigenvalue in the hypothesis of Lemma 1.1.

Example 1.1 It is essential for an eventually nonnegative matrix A to have a nonzero eigenvalue for A to be in WPF_n. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, then $A^2 = 0$.

Hence, A is eventually nonnegative. But, 0 is the only eigenvalue of A, the Jordan canonical form of A is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and all the ordinary eigenvectors of A are of the form $[\alpha \quad -\alpha]^T$ for some $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Therefore, A does not possess the strong Perron-Frobenius property nor the Perron-Frobenius property. To avoid such a situation, we have to stipulate that at least one of the eigenvalues of A is nonzero or equivalently that A is nonnilpotent.

Corollary 1.1 *Not all eventually nonnegative matrices are in* WPF_n.

In fact, we can see from Example 1.1, Lemma 1.1, and Corollary 1.1, that all eventually nonnegative matrices are inside WPF_n with the exception of nilpotent matrices. Moreover, the set of nonnilpotent eventually nonnegative matrices is a proper subset of WPF_n as we show in the following proposition.

Proposition 1.1 *The collection of eventually nonnegative matrices with at least one nonzero eigenvalue is properly contained in* WPF_n.

Proof. It suffices to find a matrix A in WPF_n which is not eventually nonnegative. Consider the matrix $A = E \oplus [-1]$ where E is the matrix of dimension $(n-1)$ having all its entries equal to 1. Then, $A^k = [(n-1)^{(k-1)}E] \oplus [(-1)^k]$. Clearly, A is not eventually nonnegative because the (n,n)-entry of A keeps alternating signs. However, $A \in WPF_n$ since $\rho(A) = n-1$ and there is a semipositive vector $v = [1 \ \cdots \ 1 \ 0]^T \in \mathbb{R}^n$ satisfying $v^T A = \rho(A) v^T$ and $Av = \rho(A)v$. □

Remark 1.1 *This case is to be taken in contrast with eventually positive matrices which fill all of* PF_n.

Thus, Proposition 1.1 tells us that if we exclude nilpotent matrices from the collection of eventually nonnegative matrices, then still we do not cover all of WPF_n. Hence, Proposition 1.1 establishes that all the containments are proper in the following statement:

$$\begin{aligned} PF_n &= \{\text{Eventually Positive Matrices}\} \\ &\subset \{\text{Nonnilpotent eventually nonnegative matrices}\} \\ &\subset WPF_n. \end{aligned}$$

Moreover, it turns out that an irreducible matrix in WPF_n does not have to be eventually nonnegative as the following example inspired by [5, Example 3.1] shows.

Example 1.2 Let

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that A is an irreducible matrix. Also, note that $\rho(A) = 2$ and that if $v = [2,2,1,1]^T$ and $w = [1,1,0,0]^T$ then $Av = \rho(A)v$ and $w^TA = \rho(A)w^T$. Thus, A is an irreducible matrix in WPF_n. Furthermore, it is easy to see that $A = B + C$ and that $BC = CB = C^2 = 0$. Hence, $A^j = B^j$ for all $j \geq 2$. But, using an induction argument, it is easy to check that $B^{2j+1} = \begin{bmatrix} 2^{2j} & 2^{2j} & 0 & 0 \\ 2^{2j} & 2^{2j} & 0 & 0 \\ 2^{2j} & 2^{2j} & -2^{2j} & -2^{2j} \\ 2^{2j} & 2^{2j} & -2^{2j} & -2^{2j} \end{bmatrix}$ for all $j \geq 1$. Hence, B is not eventually nonnegative, and thus, A is not eventually nonnegative.

CHAPTER 2

CHARACTERIZATIONS

2.1 Spectral Characterizations

In this section, we give characterizations of all matrices in PF_n and some matrices in WPF_n in terms of the positivity or nonnegativity of their spectral projectors. We mention first Theorem 2.1, which is a known result that can be derived from the usual spectral decomposition that can be found, e.g., in [6, page 27] or [46, pages 114, 225] and its method of proof is similar to that of [52, Theorem 3.6]. After that, we prove, using the spectral decomposition, our main results in this section, Theorems 2.2 and 2.3, which say that if $A \in \mathbb{R}^{n\times n}$ has the spectral decomposition $A = \rho(A)P + Q$, then the following statements are true:

1. $A \in PF_n \Leftrightarrow P > 0$, rank $P = 1$, and $\rho(Q) < \rho(A)$.
2. $A \in WPF_n$ and $\rho(A)$ is simple, positive, and strictly dominant
   $\Leftrightarrow P \geq 0$, rank $P = 1$, and $\rho(Q) < \rho(A)$.

We present a number of preliminary results leading to the main results. For clarity of exposition, we postpone the proofs of the preliminary results (Theorem 2.1 through Lemma 2.2) until Appendix A.

Theorem 2.1 *If $A \in \mathbb{C}^{n\times n}$ has d distinct eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_d|$ then A has a decomposition: $A = \lambda_1 P + Q$ satisfying the following:*

(i) P *is the projection matrix onto* $G_{\lambda_1}(A)$ *along* $\bigoplus_{j=2}^{d} G_{\lambda_j}(A)$.

(ii) $PQ = QP$.

(iii) $\rho(Q) \leq \rho(A)$.

(iv) If $index_{\lambda_1}(A) = 1$ *then* $PQ = QP = 0$.

Remark 2.1 *Let X be the similarity matrix that gives the Jordan canonical form of A, $J(A) = X^{-1}AX$, in which the Jordan blocks corresponding to λ_1, a dominant eigenvalue of A, appear first on the diagonal. For $m_1 = mult_{\lambda_1}(A)$, the projection matrix P, which appears in Theorem 2.1, can be expressed in terms of the columns of X and rows of X^{-1}:*

$$P = X\,[I_{m_1} \oplus O_{n-m_1}]\,X^{-1} = [Xe_1 \cdots Xe_{m_1}] \begin{bmatrix} e_1^T X^{-1} \\ \vdots \\ e_{m_1}^T X^{-1} \end{bmatrix}$$

The following lemma says that every Jordan block is permutationally similar to its transpose via a symmetric involutory permutation matrix.

Lemma 2.1 *For any Jordan block $J_s(\lambda_j)$, there exists a permutation matrix R_{js} such that:*
(i) $R_{js} = R_{js}^{-1} = R_{js}^T$, *and*
(ii) $[J_s(\lambda_j)]^T = R_{js}\; J_s(\lambda_j)\; R_{js}$

Corollary 2.1 *$Box(\lambda_j)$ is permutationally similar to its transpose $[Box(\lambda_j)]^T$.*

Corollary 2.2 *For any matrix $A \in \mathbb{C}^{n \times n}$, we have the following:*

(i) $J(A)$, the Jordan form of matrix A, is permutationally similar to its transpose by means of a permutation matrix R satisfying the property: $R = R^{-1} = R^T$.

(ii) A^T is similar to A.

The existence of the matrix R in Corollary 2.2 was already noted by Noutsos [31].

Lemma 2.2 *Let A be a matrix in $\mathbb{C}^{n\times n}$ such that $index_{\lambda_1}(A) = 1$, $mult_{\lambda_1}(A) = m_1$, and $J(A) = X^{-1}AX = Box(\lambda_1) \oplus Box(\lambda_2) \oplus \cdots \oplus Box(\lambda_d)$. Then, $\left\{ [e_i^T X^{-1}]^T : 1 \le i \le m_1 \right\}$ is a basis for $G_{\lambda_1}(A^T)$, i.e., transposing the first m_1 rows of X^{-1} gives a basis for $G_{\lambda_1}(A^T)$.*

Theorem 2.2 *The following statements are equivalent:*

(i) $A \in PF_n$.

(ii) $\rho(A)$ *is an eigenvalue of* A *and in the spectral decomposition*

$A = \rho(A)P + Q$ *we have* $P > 0$, *rank* $P = 1$ *and* $\rho(Q) < \rho(A)$.

Proof. Suppose that $A \in PF_n$. Then, each of A and A^T has a positive (or negative) eigenvector corresponding to a simple, positive, and strictly dominant eigenvalue $\rho = \rho(A)$. We use some expressions from the Appendix. Let $J(A) = XAX^{-1}$ be the Jordan decomposition of A as in (A.1), in which the Jordan box corresponding to $\rho = \lambda_1$ appears first on the diagonal of $J(A)$. Moreover, let $v = Xe_1$ and let $w = [e_1^T X^{-1}]^T$. Then, v and w are respectively right and left eigenvectors (each of which is either positive or negative) corresponding to ρ and $v^T w = 1$. Then, by (A.2), we have $P = vw^T$. Note that P is either positive or negative since each of v and w is either positive or negative. Since $w^T v = 1$, it follows that the vectors v and w are either both positive or both negative. Therefore, $P = vw^T > 0$. Moreover, since the range of $P = vw^T$ is spanned by v, it follows that rank $P = 1$. By (A.2) and (A.3), we have $\rho(Q) < \rho(A)$ in the spectral decomposition $A = \rho(A)P + Q$. Conversely, suppose that $\rho = \rho(A)$ is an eigenvalue of A and that in the spectral decomposition $A = \rho(A)P + Q$, we have $P > 0$, rank P = 1 and $\rho(Q) < \rho(A)$. Since rank P =1, it follows that the algebraic multiplicity of ρ is 1. Thus, $index_\rho(A) = 1$ and by Theorem 2.1

we conclude that $PQ = QP = O$. Therefore,

$$\left(\frac{1}{\rho}A\right)^k = \left(P + \frac{1}{\rho}Q\right)^k = P^k + \left(\frac{1}{\rho}Q\right)^k = P + \left(\frac{1}{\rho}Q\right)^k,$$

and consequently,

$$\lim_{k\to\infty}\left(\frac{1}{\rho}A\right)^k = P + \lim_{k\to\infty}\left(\frac{1}{\rho}Q\right)^k = P > 0.$$

Since $\rho > 0$ and the matrix $\frac{1}{\rho}A$ is real and eventually positive, it follows that the matrix A is also real and eventually positive. By (1.1), $A \in PF_n$. □

The proofs of the following two results are very similar to that of Theorem 4.2, and are therefore omitted.

Theorem 2.3 *The following statements are equivalent:*

(i) $A \in WPF_n$ *has a simple, positive, and strictly dominant eigenvalue.*

(ii) $\rho(A)$ *is an eigenvalue of* A *and in the spectral decomposition*

$A = \rho(A)P + Q$ *we have* $P \geq 0$, *rank* $P = 1$ *and* $\rho(Q) < \rho(A)$.

Theorem 2.4 *Let one of the two real matrices* A *and* A^T *possess the strong Perron-Frobenius property but not the other. Then, the projection matrix* P *in the spectral decomposition of* A *satisfies the relation* $P = vw^T$ *where one of the vectors* v *and* w *is positive while the other is neither positive nor negative.*

Corollary 2.3 *If one of the two real matrices* A *and* A^T *has a Perron-Frobenius eigenpair of a strictly dominant simple positive eigenvalue and a nonnegative eigenvector but the other matrix does not, then the projection matrix* P *in the spectral decomposition of* A *has positive and negative entries, and rank* $P = 1$.

Corollary 2.4 *If one of the two real matrices* A *and* A^T *has the strong Perron-Frobenius property but not the other, then the projection matrix* P *in the spectral decomposition of* A *is neither positive nor negative, and rank* $P = 1$.

Example 2.1 Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, then, the Jordan canonical form of A is given by: $X^{-1}AX = J(A) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ where $X = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix}$ and, as a result, $X^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$. Thus, $\rho(A) = 2$ is a simple, positive, and strictly dominant eigenvalue of A with a corresponding eigenvector $v = Xe_1 = [1\ 1\ 1]^T$. Hence, A has the strong Perron-Frobenius property. The matrix A^T also has $\rho(A^T) = \rho(A) = 2$ as a simple, positive, and strictly dominant eigenvalue but with a corresponding eigenvector $w = [e_1^T X^{-1}]^T = [1\ 1\ -1]^T$. Thus, A^T does not have the strong Perron-Frobenius property, i.e., $A \notin PF_n$. From Remark 2.1, the spectral projector P is given by $P = [Xe_1][e_1^T X^{-1}] = vw^T = [1\ 1\ 1]^T [1\ 1\ -1] = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$, which shows that P is neither positive nor negative, rank $P = 1$, and $P = vw^T$ where $v > 0$ and w is neither positive nor negative, and this is consistent with Theorem 2.4 and Corollary 2.4.

2.2 Combinatorial Characterizations

In this section, we focus on the combinatorial properties of eventually nonnegative matrices. We look at the necessary and sufficient conditions for a matrix to be eventually nonnegative. In particular, we look at how eventual nonnegativity is reflected in the walks of the graph, in the graphs of the positive and negative parts of a matrix, etc. In subsections 2.2.1, 2.2.2, and 2.2.3 our main focus will be on the combinatorial properties of eventually nonnegative matrices, in general, which may include nilpotent matrices. In subsection 2.2.4, we shift our focus to the combinatorial properties of WPF_n which includes the collection of nonnilpotent eventually nonnegative matrices. We begin first by recalling some basic definitions which can be found, e.g., in [2], [4].

For an $n \times n$ matrix A, we define the *(directed) graph* $G(A)$ to be the graph with vertices $1, 2, \ldots, n$ in which there is an edge (i, j) if and only if $a_{ij} \neq 0$. If $a_{ij} \neq 0$ then we call a_{ij} the *weight of the edge* (i, j). A *walk* from i to j of length k is a (finite) sequence of vertices $v_1, \ldots, v_{k+1}$ where $v_1 = i$, $v_{k+1} = j$, and (v_i, v_{i+1}) is an edge in $G(A)$ for $i = 1, \ldots, k$. We define the *weight of a walk* in $G(A)$ to be the product of the weights of the edges in this walk. We say that a walk is positive (negative, respectively) if its weight is positive (negative, respectively). We define the *total weight of a collection of walks* from vertex i to vertex j in $G(A)$ to be the sum of the weights of each of the walks in this collection. We say vertex i has access to vertex j if $i = j$ or else if there is a walk from i to j. If i has access to j and j has access to i then we say i and j communicate. Equivalence classes under the communication relation on the set of vertices of $G(A)$ are called the *classes of A*. By $A[\alpha]$ we denote the principal submatrix of $A \in \mathbb{R}^{n \times n}$ indexed by $\alpha \subseteq \{1, 2, \ldots, n\}$. The graph $G(A[\alpha])$ is called a *strong component* of $G(A)$ whenever α is a class of A. We say that $G(A)$ is *strongly connected* whenever A has one class, or equivalently, whenever A is irreducible. We call a class α *basic* if $\rho(A[\alpha]) = \rho(A)$. We call a class α *initial* if no vertex in any other class β has access to any vertex in α. We call a class α *final* if no vertex in α has access to any vertex in any other class β.

2.2.1 Eventual Nonnegativity and Walks in the Graph

We begin with a theorem that characterizes eventual nonnegativity of a matrix in terms of walks in the graph of that matrix. We say that a sequence of real numbers $\{x_k\}_{k=1}^{\infty}$ eventually majorizes (eventually majorizes and strictly dominates, respectively) another sequence of real numbers $\{y_k\}_{k=1}^{\infty}$ if $x_k \geq y_k$

($x_k > y_k$, respectively) for all $k \geq k_0$ for some positive integer k_0.

Theorem 2.5 *A matrix $A = (a_{ij})$ in $\mathbb{R}^{n\times n}$ is eventually nonnegative if and only if for any fixed pair of vertices i and j the total weight of positive walks from i to j of length k eventually majorizes the absolute value of the total weight of the negative walks from i to j having the same length in $G(A)$.*

Proof. Suppose $A \in \mathbb{R}^{n\times n}$ is eventually nonnegative, i.e., there exists $k_0 \in \mathbb{N}$ such that $A^k \geq 0$ for all $k \geq k_0$. Equivalently, for every $k \geq k_0$ and every $i, j \in \{1, 2, \ldots, n\}$, we have $A^k_{i,j} \geq 0$, where $A^k_{i,j}$ denotes the (i, j)-entry of A^k, which can be written as

$$A^k_{i,j} = \sum_{l_{k-1}=1}^{n} \cdots \sum_{l_2=1}^{n} \sum_{l_1=1}^{n} a_{l_0 l_1} a_{l_1 l_2} \cdots a_{l_{k-1} l_k}, \tag{2.1}$$

where $l_0 = i$ and $l_k = j$. Let us define

$$S_{k,n} = \underbrace{\{1, 2, \ldots, n\} \times \cdots \times \{1, 2, \ldots, n\}}_{k-1 \text{ times}}.$$

In other words, the set $S_{k,n}$ is the Cartesian product of $k - 1$ copies of the set $\{1, 2, \ldots, n\}$. And, for every $\alpha = (l_1, l_2, \ldots, l_{k-1}) \in S_{k,n}$ we define $A_\alpha(i, j) := a_{l_0 l_1} a_{l_1 l_2} \cdots a_{l_{k-1} l_k}$, where $l_0 = i$ and $l_k = j$. Note that $A_\alpha(i, j)$ is the weight of the walk i, l_1, l_2, …, l_{k-1}, j going from i to j in $G(A)$.
Thus, with this notation, saying that A is eventually nonnegative is equivalent to saying that there exists $k_0 \in \mathbb{N}$ such that $\sum_{\alpha\in S_{k,n}} A_\alpha(i, j) \geq 0$ for all $k \geq k_0$ and all $i, j \in \{1, 2, \ldots, n\}$.

Let

$$S^+_{k,n}(i,j) := \{\alpha \in S_{k,n} | A_\alpha(i,j) > 0\}, \quad S^0_{k,n}(i,j) := \{\alpha \in S_{k,n} | A_\alpha(i,j) = 0\},$$

and

$$S^-_{k,n}(i,j) := \{\alpha \in S_{k,n} | A_\alpha(i,j) < 0\},$$

we can then write

$$\begin{aligned}\sum_{\alpha\in S_{k,n}} A_\alpha(i,j) &= \sum_{\alpha\in S^+_{k,n}(i,j)} A_\alpha(i,j) + \sum_{\alpha\in S^0_{k,n}(i,j)} A_\alpha(i,j) + \sum_{\alpha\in S^-_{k,n}(i,j)} A_\alpha(i,j)\\ &= \sum_{\alpha\in S^+_{k,n}(i,j)} A_\alpha(i,j) + \sum_{\alpha\in S^-_{k,n}(i,j)} A_\alpha(i,j),\end{aligned}$$

wherein a sum is equal to zero if it is taken over an empty set. Hence, A is eventually nonnegative if and only if there is an integer $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and all $i, j \in \{1, 2, \ldots, n\}$

$$\sum_{\alpha\in S^+_{k,n}(i,j)} A_\alpha(i,j) \geq -\sum_{\alpha\in S^-_{k,n}(i,j)} A_\alpha(i,j) = \sum_{\alpha\in S^-_{k,n}(i,j)} |A_\alpha(i,j)|$$

which means that the total weight of positive walks of length k from i to j in $G(A)$ majorizes the absolute value of the total weight of negative walks of length k from i to j in $G(A)$ for all $k \geq k_0$. □

Using the same technique in the proof of Theorem 2.5 and in light of Remark 1.1, we have the following characterization of PF_n.

Theorem 2.6 *If $A \in \mathbb{R}^{n\times n}$ then the following statements are equivalent:*

(i) $A \in PF_n$.

(ii) A is eventually positive.

(iii) For any fixed pair of vertices i and j, the total weight of positive walks from i to j of length k eventually majorizes and strictly dominates the absolute value of the total weight of the negative walks from i to j of the same length in $G(A)$.

Example 2.2 Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Then, $A^k \geq 0$ for all $k \geq 4$. Hence, A is eventually nonnegative. Theorem 2.5 indicates that for any fixed pair of vertices i and j the total weight of positive walks from i to j in $G(A)$ majorizes the absolute value of the total weight of the negative walks from i to j of length $k \geq 4$. In particular, the walks of length 4 from vertex 2 to vertex 1 must satisfy this property. We list, respectively, all positive walks and all negative walks of length 4 from vertex 2 to vertex 1 in Tables 2.1 and 2.2. In these two tables, an edge is represented by an arrow and the weight of an edge is placed over the arrow. Hence, we see from Table 2.1 and Table 2.2 that the total

Table 2.1: Positive walks of length 4 from vertex 2 to vertex 1

Positive Walk	*Corresponding Weight*
$2 \xrightarrow{1} 1 \xrightarrow{2} 1 \xrightarrow{2} 1 \xrightarrow{2} 1$	8
$2 \xrightarrow{1} 1 \xrightarrow{2} 1 \xrightarrow{1} 2 \xrightarrow{1} 1$	2
$2 \xrightarrow{1} 1 \xrightarrow{1} 2 \xrightarrow{1} 1 \xrightarrow{2} 1$	2
$2 \xrightarrow{-1} 2 \xrightarrow{-1} 2 \xrightarrow{1} 1 \xrightarrow{2} 1$	2
Total Weight	14

Table 2.2: Negative walks of length 4 from vertex 2 to vertex 1

Negative Walk	*Corresponding Weight*
$2 \xrightarrow{-1} 2 \xrightarrow{1} 1 \xrightarrow{2} 1 \xrightarrow{2} 1$	-4
$2 \xrightarrow{-1} 2 \xrightarrow{1} 1 \xrightarrow{1} 2 \xrightarrow{1} 1$	-1
$2 \xrightarrow{1} 1 \xrightarrow{1} 2 \xrightarrow{-1} 2 \xrightarrow{1} 1$	-1
$2 \xrightarrow{-1} 2 \xrightarrow{-1} 2 \xrightarrow{-1} 2 \xrightarrow{1} 1$	-1
Total Weight	-7

weight of positive walks of length 4 from vertex 2 to vertex 1 in $G(A)$ majorizes the total weight of negative walks of length 4 from vertex 2 to vertex 1 because $14 \geq |-7|$.

Example 2.3 Let $A = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 6 & -1 \\ -1 & 2 & 7 \end{bmatrix}$. Then, $A^k > 0$ for all $k \geq 3$. Hence, A is eventually positive. Theorem 2.6 indicates that for any fixed pair of vertices i and j the total weight of positive walks from i to j majorizes and strictly dominates the absolute value of the total weight of the negative walks from i to j of length $k \geq 3$. In particular, the walks of length 3 from vertex 3 to vertex 1 must satisfy this property. And, just like in Example 2.2, we can see that the total weight of positive walks from vertex 3 to vertex 1 is 101, while the total weight of negative walks is -85. And thus, the total weight of positive walks strictly dominates the corresponding total weight of negative walks because $101 > |-85|$.

For completeness, we make now a couple of observations that relate $G(A^k)$ to $G(A)$. They follow using the same tools as in the proof of Theorem 2.5.

Proposition 2.1 *For any $A \in \mathbb{R}^{n \times n}$ and any $k \geq 1$, the graph $G(A^k)$ contains an edge (i, j) if and only if the total weight of positive walks of length k from i to j in $G(A)$ is not equal to the absolute value of the total weight of the negative walks from i to j of length k in $G(A)$.*

Corollary 2.5 *Let A be an arbitrary matrix in $\mathbb{R}^{n \times n}$ and let $A^k_{i,j}$ denote the (i, j)-entry of A^k. Then, the following statements are true:*

If $A^k_{i,j} \neq 0$, then there is a walk of length k from i to j in $G(A)$.

If $A^k_{i,j} = 0$, then either

- *there is no walk of length k from i to j in $G(A)$, or*
- *there is a collection of walks of length k from i to j in $G(A)$ containing at least two walks whose total weight of positive walks is equal to the absolute value of its total weight of negative walks.*

2.2.2 Eventually Nonnegative Product of Two Matrices

We next characterize when a product of two matrices is eventually nonnegative, and in particular, when the product of two eventually nonnegative (or eventually positive) matrices maintains this property.

We begin by defining an AB-alternating walk for a pair of square matrices A and B. Let A and B be two matrices in $\mathbb{R}^{n\times n}$ and let $G(A)$ and $G(B)$ be their respective graphs. We define the graph $G(A) \cup G(B)$ to be the graph on the set of vertices $\{1, 2, \ldots, n\}$ satisfying $(i, j) \in G(A) \cup G(B)$ if and only if $(i, j) \in G(A)$ or $(i, j) \in G(B)$. We call a walk l_0, l_1, $l_2, \ldots, l_{2k}$ of length $2k$ in $G(A) \cup G(B)$ an AB-alternating walk if the edges in the odd positions are in $G(A)$, while the edges in the even positions are in $G(B)$. The following theorem gives a necessary and sufficient condition for a product of two matrices to be eventually nonnegative.

Theorem 2.7 *Let $A, B \in \mathbb{R}^{n\times n}$ then the following are equivalent:*

(i) The product AB is eventually nonnegative.

(ii) For all $1 \leq i, j \leq n$, the total weight of positive AB-alternating walks of length k from i to j in $G(A) \cup G(B)$ eventually majorizes the absolute value of total weight of negative AB-alternating walks from i to j of the same length in $G(A) \cup G(B)$.

Proof. Let $C = AB$. Furthermore, let a_{ij}, b_{ij}, and c_{ij}, denote the (i, j)-entries of A, B, and C, respectively, and let $C^k_{i,j}$ denote the (i, j)-entry of C^k. Then, AB is eventually nonnegative if and only if there is an integer $k_0 \geq 1$ such that for all $1 \leq i, j \leq n$ and for all $k \geq k_0$, we have $C^k_{i,j} \geq 0$, where, as in (2.1), we write

$$C^k_{i,j} = \sum_{l_{k-1}=1}^{n} \cdots \sum_{l_2=1}^{n} \sum_{l_1=1}^{n} c_{l_0 l_1} c_{l_1 l_2} \cdots c_{l_{k-1} l_k},$$

with $l_0 = i$ and $l_k = j$. Then, writing $c_{l_w l_{w+1}} = \sum_{u_w=1}^{n} a_{l_w u_w} b_{u_w l_{w+1}}$ for $0 \le w \le k-1$, we have:

$$\begin{aligned} C^k_{i,j} &= \sum_{l_{k-1}=1}^{n} \cdots \sum_{l_1=1}^{n} (\sum_{u_0=1}^{n} a_{l_0 u_0} b_{u_0 l_1}) \cdots (\sum_{u_{k-1}=1}^{n} a_{l_{k-1} u_{k-1}} b_{u_{k-1} l_k}) \\ &= \sum_{l_{k-1}=1}^{n} \cdots \sum_{l_1=1}^{n} \sum_{u_0=1}^{n} \cdots \sum_{u_{k-1}=1}^{n} (a_{l_0 u_0} b_{u_0 l_1} \cdots a_{l_{k-1} u_{k-1}} b_{u_{k-1} l_k}) \end{aligned}$$

Note that all the indices in the above summation are between 1 and n. Hence, after relabeling the indices and setting $l_0 = i$ and $l_{2k} = j$, the (i, j)-entry of C^k can be written as:

$$\begin{aligned} C^k_{i,j} &= \sum_{l_{2k-1}=1}^{n} \cdots \sum_{l_1=1}^{n} a_{l_0 l_1} b_{l_1 l_2} a_{l_2 l_3} b_{l_3 l_4} \cdots a_{l_{2k-2} l_{2k-1}} b_{l_{2k-1} l_{2k}} \\ &= \sum_{1 \le l_1, \cdots, l_{2k-1} \le 1}^{n} a_{l_0 l_1} b_{l_1 l_2} a_{l_2 l_3} b_{l_3 l_4} \cdots a_{l_{2k-2} l_{2k-1}} b_{l_{2k-1} l_{2k}} \end{aligned}$$

Note that $a_{l_0 l_1} b_{l_1 l_2} a_{l_2 l_3} b_{l_3 l_4} \cdots a_{l_{2k-2} l_{2k-1}} b_{l_{2k-1} l_{2k}}$ is the weight of the AB-alternating walk $i = l_0, l_1, l_2, \cdots, l_{2k} = j$ from i to j of length $2k$ in $G(A) \cup G(B)$. Having noted that, consider the following sets

$$\begin{aligned} T^+ &= \{(l_1, l_2, \cdots, l_{2k-1}) \mid a_{l_0 l_1} b_{l_1 l_2} a_{l_2 l_3} b_{l_3 l_4} \cdots a_{l_{2k-2} l_{2k-1}} b_{l_{2k-1} l_{2k}} > 0\} \\ T^0 &= \{(l_1, l_2, \cdots, l_{2k-1}) \mid a_{l_0 l_1} b_{l_1 l_2} a_{l_2 l_3} b_{l_3 l_4} \cdots a_{l_{2k-2} l_{2k-1}} b_{l_{2k-1} l_{2k}} = 0\} \\ T^- &= \{(l_1, l_2, \cdots, l_{2k-1}) \mid a_{l_0 l_1} b_{l_1 l_2} a_{l_2 l_3} b_{l_3 l_4} \cdots a_{l_{2k-2} l_{2k-1}} b_{l_{2k-1} l_{2k}} < 0\}. \end{aligned}$$

And thus, $C^k_{i,j} > 0$ if and only if the following condition holds,

$$\begin{aligned} \sum_{T^+} a_{l_0 l_1} b_{l_1 l_2} \cdots a_{l_{2k-2} l_{2k-1}} b_{l_{2k-1} l_{2k}} &\ge -\sum_{T^-} a_{l_0 l_1} b_{l_1 l_2} \cdots a_{l_{2k-2} l_{2k-1}} b_{l_{2k-1} l_{2k}} \\ &= |\sum_{T^-} a_{l_0 l_1} b_{l_1 l_2} \cdots a_{l_{2k-2} l_{2k-1}} b_{l_{2k-1} l_{2k}}|, \end{aligned}$$

which is what we wanted to show. □

A similar proof leads to the following result.

Theorem 2.8 *Let $A, B \in \mathbb{R}^{n \times n}$. Then, the following are equivalent:*

(i) The product AB is eventually positive.

(ii) For all $1 \le i, j \le n$, the total weight of positive AB-alternating walks from i to j in $G(A) \cup G(B)$ eventually majorizes and strictly dominates the absolute value of total weight of negative AB-alternating walks from i to j of the same length in $G(A) \cup G(B)$.

Remark 2.2 *It is easy to see that if $A, B \in \mathbb{R}^{n \times n}$ are eventually nonnegative (eventually positive, respectively) and $AB = BA$ then AB is also eventually nonnegative (eventually positive, respectively). However, if $AB \neq BA$ then AB does not have to be eventually nonnegative (eventually positive, respectively). Moreover, AB might be eventually nonnegative (eventually positive, respectively) yet neither A nor B is so.*

We illustrate this in the following examples.

Example 2.4 Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and Let $B = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 6 & -1 \\ -1 & 2 & 7 \end{bmatrix}$. Then, both A and B are eventually nonnegative. In fact, $A^k \geq 0$ for all $k \geq 4$ and $B^k \geq 0$ for all $k \geq 3$. However,

$$\begin{bmatrix} 9 & 10 & 5 \\ 0 & -4 & 4 \\ -1 & 2 & 7 \end{bmatrix} = AB \neq BA = \begin{bmatrix} 8 & 1 & 3 \\ 12 & -3 & -1 \\ 0 & -3 & 7 \end{bmatrix},$$

and neither AB nor BA is eventually nonnegative because, by Lemma 1.1, they have nonzero eigenvalues yet neither AB nor BA has the Perron-Frobenius property.

Example 2.5 Let $A = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 6 & -1 \\ -1 & 2 & 7 \end{bmatrix}$ and let $B = \begin{bmatrix} 20 & 1 & 1 \\ 1 & -10 & 1 \\ 1 & 1 & -10 \end{bmatrix}$ (matrix B is from [26]). Then, $A^k > 0$ for all $k \geq 3$ and $B^k > 0$ for all $k \geq 10$.

Hence, both A and B are eventually positive. However,

$$\begin{bmatrix} 65 & -14 & -25 \\ 65 & -58 & 19 \\ -11 & -14 & -69 \end{bmatrix} = AB \neq BA = \begin{bmatrix} 62 & 48 & 66 \\ -28 & -56 & 20 \\ 16 & -12 & -68 \end{bmatrix},$$

and neither AB nor BA is eventually positive because neither possesses the strong Perron-Frobenius property.

Example 2.6 If $A = -I$ and $B = -2I$, where I is the identity matrix, then neither A nor B is eventually nonnegative, yet their product $AB = 2I$ is nonnegative, thus eventually nonnegative. Another nontrivial example is the following: let $A = \begin{bmatrix} -2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and let $B = \begin{bmatrix} -5 & -1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Then, neither A nor B is eventually nonnegative because the $(3,3)$-entry in both of them keeps on alternating signs yet their product $AB = \begin{bmatrix} 10 & 5 & 0 \\ 5 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is eventually nonnegative. In fact, $(AB)^k \geq 0$ for all $k \geq 4$. Theorem 2.7 indicates that for all $1 \leq i, j \leq n$, the total weight of positive AB-alternating walks from i to j in $G(A) \cup G(B)$ of length $2k$ majorizes the absolute value of total weight of negative AB-alternating walks from i to j of the same length for all $k \geq 4$. In particular, for $k = 4$ there is only one AB-alternating walk of length 8 from vertex 3 to itself and the weight of such a walk is 1. Hence, it is a positive walk. On the other hand, there are 74 AB-alternating walks of length 8 from vertex 1 to vertex 2. Among these 74 walks, the total weight of positive AB-alternating walks of length 8 majorizes the absolute value of the total weight of the corresponding negative AB-alternating walks, and these positive walks exceed in weight the absolute value of their corresponding negative walks by 4375.

2.2.3 Eventual Nonnegativity in $G(A^+)$ and $G(A^-)$

We study the eventual nonnegativity of a matrix using the graphs of the positive part and the negative part of the matrix. We begin with some preliminary material followed by some lemmas on products and unions of graphs that will be needed for Theorem 2.9, which is the main theorem of this subsection. Let A be a matrix in $\mathbb{R}^{n\times n}$. We define the matrix A^+, the positive part of A, as the matrix obtained from A by replacing the negative entries with zeroes. Similarly, we define A^-, the negative part of A, as the matrix obtained from A by replacing the positive entries with zeroes. And thus, $A = A^+ + A^-$, where $A^+ \geq 0$ and $A^- \leq 0$. For every $k \geq 2$, we look at the collection of maps from the set of integers $\{1, 2, \dots, k\}$ to the set of symbols $\{+, -\}$ and we divide this collection of maps into two sub-collections: a sub-collection of maps that take an *even* number of integers to the symbol "$-$" and a sub-collection of maps that take an *odd* number of integers to the symbol "$-$". In other words, we define:

- $Even_k = \{\tau \mid \tau : \{1, 2, \dots, k\} \to \{+, -\}$ and the cardinality of $\tau^{-1}\{-\}$ is even$\}$, and
- $Odd_k = \{\tau \mid \tau : \{1, 2, \dots, k\} \to \{+, -\}$ and the cardinality of $\tau^{-1}\{-\}$ is odd$\}$.

If $A \in \mathbb{R}^{n\times n}$ and τ is any map from the set $\{1, 2, \dots, k\}$ to the set $\{+, -\}$, then for any $i \in \{1, 2, \dots, k\}$ we define the following:

$$A^{\tau(i)} = \begin{cases} A^+ & \text{(the positive part of } A\text{)} \quad \text{if } \tau(i) = + \\ A^- & \text{(the negative part of } A\text{)} \quad \text{if } \tau(i) = - \end{cases}.$$

Note that the product $(A^-)(A^-) = (A^-)^2$ is nonnegative. Similarly, A^+A^- and A^-A^+ are nonpositive. In general, if τ is any map from the set $\{1, 2, \dots, k\}$ to the set $\{+, -\}$, we have that if $\tau \in Even_k$, then $A^{\tau(1)}A^{\tau(2)} \cdots A^{\tau(k)} \geq 0$, and if $\tau \in Odd_k$, then $A^{\tau(1)}A^{\tau(2)} \cdots A^{\tau(k)} \leq 0$.

Let G_1 and G_2 be two graphs on the set of vertices $\{1, 2, \ldots, n\}$. We say G_2 dominates G_1 in weight (or G_2 is weight-dominant over G_1), denoted by $G_2 \succ G_1$, if for all $1 \le i, j \le n$,

1. whenever an edge (i, j) is in G_1 then (i, j) is also in G_2, and

2. the weight of (i, j) in G_1 does not exceed the weight of (i, j) in G_2.

Note that if (i, j) is not an edge in G_2 and $G_2 \succ G_1$ then (i, j) is not an edge in G_1.

If A, B are either nonnegative or nonpositive, then the "product" graph $G(A)G(B)$ is the graph on the set of vertices $\{1, 2, \ldots, n\}$ defined by $(i, j) \in G(A)G(B)$ if and only if there is an $m \in \{1, 2, \ldots, n\}$ such that $(i, m) \in G(A)$ and $(m, j) \in G(B)$.

The following three results follow immediately from the definitions.

Lemma 2.3 *Let $A, B \in \mathbb{R}^{n \times n}$ be nonnegative and let γ be a nonzero scalar, then $G(\gamma A) = G(A)$, and $G(A + B) = G(A) \cup G(B)$.*

Lemma 2.4 *If $A_1, A_2, A_3 \in \mathbb{R}^{n \times n}$ and A_i is either nonnegative or nonpositive for $i = 1, 2, 3$, then $G(A_1 A_2) = G(A_1)G(A_2)$, and $[G(A_1)G(A_2)]G(A_3) = G(A_1)[G(A_2)G(A_3)]$.*

Corollary 2.6 *If $\{A_1, A_2, \ldots, A_k\}$ is any collection consisting of k matrices in $\mathbb{R}^{n \times n}$ with the property that each A_i is either nonpositive or nonnegative for $i = 1, 2, \ldots, k$, then $G(A_1 A_2 \cdots A_k) = G(A_1)G(A_2) \cdots G(A_k)$.*

Theorem 2.9 *$A \in \mathbb{R}^{n \times n}$ is eventually nonnegative if and only if there is a $k_0 \in \mathbb{N}$ such that $\bigcup_{\tau \in Even_k} \prod_{i=1}^{k} G(A^{\tau(i)}) \succ \bigcup_{\tau \in Odd_k} \prod_{i=1}^{k} G(|A^{\tau(i)}|)$ for all $k \ge k_0$.*

Proof. $A \in \mathbb{R}^{n \times n}$ is eventually nonnegative if and only if there is a $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$ we have $A^k = (A^+ + A^-)^k \ge 0$. Note that $(A^+ + A^-)^k$ equals a sum of products of k matrices each of which is either A^+ or A^-.

And thus, $(A^+ + A^-)^k = \sum_\tau A^{\tau(1)} \cdots A^{\tau(k)}$, where the sum runs over all maps $\tau : \{1, 2, \ldots, k\} \to \{+, -\}$. Such maps have either an even or an odd number of integers mapping to "$-$", i.e., $\tau \in Even_k$ or $\tau \in Odd_k$. Thus, for all $k \geq k_0$, we have

$$\begin{aligned} A^k &= \sum_{\tau \in Even_k} A^{\tau(1)} A^{\tau(2)} \cdots A^{\tau(k)} + \sum_{\tau \in Odd_k} A^{\tau(1)} A^{\tau(2)} \cdots A^{\tau(k)} \geq 0 \\ &\Leftrightarrow \sum_{\tau \in Even_k} A^{\tau(1)} A^{\tau(2)} \cdots A^{\tau(k)} - \sum_{\tau \in Odd_k} |A^{\tau(1)} A^{\tau(2)} \cdots A^{\tau(k)}| \geq 0 \\ &\Leftrightarrow G\left(\sum_{\tau \in Even_k} A^{\tau(1)} A^{\tau(2)} \cdots A^{\tau(k)}\right) \succ G\left(\sum_{\tau \in Odd_k} |A^{\tau(1)} A^{\tau(2)} \cdots A^{\tau(k)}|\right) \\ &\Leftrightarrow \bigcup_{\tau \in Even_k} \prod_{i=1}^{k} G(A^{\tau(i)}) \succ \bigcup_{\tau \in Odd_k} \prod_{i=1}^{k} G(|A^{\tau(i)}|). \qquad \square \end{aligned}$$

Example 2.7 Let $A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$, then $A^+ = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, and $A^- = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$. It is easy to verify that $A^4 > 0$ and $A^5 > 0$. Thus, by [26, Theorem 1], A is eventually positive (hence eventually nonnegative) and $A^k \geq 0$ for all $k \geq 4$. From Theorem 2.9, we have

$$\bigcup_{\tau \in Even_k} \prod_{i=1}^{k} G(A^{\tau(i)}) \succ \bigcup_{\tau \in Odd_k} \prod_{i=1}^{k} G(|A^{\tau(i)}|) \qquad \text{for all } k \geq 4.$$

In particular, for $k = 4$, we have

$$\bigcup_{\tau \in Even_4} \prod_{i=1}^{4} G(A^{\tau(i)}) \succ \bigcup_{\tau \in Odd_4} \prod_{i=1}^{4} G(|A^{\tau(i)}|).$$

We can see in Figure 2.1 that the graph of $\bigcup_{\tau \in Even_4} \prod_{i=1}^4 G(A^{\tau(i)})$ (on the left) is weight-dominant over the graph of $\bigcup_{\tau \in Odd_4} \prod_{i=1}^4 G(|A^{\tau(i)}|)$ (on the right).

2.2.4 The Classes of Matrices in PF_n and WPF_n

It was shown in [2, Chapter 2, Section3] that a nonnegative matrix A has positive left and right eigenvectors corresponding to $\rho(A)$ if and only if all

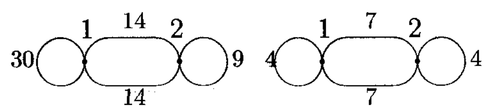

Figure 2.1: The graph on the left dominates in weight the graph on the right.

classes of A are basic and final. Note that the latter statement is not equivalent to saying that $A \in PF_n$ because $\rho(A)$ may not be a simple eigenvalue of A. In this subsection, we consider arbitrary real matrices and try to obtain analogous results for WPF_n. We study the necessary and sufficient conditions on the classes of a matrix so that it is in WPF_n. However, we note that we do not present a full characterization of WPF_n in terms of classes. Instead, we present necessary conditions in Theorem 2.11 and sufficient conditions in Theorem 2.12. See also the end of Section 3.1 for a special case. We first review some definitions and well-known results, which can be found, e.g., in [2], [4].

We call a collection $\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ of subsets of $\{1, 2, \ldots, n\}$ a partition of $\{1, 2, \ldots, n\}$ if $\cup_{i=1}^{m} \alpha_i = \{1, 2, \ldots, n\}$ and $\alpha_i \cap \alpha_j = \phi$ whenever $i \neq j$. Moreover, we call the m-tuple $(\alpha_1, \alpha_2, \cdots, \alpha_m)$ an ordered partition of $\{1, 2, \ldots, n\}$. If $A \in \mathbb{R}^{n \times n}$, $v \in \mathbb{R}^n$, and $\alpha, \beta \subset \{1, 2, \ldots, n\}$, then $A[\alpha, \beta]$ denotes the submatrix of A whose rows are indexed by α and whose columns are indexed by β. If $\alpha = \beta$, then we write $A[\alpha]$ for the principal submatrix of A whose rows and columns are indexed by α. Moreover, by $v[\alpha]$ we denote the subvector of v indexed by α. If A is in $\mathbb{R}^{n \times n}$ and $\kappa = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ is an ordered partition of $\{1, 2, \ldots, n\}$, then A_κ denotes the block matrix whose $(i, j)^{th}$ block is

$A[\alpha_i, \alpha_j]$. In other words, we have a representation of the following form:

$$A_\kappa = \begin{array}{c} \begin{array}{ccc} \alpha_1 & \alpha_j & \alpha_m \end{array} \\ \left[\begin{array}{cccc} A[\alpha_1] & A[\alpha_1, \alpha_j] & \dots & A[\alpha_1, \alpha_m] \\ \vdots & A[\alpha_i, \alpha_j] & & \vdots \\ A[\alpha_m, \alpha_1] & \dots & & A[\alpha_m] \end{array} \right] \begin{array}{c} \alpha_1 \\ \alpha_i \\ \alpha_m \end{array} \end{array}$$

Lemma 2.5 *For any ordered partition* $\kappa = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ *of* $\{1, 2, \ldots, n\}$ *and any matrix* $A \in \mathbb{R}^{n \times n}$*, the matrix* A_κ *is permutationally similar to the matrix* A*.*

Lemma 2.6 *For any matrix* $A \in \mathbb{R}^{n \times n}$ *there is an ordered partition* $\kappa = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ *of* $\{1, 2, \ldots, n\}$ *such that* A_κ *is a block lower triangular matrix with* m *diagonal blocks. Moreover, each of the* m *diagonal blocks is either an irreducible block or a* 1×1 *zero block. Such a form is known as the (lower triangular) Frobenius normal form of* A*.*

Theorem 2.10 *If* A *is a matrix in* PF_n*, then* A *is irreducible. Hence,* A *has one class, which is basic, final, and initial.*

Proof. Let $\kappa = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be an ordered partition of the set of vertices $\{1, 2, \ldots, n\}$ that gives the Frobenius normal form of A. It is enough to show that the Frobenius normal form of A has only one class, i.e., $m = 1$. Assume with the hope of getting a contradiction that $m > 1$, then a lower triangular Frobenius normal form of A is given by the partition κ as follows:

$$A_\kappa = \begin{bmatrix} A_1 & & 0 \\ \vdots & \ddots & \\ * & \cdots & A_m \end{bmatrix}$$

where $A_i = A[\alpha_i]$. By (1.1), A is in PF_n if and only if A is eventually positive if and only if A_κ is eventually positive. If $m > 1$ then for all $s \in \{1, 2, 3, \ldots\}$

the matrix $(A_\kappa)^s$ will always have a zero in the $(1,m)$-block. And thus, A_κ can not be eventually positive, a contradiction. Hence, A has only one class, which is basic, final, and initial. □

Theorem 2.11 *Let A be a matrix in WPF_n. Then,*

(i) If α is a final class of A and $v[\alpha]$ is nonzero for some right eigenvector v of A corresponding to $\rho(A)$, then α is a basic class.

(ii) If α is an initial class of A and $w[\alpha]$ is nonzero for some left eigenvector w of A corresponding to $\rho(A)$, then α is a basic class.

Proof. In general, for any class α of A, we have

$$(Av)[\alpha] = A[\alpha]v[\alpha] + \sum_{\beta} A[\alpha,\beta]v[\beta],$$

where the sum on the right side is taken over all classes β that have access from α but are different from α. When α is a final class and v is an eigenvector of A corresponding to $\rho(A)$, we have

$$\rho(A)v[\alpha] = (Av)[\alpha] = A[\alpha]v[\alpha]$$

If, in addition, $v[\alpha]$ is nonzero, we can conclude that α is a basic class.

This proves (i) and the proof of (ii) is analogous. □

Theorem 2.12 *If $A \in \mathbb{R}^{n\times n}$ has two classes α and β, not necessarily distinct, such that:*

(i) α is basic, initial, and $A[\alpha]$ has a right Perron-Frobenius eigenvector

and

(ii) β is basic, final, and $A[\beta]$ has a left Perron-Frobenius eigenvector,

then $A \in WPF_n$.

Proof. There is a semipositive vector v such that $A[\alpha]v = \rho(A)v$. Define the vector $\tilde{v} \in \mathbb{R}^n$ as follows: for any class γ of A, $\tilde{v}[\gamma] = v$ if $\gamma = \alpha$, and $\tilde{v}[\gamma] = 0$ if $\gamma \neq \alpha$. It is easily seen that $\tilde{v}$ is semipositive and that $A\tilde{v} = \rho(A)\tilde{v}$. Similarly, using the class β, there is a semipositive vector $\tilde{w}$ for which $\tilde{w}^T A = \rho(A)\tilde{w}^T$. Hence, $A \in WPF_n$. □

2.3 Eventual Nonnegativity and Convexity

In this section, we study eventual nonnegativity in terms of the relations that exist between certain convex subsets obtained from the rows and columns of the matrix and we use these relations to characterize eventual nonnegativity of a matrix. First, let us introduce some notation needed in this section. The ith row of matrix A is denoted by A_{i*}. The j^{th} column of matrix A is denoted by A_{*j}. By $Hull(A)$ we denote the convex hull of the transposed rows of matrix A, i.e., $Hull(A)$ is the convex hull of the (column) vectors $\{(A_{i*})^T\}_{i=1}^n$. If y is a vector in $\mathbb{R}^n$, then $H(y)$ denotes the closed half-space consisting of vectors that are orthogonal to y or making an acute angle with y, i.e., $H(y) = \{x \in \mathbb{R}^n \mid x^T y \geq 0\}$.

Theorem 2.13 *Let A be an $n \times n$ real matrix and suppose that k is a positive integer. Then, the following statements are equivalent:*

(i) $A^{k+1} \geq 0$.

(ii) $Hull(A^k) \subset \bigcap_{j=1}^n H(A_{*j})$.

(iii) $Hull(A) \subset \bigcap_{j=1}^n H((A^k)_{*j})$.

Proof. Let $A_{i,j}^{k+1}$ denote (i,j)-entry of A^{k+1} and note that $A^{k+1} = A^k A$. Thus, $A_{i,j}^{k+1}$ is the ith row of A^k multiplied by the j^{th} column of A, i.e., $A_{i,j}^{k+1} =$

$(A^k)_{i*}A_{*j}$. Therefore,

$$\begin{aligned} A^{k+1} \geq 0 &\Leftrightarrow A^{k+1}_{i,j} \geq 0 \text{ for all } 1 \leq i,j \leq n \\ &\Leftrightarrow (A^k)_{i*}A_{*j} \geq 0 \text{ for all } 1 \leq i,j \leq n \\ &\Leftrightarrow ((A^k)_{i*})^T \in H(A_{*j}) \text{ for all } 1 \leq i,j \leq n \\ &\Leftrightarrow \{((A^k)_{i*})^T\}_{i=1}^n \subset \bigcap_{j=1}^n H(A_{*j}) \end{aligned}$$

But $\bigcap_{j=1}^n H(A_{*j})$ is a convex set since it is the intersection of convex sets. Hence, the later statement is equivalent to $Hull(A^k) \subset \bigcap_{j=1}^n H(A_{*j})$. This establishes the equivalence of (i) and (ii). The equivalence of (i) and (iii) is shown in a similar manner by noting that $A^{k+1} = AA^k$ and using the fact that the (i,j)-entry of A^{k+1} can be alternatively written as $A^{k+1}_{i,j} = A_{i*}(A^k)_{*j}$. □

Corollary 2.7 *Let A be an $n \times n$ real matrix and suppose that k is a positive integer. Then, the following statements are equivalent:*

(i) $A^{k+1} \geq 0$ *and* $A^{k+2} \geq 0$.

(ii) $Hull(A^k) \cup Hull(A^{k+1}) \subset \bigcap_{j=1}^n H(A_{*j})$.

(iii) $Hull(A) \subset \bigcap_{j=1}^n \left(H((A^{k+1})_{*j}) \cap H((A^{k+2})_{*j})\right)$.

Corollary 2.8 *Let A be an $n \times n$ real matrix. Then, the following statements are equivalent:*

(i) A is eventually nonnegative.

(ii) $\bigcup_{l=k_0}^{\infty} Hull(A^l) \subset \bigcap_{j=1}^n H(A_{*j})$ *for some* $k_0 \geq 0$.

(iii) $Hull(A) \subset \bigcap_{l=k_0}^{\infty} \bigcap_{j=1}^n H((A^l)_{*j})$ *for some* $k_0 \geq 0$.

Similar results hold for eventually positive matrices by replacing the closed half-spaces with open half spaces.

CHAPTER 3

SPECTRAL, COMBINATORIAL, AND TOPOLOGICAL PROPERTIES

3.1 The Classes of an Eventually Nonnegative Matrix and Its Algebraic Eigenspace

Carnochan Naqvi and McDonald [5] showed that the matrices A and A^g share some combinatorial properties for large prime numbers g if A is eventually nonnegative and $index_0(A) \in \{0, 1\}$. In this section, we give slight improvements of their result by expanding the set of powers g for which their result is true and by using this set of powers to prove our main theorem in this section, Theorem 3.3, which generalizes Rothblum's Theorem [35] about the algebraic eigenspace of a nonnegaive matrix and its basic classes. First, we begin by some definitions.

By $R(A)$ we denote the reduced graph of $G(A)$, i.e., the graph whose vertices are the strong components of $G(A)$ and there is an edge from vertex

a to vertex b in $R(A)$ if and only if there is an edge from the component that a represents to the component that b represents in $G(A)$. By $\overline{R(A)}$ we denote the (reflexive) transitive closure of $R(A)$, i.e., the graph whose vertices are the same as those of $R(A)$ and there is an edge from vertex a to vertex b in $\overline{R(A)}$ if and only if vertex a has access to vertex b in $R(A)$.

Following the notation of [5], for any real matrix A, we define a set of integers D_A (the *denominator set* of the matrix A) as follows:

$$\begin{aligned} D_A &= \{d \mid \theta - \alpha = \tfrac{c}{d}, \text{ where } re^{2\pi i\theta}, re^{2\pi i\alpha} \in \sigma(A),\ r > 0,\ c \in \mathbb{Z}^+, \\ &\qquad d \in \mathbb{Z}\backslash\{0\},\ \gcd(c,d) = 1, \text{ and } |\theta - \alpha| \notin \{0,1,2,\ldots\}\}. \end{aligned}$$

The set D_A captures the denominators of those lowest term rational numbers that represent the argument differences (normalized by a factor of $\frac{1}{2\pi}$) of two distinct eigenvalues of A lying on the same circle in the complex plane. In other words, if two distinct eigenvalues of A lie on the same circle in the complex plane and their argument difference is a rational multiple of 2π, then the denominator of this rational multiple in the lowest terms belongs to D_A. Note that the set D_A defined above is empty if and only if *one* of the following statements is true:

1. A has no distinct eigenvalues lying on the same circle in the complex plane.

2. The argument differences of the distinct eigenvalues of A that lie on the same circle in the complex plane are irrational multiples of 2π.

Note also that D_A is always a finite set and that 1 is never an element of D_A. Moreover, $d \in D_A$ if and only if $-d \in D_A$.

We define now the following sets of integers:

$$P_A = \{kd \mid k \in \mathbb{Z},\ d > 0, \text{ and } d \in D_A\} \quad (\textit{Problematic Powers} \text{ of } A)$$

$$N_A = \{1,2,3,\ldots\}\backslash P_A \quad (\textit{Nice Powers} \text{ of } A)$$

Since D_A is finite and 1 is never an element of D_A, N_A is always an infinite set. In particular, N_A contains all the prime numbers that are larger than the maximum of D_A.

Lemma 3.1 *Let $A \in \mathbb{C}^{n\times n}$ and let $\lambda, \mu \in \sigma(A)$, then for all $k \in N_A$, $\lambda^k = \mu^k$ if and only if $\lambda = \mu$.*

Proof. The necessity is trivial. For the sufficiency, pick any $k \in N_A$ and suppose that $\lambda^k = \mu^k$ for some $\lambda, \mu \in \sigma(A)$. If $\lambda^k = \mu^k = 0$ then obviously $\lambda = \mu = 0$. Suppose that $\lambda^k = \mu^k \neq 0$. Then, in such a case, there is an $r > 0$ such that $\lambda = re^{2\pi i\theta}$, $\mu = re^{2\pi i\alpha}$ for some $\theta, \alpha \in [0, 1)$. In such a case, $\lambda^k = \mu^k \Leftrightarrow r^k e^{2\pi ik\theta} = r^k e^{2\pi ik\alpha} \Leftrightarrow e^{2\pi ik(\theta-\alpha)} = 1 \Leftrightarrow k(\theta - \alpha) = m$ for some $m \in \mathbb{Z}$. Assume (with the hope of getting a contradiction) that $m \neq 0$. It is enough to consider the case when $m > 0$, since the other case is analogous. If $d = gcd(k, m)$ then we have two cases. Either $d = k$ or $d < k$. If $d = k$ then $\theta - \alpha = \frac{m}{k} \in \mathbb{Z}$. But, θ and α are in $[0, 1)$. Hence $\theta - \alpha = 0 \Leftrightarrow m = 0$, a contradiction. Suppose now that $gcd(k, m) = d < k$, , then $\theta - \alpha = \frac{m/d}{k/d} \in \mathbb{Z}$ and $gcd(\frac{k}{d}, \frac{m}{d}) = 1$. Hence, $\frac{k}{d} \in D_A \Rightarrow k \in P_A \Leftrightarrow k \notin N_A$, a contradiction. □

Lemma 3.2 *Let $A \in \mathbb{C}^{n\times n}$ and let $\lambda \in \sigma(A)$, $\lambda \neq 0$, then for all $k \in N_A$ we have $E_\lambda(A) = E_{\lambda^k}(A^k)$ and the Jordan box of λ^k in $J(A^k)$ is obtained from the Jordan box of λ in $J(A)$ by replacing λ with λ^k.*

Proof. Since $E_\lambda(A) \subset E_{\lambda^k}(A^k)$, it suffices to show that dim $E_\lambda(A) =$ dim $E_{\lambda^k}(A^k)$. To prove the latter statement and the claim of this lemma, it is enough to show that there is a one-to-one correspondence between the collection of Jordan blocks of λ in A and the collection of Jordan blocks of λ^k in A^k that respects the multiplicity and the order of the Jordan block. Suppose that $J_s(\lambda) = \lambda I_s + N_s$ is an $s \times s$ Jordan block of A corresponding to λ and suppose that the Jordan canonical form of A is given for some X in $Gl(n, \mathbb{C})$ by

$$J(A) = X^{-1}AX = J_s(\lambda) \oplus \cdots \oplus J_r(\mu).$$

Then,

$$[J(A)]^k = X^{-1}A^kX \quad = \quad [J_s(\lambda)]^k \oplus \cdots \oplus [J_r(\mu)]^k. \tag{3.1}$$

If $0 \in \sigma(A)$ and $J_{s'}(0)$ is an $s' \times s'$ Jordan block corresponding to 0 that appears in $J(A)$, then whenever $J_{s'}(0)$ is raised to the power k then it becomes $[J_{s'}(0)]^k$ which is either a block whose k^{th} superdiagonal consists entirely of ones and all other entries are zeroes (if $k \leq s'$) or it becomes a block consisting entirely of zeroes (if $k > s'$). In all cases, $[J_{s'}(0)]^k$ becomes either zero or permutationally similar to a direct sum of Jordan blocks corresponding to 0 of smaller order, i.e., there is a similarity matrix which is also a permutation matrix that gives the Jordan form of $[J_{s'}(0)]^k$ and it is a direct sum of $J_{r'}(0)$ for some $r' < s'$.

If the Jordan block $J_s(\lambda)$ of a nonzero eigenvalue λ is raised to the power k then it becomes

$$[J_s(\lambda)]^k = \sum_{m=0}^{k} \binom{k}{m} \lambda^{k-m} I_s^{k-m} N_s^m,$$

where $\binom{k}{m}$ denotes k combinations taken m at a time. Hence,

$$[J_s(\lambda)]^k = \begin{bmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \cdots & \cdots \\ 0 & \lambda^k & \binom{k}{1}\lambda^{k-1} & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & \lambda^k & \binom{k}{1}\lambda^{k-1} \\ 0 & 0 & \cdots & 0 & \lambda^k \end{bmatrix}.$$

Since, the first superdiagonal of $[J_s(\lambda)]^k$ consists of nonzero entries, it follows that the Jordan canonical form of $[J_s(\lambda)]^k$ is $J_s(\lambda^k)$. Hence by looking at the k^{th} power of the Jordan canonical form of A in (3.1), one can see that there is a matrix S consisting of a direct sum of similarity matrices for each of the individual blocks $[J_s(\lambda)]^k, \cdots, [J_r(\mu)]^k$ that appear in $[J(A)]^k$ such that

$$S^{-1}X^{-1}A^kXS = J_s(\lambda^k) \oplus \cdots = J(A^k) = \text{ the Jordan canonical form of } A^k.$$

Hence, if $\lambda \in \sigma(A)$, $\lambda \neq 0$, and $J_s(\lambda)$ is a Jordan block for A, then $J_s(\lambda^k)$ is also a Jordan block for A^k and it appears in $J(A^k)$ at least as many times as $J_s(\lambda)$ appears in $J(A)$. Moreover, suppose (with the hope of getting a contradiction) that the collection of Jordan blocks corresponding to λ^k in $J(A^k)$ has more blocks than the collection of Jordan blocks corresponding to λ in $J(A)$. Then, this could only happen if there is a $\mu \in \sigma(A)$ such that $\mu^k = \lambda^k$ but $\mu \neq \lambda$. Since $k \in N_A$, it follows from Lemma 3.1 that $\mu = \lambda$, a contradiction. Hence, $J(A^k)$ has the same number of of Jordan blocks for λ^k as $J(A)$ has for λ with the same orders and multiplicities. The only difference is that in $J(A^k)$ the eigenvalue λ^k appears instead of λ. □

The following corollaries follow directly from Lemma 3.2 with the same proofs as in [5].

Corollary 3.1 *Suppose that $A \in \mathbb{R}^{n\times n}$, $index_0(A) \in \{0,1\}$, and $A^s \geq 0$ for all $s \geq m$. Then, for all $g \in N_A \cap \{m, m+1, m+2, \ldots\}$, if for some ordered partition $\kappa = (\alpha_1, \alpha_2)$ of $\{1, 2, \ldots, n\}$ we have $(A^g)[\alpha_1, \alpha_2] = 0$ and $(A^g)[\alpha_2]$ is irreducible or a 1×1 zero block, then $A[\alpha_1, \alpha_2] = 0$.*

Corollary 3.2 *Suppose that $A \in \mathbb{R}^{n\times n}$, $index_0(A) \in \{0,1\}$, $A^s \geq 0$ for all $s \geq m$. Then, for all $g \in N_A \cap \{m, m+1, m+2, \ldots\}$, if $(A^g)_\kappa$ is in the Frobenius normal form for some ordered partition κ, then A_κ is also in the Frobenius normal form.*

Corollary 3.3 *Suppose that $A \in \mathbb{R}^{n\times n}$, $index_0(A) \in \{0,1\}$, and $A^s \geq 0$ for all $s \geq m$. Then, for all $g \in N_A \cap \{m, m+1, m+2, \ldots\}$, the transitive closures of the reduced graphs of A and A^g are the same, i.e., $\overline{R(A)} = \overline{R(A^g)}$.*

Theorem 3.1 *Suppose that $A \in \mathbb{R}^{n\times n}$, $index_0(A) = \nu$, and $A^s \geq 0$ for all $s \geq m$. Let $k = \lceil \frac{m}{\nu} \rceil = inf\{s \in \mathbb{N} \mid \nu s \geq m\}$. Then, for all*

$$g \in N_{A^\nu} \cap \{k, k+1, k+2, \ldots\},$$

we have $\overline{R(A^{\nu g})} = \overline{R(A^\nu)}$.

Proof. Consider the matrix $B = A^\nu$. Then B is eventually nonnegative. In fact, $B^g \geq 0$ for all $g \geq k$ where $k = inf\{s \in \mathbb{N} \mid \nu s \geq m\}$. It is easy to see that k is the smallest integer larger than $\frac{m}{\nu}$, denoted by $\lceil \frac{m}{\nu} \rceil$. By Corollary 3.3, for all $g \in N_B \cap \{k, k+1, k+2, \ldots\}$, we have $\overline{R(B^g)} = \overline{R(B)}$. In other words, for all $g \in N_{A^\nu} \cap \{k, k+1, k+2, \ldots\}$, we have $\overline{R(A^{\nu g})} = \overline{R(A^\nu)}$. □

Corollary 3.4 *Suppose that $A \in \mathbb{R}^{n\times n}$, $index_0(A) = \nu$, and $A^s \geq 0$ for all $s \geq m$. Let $k = \lceil \frac{m}{\nu} \rceil$. Then,*

> *if A does not have distinct eigenvalues with the same modulus, or*
>
> *if the argument differences of the distinct eigenvalues of A having the same modulus are irrational multiples of 2π,*

then for all $g \geq k$, we have $\overline{R(A^{\nu g})} = \overline{R(A^\nu)}$.

Proof. If A does not have distinct eigenvalues with the same modulus then neither does A^ν. Also, if the argument differences of the distinct eigenvalues of A having the same modulus are irrational multiples of 2π then so do the argument differences of the distinct eigenvalues of A^ν that have the same modulus. Hence, if either of these conditions is satisfied, then

$$D_{A^\nu} = D_A = \phi \Rightarrow P_{A^\nu} = P_A = \phi \Rightarrow N_{A^\nu} = N_A = \{1, 2, 3, \ldots\}.$$

Thus, by Theorem 3.1, for all $g \geq k$, we have $\overline{R(A^{\nu g})} = \overline{R(A^\nu)}$, where $k = \lceil \frac{m}{\nu} \rceil$. □

Recall that $v \in \mathbb{C}^n$ is a generalized eigenvector of $A \in \mathbb{C}^{n\times n}$ having order $m \geq 1$ and corresponding to $\lambda \in \mathbb{C}$ if

$$(A - \lambda I)^m v = 0 \qquad \text{but} \quad (A - \lambda I)^{m-1} v \neq 0.$$

In the following lemma, we collect some known properties of generalized eigenvectors, and then we prove a result needed for our main theorem.

Lemma 3.3 *Let $A \in \mathbb{C}^{n\times n}$.*

(i) A vector $v \in G_\lambda(A)$ is a generalized eigenvector of order $m \geq 2$ if and only if there is a generalized eigenvector $w \in G_\lambda(A)$ of order $m-1$ such that $Av = \lambda v + w$.

(ii) Let $A \in \mathbb{C}^{n\times n}$. If v is a generalized eigenvector in $G_\lambda(A)$ of order m, then Av is also a generalized eigenvector in $G_\lambda(A)$ of order m.

(iii) Let $A \in \mathbb{C}^{n\times n}$. If v and w are generalized eigenvectors in $G_\lambda(A)$ having orders m and l, respectively, and $1 \leq l \leq m$, then $v+w$ is a generalized eigenvector that has an order m corresponding to λ.

Lemma 3.4 *Let $A \in \mathbb{C}^{n\times n}$ and let $\lambda \in \sigma(A)$, $\lambda \neq 0$, then $G_\lambda(A) = G_{\lambda^k}(A^k)$ for all $k \in N_A$.*

Proof. We know from Lemma 3.2 that the Jordan box corresponding to λ^k in $J(A^k)$ is obtained from the Jordan box corresponding to λ in $J(A)$ by replacing λ with λ^k. And thus, $\dim G_\lambda(A) = mult_\lambda(A) = mult_{\lambda^k}(A^k) = \dim G_{\lambda^k}(A^k)$. Hence, to prove that $G_\lambda(A) = G_{\lambda^k}(A^k)$, it is enough to show that $G_\lambda(A) \subseteq G_{\lambda^k}(A^k)$. To do that, it is enough to show that v is a generalized eigenvector of order m in $G_{\lambda^k}(A^k)$ whenever v is a generalized eigenvector of order m in $G_\lambda(A)$ for all $m \in \{1, 2, \ldots, index_\lambda(A)\}$. We prove the latter statement by induction on m, the order of v. If $m = 1$, then $v \in G_\lambda(A)$ is an ordinary eigenvector of A corresponding to λ. Hence, $Av = \lambda v$ which implies $A^k v = \lambda^k v$. And thus, $v \in G_{\lambda^k}(A^k)$ is a generalized eigenvector of A^k of order 1. Suppose that for all $1 \leq l < m$ whenever $v \in G_\lambda(A)$ is a generalized eigenvector of order l, then $v \in G_{\lambda^k}(A^k)$ is a generalized eigenvector of A^k of order l. Let $v \in G_\lambda(A)$ be a generalized eigenvector of order m. By Lemma 3.3 (i), there is a generalized eigenvector $w \in G_\lambda(A)$ of order $m-1$

such that $Av = \lambda v + w$. And thus,

$$\begin{aligned} Av &= \lambda v + w, \\ A^2 v &= \lambda^2 v + \lambda w + Aw, \\ A^3 v &= \lambda^3 v + \lambda^2 w + \lambda A w + A^2 w, \\ &\vdots \\ A^k v &= \lambda^k v + \lambda^{k-1} w + \lambda^{k-2} A w + \cdots + \lambda A^{k-2} w + A^{k-1} w. \end{aligned}$$

By Lemma 3.3 (ii), the vectors $Aw, A^2w, \cdots, A^{k-1}w$ are all generalized eigenvectors in $G_\lambda(A)$ having order $m-1$. Hence, by Lemma 3.3 (iii), the vector $\lambda^{k-1}w + \lambda^{k-2}Aw + \cdots + \lambda A^{k-2}w + A^{k-1}w$ is a generalized eigenvector in $G_\lambda(A)$ having order $m-1$. By the induction hypothesis, the vector $\lambda^{k-1}w + \lambda^{k-2}Aw + \cdots + \lambda A^{k-2}w + A^{k-1}w$ is a generalized eigenvector in $G_{\lambda^k}(A^k)$ of order $m-1$. But, in such a case Lemma 3.3 (i), implies that v is a generalized eigenvector in $G_{\lambda^k}(A^k)$ of order m. □

Rothblum [35, Theorem 3.1] proved the following result:

Theorem 3.2 *Let $A \in \mathbb{R}^{n \times n}$ be nonnegative and let $\mathcal{N}(A - \rho(A)I)^k$ with $k = index_{\rho(A)}(A)$ being the algebraic eigenspace corresponding to $\rho(A)$. Assume that A has m basic classes $\alpha_1, \ldots, \alpha_m$. Then, $k = m$ and the algebraic eigenspace $\mathcal{N}(A - \rho(A)I)^m$ contains nonnegative vectors $v^{(1)}, \cdots, v^{(m)}$, such that $v_j^{(i)} > 0$ if and only if the index j has access to α_i in $G(A)$, the graph of A. Furthermore, any such collection is a basis of $\mathcal{N}(A - \rho(A)I)^m$.*

We now show that the latter theorem holds for eventually nonnegative matrices A whose $\text{index}_0(A) \in \{0, 1\}$.

Theorem 3.3 *Suppose that $A \in \mathbb{R}^{n \times n}$ is eventually nonnegative with $index_0(A) \in \{0, 1\}$ and let $\mathcal{N}(A - \rho(A)I)^k$ with $k = index_{\rho(A)}(A)$ being the algebraic eigenspace corresponding to $\rho(A)$. Assume that A has m basic classes $\alpha_1, \ldots, \alpha_m$. Then $k = m$ and the algebraic eigenspace $\mathcal{N}(A - \rho(A)I)^m$ contains nonnegative vectors $v^{(1)}, \cdots, v^{(m)}$, such that $v_j^{(i)} > 0$ if and only if the index j has access to α_i in $G(A)$, the graph of A. Furthermore, any such collection is a basis of $\mathcal{N}(A - \rho(A)I)^m$.*

Proof. Since A is eventually nonnegative, it follows that there is $p \in N_A$ such that $A^s \geq 0$ for all $s \geq p$. Let $k' = index_{\rho(A^p)}(A^p)$ and let $\kappa = (\alpha_1, \cdots, \alpha_{m'})$ be an ordered partition of $\{1, 2, \ldots, n\}$ that gives the Frobenius normal form of A^p. By Theorem 3.2, $k' = m'$ and the algebraic eigenspace $\mathcal{N}(A^p - \rho(A^p)I)^{k'}$ contains nonnegative vectors $v^{(1)}, \cdots, v^{(m')}$, such that $v_j^{(i)} > 0$ if and only if the index j has access to α_i in $G(A^p)$. By Lemma 3.2, $k' = k$. Moreover, Corollary 3.2 implies that $m' = m$ and the ordered partition κ also gives the Frobenius normal form of A. Hence, $k = m$ and the classes of A are the same as the classes of A^p. Moreover, we know from Lemma 3.4 that $\mathcal{N}(A - \rho(A)I)^k = \mathcal{N}(A^p - \rho(A^p)I)^k$. Thus, $v^{(1)}, \cdots, v^{(m)}$ is a basis of $\mathcal{N}(A - \rho(A)I)^k$. Furthermore, we claim that j has access to α_i in $G(A^p)$ if and only if j has access to α_i in $G(A)$. To prove the latter claim, let β denote the class to which the index j belongs and consider the reduced graphs of A and A^p. By Corollary 3.3, the transitive closures of the reduced graphs of A and A^p are the same. Hence, the reduced graphs of A and A^p have the same access relations. Thus, β has access to α_i in the reduced graph of A if and only if β has access to α_i in the reduced graph of A^p. Since j communicates with any vertex in β, it follows that j has access to α_i in $G(A^p)$ if and only if j has access to α_i in $G(A)$, and thus, the claim of this theorem is true. □

Corollary 3.5 *Suppose that $A \in \mathbb{R}^{n \times n}$ is an eventually nonnegative matrix with $index_0(A) \in \{0, 1\}$. Then, there is a positive eigenvector corresponding to $\rho(A)$ if and only if the final classes of A are exactly its basic ones.*

Corollary 3.6 *Suppose that $A \in \mathbb{R}^{n \times n}$ is an eventually nonnegative matrix with $index_0(A) \in \{0, 1\}$. Then, there are positive right and left eigenvectors corresponding to $\rho(A)$ if and only if all the classes of A are basic and final, i.e., A is permutationally similar to a direct sum of irreducible matrices having the same spectral radius.*

3.2 Matrices That Are Eventually in WPF_n and PF_n

As we have seen, nonnilpotent eventually nonnegative matrices have the Perron-Frobenius property. It is natural then to ask, what can we say of matrices whose powers eventually belong to WPF_n (or PF_n). We show in this short section that these matrices must belong to WPF_n (or PF_n).

Theorem 3.4 *$A \in WPF_n$ if and only if for some integer m, $A^k \in WPF_n$, for all $k \geq m$.*

Proof. Suppose that $A \in WPF_n$. For any $\lambda \in \sigma(A)$, $\lambda \neq 0$, and all $k \geq 1$, we have $E_\lambda(A) \subseteq E_{\lambda^k}(A^k)$. In particular, this is true for $\lambda = \rho(A) > 0$. Using the fact that $\rho(A^k) = (\rho(A))^k$, we see that $E_{\rho(A)}(A) \subseteq E_{\rho(A)^k}(A^k)$. Thus, if A has the Perron-Frobenius property, then so does A^k for all $k \geq 1$. Likewise, $(A^T)^k$ has the Perron-Frobenius property for all $k \geq 1$. Thus, $A^k \in WPF_n$ for all $k \geq 1$. Conversely, suppose that there is a positive integer m such that $A^k \in WPF_n$ for $k \geq m$. Since the eigenvalues of A are the k^{th} roots of the eigenvalues of A^k for all $k \geq m$, it follows that $0 \neq \rho(A) \in \sigma(A)$. Moreover, by picking $k \in N_A \cap \{m, m+1, m+2, \ldots\}$ ("nice powers" k of A that are larger than m), we have $E_{\rho(A)}(A) = E_{\rho(A)^k}(A^k)$ (this follows from Lemma 3.2). Hence, we can choose a nonnegative eigenvector of A corresponding $\rho(A)$. So, A has the Perron-Frobenius property. Similarly, A^T has the Perron-Frobenius property. Thus, $A \in WPF_n$. □

Similarly, we obtain the following result.

Theorem 3.5 *$A \in PF_n$ if and only if for some integer m, $A^k \in PF_n$, for all $k \geq m$.*

3.3 Similarity Matrices Preserving the Perron-Frobenius Property

If S is a positive diagonal matrix or a permutation matrix then clearly $S^{-1}AS$ possesses the Perron-Frobenius property whenever A does. This observation leads to the following question, which we answer in this section: which similarity matrices S preserve the Perron-Frobenius property, the strong Perron-Frobenius property, or being in WPF_n, or in PF_n? We first prove a preliminary lemma that leads to answering the latter question.

Lemma 3.5 *Let S be an $n \times n$ real matrix which has a positive entry and a negative entry. If S is of rank one but not expressible as xy^T with x being a nonnegative vector, or S is of rank two or more, then there is a positive vector $v \in \mathbb{R}^n$ such that Sv has a positive entry and a negative entry.*

Proof. If S is a rank-one matrix with the given property, then S is expressible as xy^T, where x is a vector which has a positive entry and a negative entry. Choose any positive vector v such that $y^Tv \neq 0$. Then Sv, being a nonzero multiple of x, clearly has a positive entry and a negative entry.

Suppose that S is of rank two or more. If S has a column which has a positive entry and a negative entry, say, the kth column, then take v to be the positive vector in $\mathbb{R}^n$ whose kth entry is 1 and all of whose other entries equal ϵ. It is readily seen that for $\epsilon > 0$ sufficiently small, Sv has a positive entry and a negative entry. It remains to consider the case when every nonzero column of S is either semipositive or seminegative. Because S is of rank two or more, it is possible to choose two linearly independent columns of S, with one semipositive and the other seminegative; say the jth column is semipositive and the kth column is seminegative. If the jth column has a zero entry such that the corresponding entry for the kth column is negative, then clearly $S(e_j + \delta e_k)$ (where e_i denotes the ith standard unit vector of $\mathbb{R}^n$) has a positive entry and a negative entry for sufficiently small $\delta > 0$, hence so does the vector Sv where v is the positive vector of $\mathbb{R}^n$ with 1 at its jth entry, δ at its kth entry and ϵ at its other entries, where $\epsilon > 0$ is sufficiently small. Similarly, if the kth column has a zero entry such that the corresponding entry for the jth column is positive, then by a similar argument we are also done. So, the jth and the kth columns of S have zeroes at exactly the same positions. Consider $S((1-\lambda)e_j + \lambda e_k)$. Let λ_0 be the largest $\lambda \in [0,1]$ such that $S((1-\lambda)e_j + \lambda e_k)$ is nonnegative. Because we assume that the jth and the kth columns of S are linearly independent, it is clear that $S((1-\lambda_0)e_j + \lambda_0 e_k)$ is in fact semipositive, i.e., a nonzero vector. Choose $\lambda_1 > \lambda_0$, sufficiently close to λ_0. Then $S((1-\lambda_1)e_j + \lambda_1 e_k)$ has a positive entry and a negative entry. Now let v be the positive vector in $\mathbb{R}^n$ whose jth entry is $1 - \lambda_1$, whose kth entry is λ_1, and all of whose other entries are ϵ. Then, for $\epsilon > 0$ sufficiently small Sv has a positive entry and a negative entry. □

We call a matrix S *monotone* if $S \in GL(n, \mathbb{R})$ and S^{-1} is nonnegative.

Theorem 3.6 *For any $S \in GL(n, \mathbb{R})$, the following statements are equivalent:*

(i) Either S or $-S$ is monotone.

(ii) $S^{-1}AS$ has the strong Perron-Frobenius property for all matrices A having the strong Perron-Frobenius property.

Proof. Suppose (i) is true. Assume without loss of generality that S is monotone. If A is a matrix with the strong Perron-Frobenius property and v is a right Perron-Frobenius eigenvector of A, then $S^{-1}v$ is an eigenvector of $S^{-1}AS$ corresponding to $\rho(A)$. The nonsingularity of S implies that none of the rows of S^{-1} is 0. Therefore, $S^{-1}v$ is a positive vector. Also, $\rho(A)$ is a simple positive and strictly dominant eigenvalue of $S^{-1}AS$ since $S^{-1}AS$ and A have the same characteristic polynomial. This shows that $(i) \Rightarrow (ii)$. Conversely, suppose (i) is not true, i.e., S and $-S$ are both not monotone. Then, in such a case, S^{-1} must have a positive entry and a negative entry. By Lemma 3.5, there is a positive vector v such that $S^{-1}v$ has a positive entry and a negative entry. For any scalar $\rho > 0$, we can construct the matrix $A = (\rho/v^T v)vv^T \in PF_n$, having v as a right Perron-Frobenius eigenvector. Moreover, for such a matrix A, we have $E_{\rho(A)}(A) = Span\{v\}$. Since the eigenvectors in $E_{\rho(A)}(S^{-1}AS)$ are of the form $S^{-1}w$ for some eigenvector $w \in E_{\rho(A)}(A)$, it follows that $E_{\rho(A)}(S^{-1}AS)$ does not have a positive vector. Thus, $S^{-1}AS$ does not have the strong Perron-Frobenius property. Hence, (ii) is not true, which shows that $(ii) \Rightarrow (i)$. □

The following results follow in the same manner.

Theorem 3.7 *For any $S \in GL(n,\mathbb{R})$, the following statements are equivalent:*

(i) Either S or $-S$ is monotone.

(ii) $S^{-1}AS$ has the Perron-Frobenius property for all matrices A having the Perron-Frobenius property.

Corollary 3.7 *For any $S \in GL(n,\mathbb{R})$, the following statements are equivalent:*

(i) S and S^{-1} are either both nonnegative or both nonpositive.

(ii) $S^{-1}AS \in PF_n$ for all $A \in PF_n$.

(iii) $S^{-1}AS \in WPF_n$ for all $A \in WPF_n$.

3.4 The Perron-Frobenius Property and Real Symmetric Matrices

In this section, S_n denotes the collection of $n \times n$ real symmetric matrices and $e \in \mathbb{R}^n$ denotes the vector that consists entirely of ones. We study the boundary of the cone in S_n of maximal angle centered at $E = ee^T$ (the matrix of ones) in which the nonnegativity of both the dominant eigenvalue and its corresponding eigenvector is retained. Tarazaga, Raydan, and Hurman studied this cone for $n \geq 3$ [48, Theorem 4.1] and showed that the angle of such a cone is

$$\alpha = \arccos\left(\frac{\sqrt{(n-1)^2+1}}{n}\right). \tag{3.2}$$

The authors of [48] do not claim that PF_n or WPF_n is a cone. In fact, it is shown by Johnson and Tarazaga [26] that PF_n is not even convex. Thus, neither PF_n nor WPF_n is necessarily a cone. We explore this further and show that there is a curve of matrices with the strong Perron-Frobenius property extending outside the cone centered at $E = ee^T$ and making an angle α given in (3.2).

Proposition 3.1 *The maximal subset of S_n ($n \geq 3$) for which there is a nonnegative Perron-Frobenius eigenpair extends outside the cone centered at $E = ee^T$ whose angle α is given by (3.2).*

Proof. Consider a matrix A in S_n ($n \geq 3$) of the form:

$$A = A(x) = \begin{bmatrix} x & x & x & \cdots & x \\ x & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x & 1 & \cdots & 1 & 1 \\ x & 1 & \cdots & 1 & x \end{bmatrix}$$

where x is a positive scalar. Obviously, the matrix A is a positive matrix in S_n, and thus, it possesses the strong Perron-Frobenius property for every positive scalar x.

We will show that there exists a $\delta > 0$ such that $Angle(A, ee^T) > \alpha$ (i.e., $\cos(A, ee^T) < \cos\alpha$) whenever $0 < x < \delta$. First, let us compute the cosine of the angle between A and ee^T. Let a_{ij} denote the (i, j)-entry of A. Then,

$$\cos(A, ee^T) = \frac{\sum_{i,j} a_{ij}}{n\sqrt{\sum_{i,j}(a_{ij})^2}} = \frac{2nx+n^2-2n}{n\sqrt{2nx^2+n^2-2n}}$$

And thus, $\cos(A, ee^T) < \cos\alpha$ if and only if $\frac{2nx+n^2-2n}{n\sqrt{2nx^2+n^2-2n}} < \frac{\sqrt{(n-1)^2+1}}{n}$. Define the functions $f_n(x)$ ($n \geq 3$) by:

$$f_n(x) = \frac{\sqrt{(n-1)^2+1}}{n} - \frac{2nx+n^2-2n}{n\sqrt{2nx^2+n^2-2n}}.$$

Then, we want to see when $f_n(x) > 0$. Note that for all $n \geq 3$:

$$f_n(0) = \frac{\sqrt{(n-1)^2+1}}{n} - \frac{n^2-2n}{n\sqrt{n^2-2n}} = \frac{\sqrt{n^2-2n+2}-\sqrt{n^2-2n}}{n} > 0$$

By continuity of the function $f_n(x)$, there exists $\delta_n > 0$ such that $f_n(x) > 0$ whenever $0 < x < \delta_n$. Hence, when $0 < x < \delta_n$ the matrix A possesses the strong Perron-Frobenius property yet it lies outside the cone centered at ee^T with angle α. Indeed, we can define a curve $A_\gamma : [0,1) \to S_n$ of the matrices $A_\gamma(t) = A(1-t)$, which lie outside the cone centered at ee^T with angle α, for $1 - \delta_n < t < 1$, while they satisfy the strong Perron-Frobenius property. Furthermore, since the eigenvalues and the eigenvectors depend continuously on the matrix entries (see, e.g., [3], [22]), it follows that there is a neighborhood of the curve $A_\gamma(t)$ defined for $1 - \delta_n < t < 1$, in which the strong Perron-Frobenius property holds as well. The intersection of this neighborhood with S_n further extends the known collection of such matrices lying outside the cone mentioned above. □

3.5 Topological Properties

In this section, we prove some topological properties of the collections of matrices with the Perron-Frobenius property and other subcollections.

The following lemma was asserted and used by Johnson and Tarazaga in the proof of [26, Theorem 2]; its proof can be found in [31].

Lemma 3.6 *Let A be a matrix in $\mathbb{R}^{n\times n}$ with the Perron-Frobenius property, and let v be its right Perron-Frobenius eigenvector. If $w \in \mathbb{R}^n$, $w \neq 0$, is such that $v^T w > 0$ then for all scalars $\epsilon > 0$ the following holds:*

(i) The matrix $B = A + \epsilon v w^T$ has the Perron-Frobenius property.

(ii) $\rho(A) < \rho(B)$.

(iii) If A has the strong Perron-Frobenius property then so does B.

Theorem 3.8 *The collection of matrices in $\mathbb{R}^{n\times n}$ with the Perron-Frobenius property is path-connected.*

Proof. Let $A \in \mathbb{R}^{n\times n}$ be any matrix with the Perron-Frobenius property. Since the collection of positive matrices is convex, it is enough to show that there is a path connecting matrix A to some positive matrix B. The proof goes as follows: connect matrix A to a matrix $\tilde{A}$ having a positive right Perron-Frobenius eigenvector and then connect $\tilde{A}$ to a positive matrix B.

If A has a positive right Perron-Frobenius eigenvector then define $\tilde{A} = A$, otherwise, consider $J(A)$, the Jordan canonical form of A. We know that $A = VJ(A)V^{-1}$ where $V = [v\ w_2\ w_3 \cdots w_n]$ and v is a right Perron-Frobenius eigenvector of A. For every scalar $t \geq 0$, we construct the vector v_t by replacing the zero entries of v by t, and we construct a new matrix $V_t = [v_t\ w_2\ w_3 \cdots w_n]$. Since $V_0 = V \in GL(n, \mathbb{C})$ and since $GL(n, \mathbb{C})$ is an open subset of $\mathbb{C}^{n\times n}$, there is a positive scalar δ such that whenever $0 \leq t \leq \delta$ we have $V_t \in GL(n, \mathbb{C})$. Define $A_t = V_tJ(A)V_t^{-1}$ for $0 \leq t \leq \delta$. Then, A_t is a path of complex matrices having a positive dominant eigenvalue $\rho(A)$ with a corresponding nonnegative eigenvector v_t. We show now that the real part of A_t is a path of real matrices connecting matrix A to our desired matrix $\tilde{A}$, which will be defined soon. Note that v_t is positive for all $0 < t \leq \delta$ and that $A_t = C_t + iD_t$ where C_t and D_t are paths of real matrices. Since $A_tv_t = C_tv_t + iD_tv_t = \rho(A)v_t \in \mathbb{R}^n$ for all $0 \leq t \leq \delta$, it follows that $D_tv_t = 0$ and that $C_tv_t = \rho(A)v_t$ for all $0 \leq t \leq \delta$. Moreover, $C_0 = A_0 = A$. Hence, C_t is a path of real matrices connecting A to C_δ, and each matrix C_t has a positive dominant eigenvalue $\rho(A)$ with a corresponding nonnegative eigenvector v_t, therefore having the Perron-Frobenius property. Let $\tilde{A} = C_\delta$ and let v_δ be its corresponding positive eigenvector.

Let w be any positive vector. For all scalars $\epsilon \geq 0$, define the path of real matrices $K_\epsilon = \tilde{A} + \epsilon v_\delta w^T$. By Lemma 3.6, K_ϵ possesses the Perron-Frobenius property for all $\epsilon \geq 0$. Since $v_\delta w^T$ is a positive matrix, K_ϵ is positive for large values of ϵ. Hence, there is a positive real number M such that K_M is positive.

Let $B = K_M$. Hence, K_ϵ is a path connecting $\tilde{A}$ to B. □

Similarly, we have the following result.

Theorem 3.9 *The collection of matrices in $\mathbb{R}^{n\times n}$ with the strong Perron-Frobenius property is path-connected.*

Corollary 3.8 *PF_n is simply connected.*

Proof. Johnson and Tarazaga proved in [26, Theorem 2] that PF_n is path-connected. Thus, it is enough to show that any loop in PF_n can be shrunk to a point. Let $A_t : [0,1] \rightarrow PF_n$ be a loop of matrices in PF_n. For all $0 \leq t \leq 1$, let v_t and w_t be respectively the right and the left Perron-Frobenius unit eigenvectors of A_t. Also, for all scalars $\epsilon \geq 0$, define the loop $B_t^\epsilon = A_t + \epsilon v_t w_t^T$. By Lemma 3.6, the loop B_t^ϵ is in PF_n for all scalars $\epsilon \geq 0$. Note that for large values of ϵ the loop B_t^ϵ is a loop of positive matrices. Hence, A_t can be continuously deformed to a loop that can be shrunk to a point. □

Corollary 3.9 *The collection of matrices in $\mathbb{R}^{n\times n}$ with the strong Perron-Frobenius property is simply connected.*

Proposition 3.2 *The closure $\overline{WPF_n} = WPF_n \cup \{$nilpotent matrices with a pair of right and left nonnegative eigenvectors$\}$.*

Proof. Since the eigenvalues and eigenvector entries are continuous functions of the matrix entries, it follows that for any matrix A in $\overline{WPF_n}$ we have $\rho(A) \geq 0$ and A has a pair of left and right nonnegative eigenvectors corresponding to $\rho(A)$. If $\rho(A) = 0$ then A is nilpotent with a pair of right and left nonnegative eigenvectors, otherwise A is in WPF_n. Conversely, suppose that A is in WPF_n or A is a nilpotent matrix with a pair of right and left nonnegative eigenvectors. If A is in WPF_n then obviously A is in $\overline{WPF_n}$. If A is a nilpotent matrix with a pair of right and left nonnegative eigenvectors v and w, respectively, then A has a Jordan canonical form $A = VJ(A)V^{-1}$, where V and V^{-1} are real matrices, all the Jordan blocks in $J(A)$ are of the form $J_s(0)$ for some $s \in \{1, \dots, n\}$, the ith column of V is v, and the j^{th} row of V^{-1} is w^T for some $i, j \in \{1, \dots, n\}$. Let e_i denote the ith standard unit vector of $\mathbb{R}^n$. For every positive scalar ϵ, let $J_\epsilon = J(A) + \epsilon(e_i e_i^T + e_j e_j^T)$, and $A_\epsilon = V J_\epsilon V^{-1}$. Note that $A_\epsilon v = \epsilon v$ and $w^T A_\epsilon = \epsilon w^T$. Hence, $A_\epsilon \in WPF_n$ for all $\epsilon > 0$. Moreover, A_ϵ converges to A as $\epsilon \to 0$. □

Lemma 3.7 *For any semipositive vector v_1 and for any scalar $\epsilon > 0$, there is an orthogonal matrix Q such that $||Q - I||_2 < \epsilon$ and $Qv_1 > 0$.*

Proof. Assume without loss of generality that v_1 is a unit vector. If v_1 is a positive vector then let $Q = I$, otherwise pick any scalar $\epsilon > 0$ and replace the zero entries of v_1 by positive entries that are small enough then normalize so that the obtained vector, say $\tilde{v}$, is a positive unit vector and $||\tilde{v} - v_1||_2 < \epsilon/n$. Let $S = I - v_1 v_1^T$ be the projection matrix onto $v_1^\perp$, the hyperplane orthogonal to v_1, and let $v_2 = S\tilde{v}/||S\tilde{v}||_2$. Then, v_2 is a unit vector which is orthogonal to v_1. Moreover, $\tilde{v}$ lies in the 2-dimensional plane determined by v_1 and v_2. Let $\theta = \text{Angle}(v_1, \tilde{v}) = \arccos(v_1^T \tilde{v})$. Extend $\{v_1, v_2\}$ to an orthonormal basis $\{v_1, v_2, \cdots, v_n\}$ of $\mathbb{R}^n$. Define Q to be the Givens rotation (see, e.g, [18]) by the angle θ in the 2-dimensional plane determined by v_1 and v_2. Then, $\tilde{v} = Qv_1$, $||Qv_2 - v_2||_2 = ||Qv_1 - v_1||_2 < \epsilon/n$, and $Qv_i = v_i$ for all $i \geq 3$. Therefore, $Qv_1 = \tilde{v} > 0$ and $||Q - I||_2 = sup_{||x||_2=1}||(Q - I)x||_2 < \epsilon$. □

Proposition 3.3 *Every normal matrix in WPF_n is the limit of normal matrices in PF_n.*

Proof. Let A be a normal matrix in WPF_n. Then, $A = VSV^T$ where V is an orthogonal matrix, $S = [\rho(A)] \oplus M_2 \cdots \oplus M_k$, and each M_i (i = 2, ..., k) is a real 1×1 block or a nonzero real 2×2 block of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $a, b \in \mathbb{R}$. Moreover, one of the columns of V, say the first column which we denote by v, is both a right and a left Perron-Frobenius eigenvector of A. For any scalar $\epsilon > 0$, consider the matrix $B = V[[\rho(A) + \epsilon] \oplus \cdots \oplus M_k]V^T$ which has a simple positive and strictly dominant eigenvalue $\rho(A) + \epsilon$. Note that B converges to A as $\epsilon \to 0$. By Lemma 10.7, there is an orthogonal matrix Q such that $Qv > 0$ and $||Q - I||_2 < \epsilon$. Let $C = QBQ^T$. Then, C is a normal matrix having $\rho(A) + \epsilon$ as simple positive and strictly dominant eigenvalue. Moreover, C satisfies the following vector equalities $CQv = (\rho(A) + \epsilon)Qv$, and $(Qv)^T C = (\rho(A) + \epsilon)(Qv)^T$. Therefore, C is a normal matrix in PF_n. Furthermore,

$$\begin{aligned} ||C - A||_2 &\leq ||C - B||_2 + ||B - A||_2 \\ &= ||QBQ^T - B||_2 + ||B - A||_2 \\ &= ||QB - BQ||_2 + ||B - A||_2 \\ &\leq ||QB - B||_2 + ||B - BQ||_2 + ||B - A||_2 \\ &\leq 2||B||_2 \, ||Q - I||_2 + ||B - A||_2 \to 0 \text{ as } \epsilon \to 0. \qquad \square \end{aligned}$$

3.6 Singular Values and Singular Vectors

We explore in this section some sign properties of the singluar value decomposition of matrices in WPF_n. Recall that every rectangular matrix in $A \in \mathbb{R}^{m \times n}$ can be expressed as $A = U \Sigma V^T$, where $U = [u_1 \cdots u_m]$ is an $m \times m$ orthogonal matrix (u_j is the j^{th} column of U), $V = [v_1 \cdots v_n]$ is an $n \times n$ orthogonal matrix (v_j is the j^{th} column V), and $\Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_p)$ is an $m \times n$ matrix satisfying $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$ and $p = min\{m, n\}$. Such a decomposition is known as the singular value decomposition of A or simply as the SVD of A. It is easy to check that $Av_i = \sigma_i u_i$ and $A^T u_i = \sigma_i v_i$ for all $i \in \{1, 2, \ldots, p\}$. The v_i's are known as the right singular vectors of A, the u_i's are known the left singular vectors of A, and the σ_i's are known as the singular values of A; see, e.g, [18], [22]. In the following three results, the SVD's of matrices that are eventually nonnegative or in general enjoying the Perron-Frobenius property are analyzed and sufficient conditions for the nonnegativity of the right and left singular vectors corresponding to the maximum singular value are given.

Theorem 3.10 *If A is a normal matrix in PF_n, then the maximum singular* *value is strictly larger than the other singular values and its right and left singular vectors are positive.*

Proof. Let $A = U\Sigma V^T = [u_1\ u_2 \cdots u_n]\ diag(\sigma_1, \sigma_2, \ldots, \sigma_n)\ [v_1\ v_2 \cdots v_n]^T$ be the singular value decomposition of A, where U and V orthogonal, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are the singular values of A. Note that v_1 and u_1 are, respectively, the right and the left singular vectors corresponding to σ_1, i.e., $Av_1 = \sigma_1 u_1$ and $A^T u_1 = \sigma_1 v_1$. Since $A \in PF_n$, then both A and A^T are commuting eventually positive matrices. Hence, $A^T A$ is eventually positive, and thus, it possesses the strong Perron-Frobenius property. But, $A^T A = V\Sigma^2 V^T = [v_1\ v_2 \cdots v_n]\ diag(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)\ [v_1\ v_2 \cdots v_n]^T$ is the Jordan decomposition of $A^T A$. Therefore, v_1 must be positive and $\sigma_1^2 > \sigma_i \geq 0$ for all $i \in \{2, \ldots, n\}$. Similarly, by noting that AA^T is eventually positive, we show that u_1 is positive. □

Proposition 3.4 *If A is a nonzero, normal and eventually nonnegative matrix, then its right and left singular vectors corresponding to its maximum singular value are nonnegative.*

Proof. If A is a nonzero, normal and eventually nonnegative matrix, then both A and A^T are commuting eventually nonnegative matrices. Hence, $A^T A$ is eventually nonnegative. Since the maximum eigenvalue of $A^T A$ is $||A||_2^2 \neq 0$ (because $A \neq 0$), it follows that $A^T A \in WPF_n$, and thus, it possesses the Perron-Frobenius property. And thus, the right singular vector corresponding to its maximum singular value must be nonnegative. Similarly, by noting that AA^T is eventually nonnegative, we show that the left singular vector corresponding to its maximum singular value is also nonnegative. □

Note that a matrix A which satisfies the conditions required by Proposition 3.4 is automatically nonnilpotent since there is no normal nonzero nilpotent matrix. The next result is a more general result that applies to WPF_n.

Theorem 3.11 *If A is a normal matrix in WPF_n, then its right and left singular vectors corresponding to its maximum singular value are nonnegative.*

Proof. If A is a normal matrix in WPF_n, then by Proposition 3.3 there is a sequence of normal matrices $\{A_k\}_{k=1}^{\infty} \subset PF_n$ that converges to A. For each $k \in \{1, 2, \ldots\}$, the matrices A_k and A_k^T are commuting eventually positive matrices. And just like in the proof of Proposition 3.10, the matrix $A_k^T A_k$ possesses the strong Perron-Frobenius property and its (positive) Perron-Frobenius eigenvector, say v_k, is the right singular vector of the maximum singular value of A_k. By continuity of eigenvector entries as functions of matrix entries and since $A_k^T A_k \rightarrow A^T A$ as $k \rightarrow \infty$, it follows that the positive unit vector v_k which is an eigenvector of $A_k^T A_k$ converges to some nonnegative unit vector v which is an eigenvector of $A^T A$, i.e., to the right singular vector of A corresponding to the maximum singular value. Similarly, we show that there is a sequence of positive unit vectors converging to the left singular vector of A corresponding to the maximum singular value. □

Example 3.1 If a matrix in PF_n or WPF_n is not normal or, equivalently, not unitarily diagonalizable, then the singular vectors may have positive and negative entries. For example, consider the matrix

$$C = \frac{1}{4}\begin{bmatrix} 11 & 30 & -9 \\ 7 & 2 & 23 \\ 7 & 30 & -5 \end{bmatrix}.$$

The matrix C is a diagonalizable matrix in PF_3, but it is not unitarily diagonalizable. In fact, the Jordan decomposition of C is given by $C = XJ(C)X^{-1}$, where

$$X = \begin{bmatrix} 1 & 1 & -3 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad J(C) = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad X^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \\ -1 & 0 & 1 \end{bmatrix}.$$

The singular value decomposition of C yields that the right singular vector corresponding to the maximum singular value is $[-0.2761, \ -0.9311, \ 0.2385]^T$ and the corresponding left singular vector is $[-0.7289, \ 0.0372, \ -0.6836]^T$, each of which has a negative entry and a positive entry. Similarly, by taking direct sums of matrix C with positive matrices, one can find counter-examples in WPF_n in which the right and left singular vectors corresponding to the maximum singular value have positive and negative entries.

CHAPTER 4

GENERALIZATIONS OF M-MATRICES

4.1 Introduction and Preliminaries

Closely related to the subject of nonnegative matrices and their generalizations is the subject of M-matrices. A matrix $A \in \mathbb{R}^{n \times n}$ is called an M-matrix if it can be expressed as $A = sI - B$ where B is nonnegative and $\rho(B) \leq s$. In this chapter, we study generalizations of M-matrices of the form $A = sI - B$ where $B \in WPF_n$ and $\rho(B) \leq s$. We call such matrices GM-matrices. We also study other generalizations of this type and present some of their properties which are counterparts to those of M-matrices. Among the generalizations of M-matrices we study are matrices of the form $A = sI - B$ with $\rho(B) \leq s$ and B being an eventually nonnegative or an eventually positive matrix. Johnson and Tarazaga [26] termed the latter class, pseudo-M-matrices. Le and McDonald [29] studied the case where B is an irreducible eventually nonnegative matrix. We mention also other generalizations of M-matrices not considered in this chapter; namely, where B leaves a cone invariant (see, e.g., [45], [50]) or for rectangular matrices; see, e.g., [34].

It is well-known that the inverse of a nonsingular M-matrix is nonnegative [2], [49]. This property leads to the natural question: for which nonnegative we study analogous questions, such as: for which matrices having the Perron-Frobenius property is the inverse a GM-matrix?

Another aspect we address in Section 4.3 is the study of splittings $A = M - N$ of a GM-matrix A with conditions for their convergence.

Recall that a matrix $A \in \mathbb{R}^{n\times n}$ is a Z-matrix if A can be expressed in the form $A = sI - B$ where s is a positive scalar and B is a nonnegative matrix. Moreover, if $A = sI - B$ is a Z-matrix such that $\rho(B) \leq s$, then we call A an M-matrix.

If $A \in \mathbb{R}^{n\times n}$ can be expressed as $A = sI - B$ where $B \in WPF_n$, then we call A

- a GZ-matrix.
- a GM-matrix if $0 < \rho(B) \leq s$.
- an EM-matrix if $0 < \rho(B) \leq s$ and B is eventually nonnegative.
- a pseudo-M-matrix if $0 < \rho(B) < s$ and $B \in PF_n$ [26].

When the inverse of a matrix C is a GM-matrix then we call C an *inverse GM-matrix.*

It follows directly from the definitions that every M-matrix is an EM-matrix, that every EM-matrix is a GM-matrix, and that every pseudo-M-matrix is an EM-matrix. We show by examples below that the converses do not hold.

Furthermore, an M-matrix may not be a pseudo-M-matrix. Consider, for example, a reducible M-matrix. We illustrate the relations among the different sets of matrices in Figure 4.1.

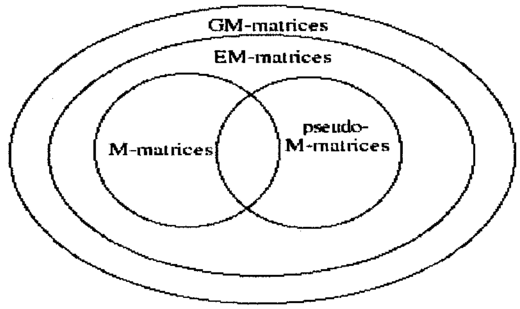

Figure 4.1: This diagram summarizes the relations between the sets of various generalizations of M-matrices using the Perron-Frobenius property.

Example 4.1

$$\text{Let } A = sI - B \text{ where } B = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & -1 & 1 & 2 & 2 \end{bmatrix} \text{ and } s > 4.$$

Note that matrix B, which is taken from [5, Example 4.8], is a reducible nonnilpotent eventually nonnegative matrix with $\rho(B) = 4$. Hence, A is an EM-matrix. Since A is reducible, it follows that, for any positive scalar δ, we have $\delta I - A$ reducible and any power of $\delta I - A$ reducible. Hence, for any positive scalar δ, the matrix $\delta I - A$ is not eventually positive (i.e. $(\delta I - A) \notin PF_6$). And thus, A is not a pseudo-M-matrix. Moreover, A is not an M-matrix because A has positive off-diagonal entries.

Example 4.2

$$\text{Let } A = sI - B \text{ where } B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \text{ and } s > 2.$$

Note that $\rho(B) = 2$ is an eigenvalue having $[1\ 1\ 0\ 0]^T$ as a right and a left eigenvector. Hence, $B \in WPF_4$ and A is a GM-matrix. However, B is not eventually nonnegative because the lower right 2×2 block of B keeps on alternating signs. Moreover, for any positive scalar δ, the lower 2×2 block of $\delta I - A$ is the matrix $C = \begin{bmatrix} \delta - s - 1 & -1 \\ -1 & \delta - s - 1 \end{bmatrix}$. Note that for any positive integer k, the lower 2×2 block of $(\delta I - A)^k$ is the matrix C^k which is, using an induction argument, the matrix $\frac{1}{2}\begin{bmatrix} (\delta - s - 2)^k + (\delta - s)^k & (\delta - s - 2)^k - (\delta - s)^k \\ (\delta - s - 2)^k - (\delta - s)^k & (\delta - s - 2)^k + (\delta - s)^k \end{bmatrix}$. It is easy to see that for any choice of a positive scalar δ the matrix $\delta I - A$ is not eventually nonnegative because the (2,1)-entry of C^k will always be negative for odd powers k.

4.2 Properties of GM-Matrices

In this section, we generalize some results known for M-matrices to GM-matrices. For example, if A is a nonsingular M-matrix, then A^{-1} is nonnegative; see, e.g., [2], [49]. We show analogous results for GM- and pseudo-M-matrices. However, we show by an example that no analogous result for EM-matrices holds.

Theorem 4.1 *Let A be a matrix in $\mathbb{R}^{n \times n}$ whose eigenvalues with multiplicity are arranged in the following manner: $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. Then the following statements are equivalent:*

(i) A is a nonsingular GM-matrix.

(ii) $A^{-1} \in WPF_n$ *and* $0 < \lambda_n < Re(\lambda_i)$ *for all* $\lambda_i \neq \lambda_n$.

Proof. Suppose first that $A = sI - B$ is a nonsingular GM-matrix ($B \in WPF_n$ and $0 < \rho(B) < s$). Then, there are semipositive vectors v and w such that $Bv = \rho(B)v$ and $w^T B = \rho(B) w^T$. This implies that $A^{-1}v = (s-\rho(B))^{-1}v$ and that $w^T A^{-1} = (s-\rho(B))^{-1} w^T$. Thus, v and w are eigenvectors of A^{-1} and furthermore $\rho(A^{-1}) = |\lambda_n|^{-1} = (s - \rho(B))^{-1} > 0$. Therefore, $A^{-1} \in WPF_n$. Moreover, $|\lambda_n| = \lambda_n$, i.e., $Re(\lambda_n) > 0$ and $Im(\lambda_n) = 0$, otherwise, if we have $Re(\lambda_n) \leq 0$, then the eigenvalue $(s - \lambda_n) \in \sigma(B)$ satisfies $|s - \lambda_n| > |s - |\lambda_n|| = \rho(B)$, which is a contradiction. Or, if $Re(\lambda_n) > 0$ but $Im(\lambda_n) \neq 0$, then again, $|s - \lambda_n| > |s - |\lambda_n|| = \rho(B)$, which is a contradiction. Therefore, $|\lambda_n| = \lambda_n > 0$. Similarly, one could show that if $|\lambda_i| = \lambda_n$ for some $i \in \{1, \ldots, n-1\}$ then $\lambda_i = \lambda_n > 0$. Furthermore, suppose that $\lambda_n \geq Re(\lambda_i)$ for some $\lambda_i \neq \lambda_n$, then $|\lambda_i| > \lambda_n$ (otherwise, $\lambda_i = \lambda_n$). If $Re(\lambda_i) = \lambda_n$, then $|\lambda_i| > Re(\lambda_i)$, therefore $|Im(\lambda_i)| > 0$. Thus,

$$|s-\lambda_i| = \sqrt{|s - Re(\lambda_i)|^2 + |Im(\lambda_i)|^2} > |s - Re(\lambda_i)| \geq |s-\lambda_n| = s-\lambda_n = \rho(B),$$

which is a contradiction because $s - \lambda_i$ is an eigenvalue of B. On the other hand, if $Re(\lambda_i) < \lambda_n$, then $s - Re(\lambda_i) > s - \lambda_n > 0$. Thus,

$$|s - \lambda_i| \geq |s - Re(\lambda_i)| > |s - \lambda_n| = s - \lambda_n = \rho(B),$$

which is again a contradiction because $s - \lambda_i$ is an eigenvalue of B. Therefore, $\lambda_n < Re(\lambda_i)$ for all $\lambda_i \neq \lambda_n$.

Conversely, suppose that $A^{-1} \in WPF_n$ and that $0 < \lambda_n < Re(\lambda_i)$ for all $\lambda_i \neq \lambda_n$. Then, there are semipositive vectors v and w such that $A^{-1}v = \rho(A^{-1})v = \lambda_n^{-1} v$ and $w^T A^{-1} = \rho(A^{-1}) w^T = \lambda_n^{-1} w$. Note that for every λ_i such that $|\lambda_i| = \lambda_n$ we have $\lambda_i = \lambda_n$ (otherwise, $0 < \lambda_n < Re(\lambda_i) \leq |\lambda_i| = \lambda_n$, which is a contradiction). Moreover, the set of complex numbers

$$\{\lambda_i \in \sigma(A) \ : \ |\lambda_i| \neq \lambda_n\} = \sigma(A) \backslash \{\lambda_n\}$$

lies completely in the set Ω defined by the intersection of the following two sets:

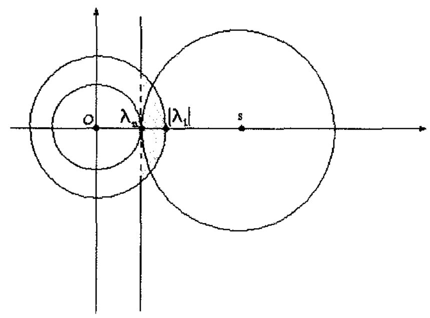

Figure 4.2: The gray region represents the set Ω, which is the intersection of the open right half-plane determined by the vertical straight line passing through λ_n and the closed annulus centered at 0 with radii λ_n and $|\lambda_1|$.

- The annulus $\{z \; : \; \lambda_n \leq |z| \leq |\lambda_1|\}$, and
- The (open) half-plane $\{z \; : \; Re(z) > Re(\lambda_n)\}$.

It is easy to see that there is a real number s large enough so that the circle centered at s of radius $s - \lambda_n$ surrounds all the complex numbers $\lambda_i \in \sigma(A)$, $\lambda_i \neq \lambda_n$ lying in Ω; see Figure 4.2. For such an s, define the matrix $B_s := sI - A$. Then the eigenvalues of B_s are $s - \lambda_1, s - \lambda_2, \ldots, s - \lambda_n$. Moreover, by our choice of s, we have the following:

$$|s - \lambda_i| < s - \lambda_n \quad \text{for all} \quad \lambda_i \neq \lambda_n.$$

Therefore, $0 < \rho(B_s) = s - \lambda_n < s$. Moreover, $B_s v = (s - \lambda_n)v$ and that $w^T B_s = (s - \lambda_n)w^T$. Thus, $B_s \in WPF_n$. And therefore, $A = sI - B_s$ is a nonsingular GM-matrix. □

In [26, Theorem 8], Johnson and Tarazaga proved that if A is a pseudo-M-matrix, then $A^{-1} \in PF_n$. We extend this theorem by giving necessary and sufficient conditions for a matrix A to be a pseudo-M-matrix. The proof is very similar to that of Theorem 4.1, and thus, it is omitted.

Theorem 4.2 *Let A be a matrix in $\mathbb{R}^{n\times n}$ whose eigenvalues with multiplicity are arranged in the following manner: $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. Then the following statements are equivalent:*

(i) A is a pseudo-M-matrix.

(ii) A^{-1} exists, A^{-1} is eventually positive, and $0 < \lambda_n < Re(\lambda_i)$, for $i = 1, \ldots, n-1$.

Remark 4.1 Since every M-matrix is a GM-matrix, it follows that condition (ii) in Theorem 4.1 can be used to check if a matrix is not an inverse M-matrix. In particular, if the real part of any eigenvalue is less than the minimum of all moduli of all eigenvalues then the given matrix is not an inverse M-matrix.

Remark 4.2 The set WPF_n in Theorem 4.1 is not completely analogous to the set of nonnegative matrices. In other words, if we replace in Theorem 4.1 WPF_n by the set of nonnegative matrices and if we replace a GM-matrix by an M-matrix, then the statement of the theorem would not be correct. Similarly, in Theorem 4.2, PF_n is not completely analogous with the set of positive matrices. For example, we may find a nonnegative matrix whose inverse is a GM-matrix but not an M-matrix. An example of the latter is the positive matrix $C = \frac{1}{36}\begin{bmatrix} 7 & 6 & 5 \\ 5 & 12 & 1 \\ 1 & 6 & 11 \end{bmatrix}$. Note that $C^{-1} = \begin{bmatrix} 7 & -2 & -3 \\ -3 & 4 & 1 \\ 1 & -2 & 3 \end{bmatrix} = sI - B$ where $s = 10$, $B = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 6 & -1 \\ -1 & 2 & 7 \end{bmatrix} \in WPF_3$, and $\rho(B) = 8$. Hence, C^{-1} is a nonsingular GM-matrix. However, C^{-1} is not an M-matrix since it has some positive off-diagonal entries.

Corollary 4.1 *A matrix $C \in \mathbb{R}^{n\times n}$ is an inverse GM-matrix if and only if $C \in WPF_n$ and $Re(\lambda^{-1}) > \rho(C)^{-1}$ for all $\lambda \in \sigma(C)$, $\lambda \neq \rho(C)$.*

Corollary 4.2 *Every real eigenvalue of a nonsingular GM-matrix is positive.*

Example 4.3 In this example, we show a nonsingular EM-matrix whose inverse is not eventually nonnegative. This implies that no result analogous to Theorems 4.1 and 4.2 holds for this case. Let $A = \begin{bmatrix} 2 & -1 & -1 & 1 \\ -1 & 2 & 1 & -1 \\ -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{bmatrix} = 3I - \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 3I - B$. Then, $\rho(B) = 2$ and, using an induction argument, $B^k = \begin{bmatrix} 2^{k-1} & 2^{k-1} & 0 & 0 \\ 2^{k-1} & 2^{k-1} & 0 & 0 \\ k2^{k-1} & k2^{k-1} & 2^{k-1} & 2^{k-1} \\ k2^{k-1} & k2^{k-1} & 2^{k-1} & 2^{k-1} \end{bmatrix} \geq 0$ for all integers $k \geq 2$. Hence, A is an EM-matrix. But, $A^{-1} = 3^{-2}(E + F)$ where $E = \begin{bmatrix} 6 & 3 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 9 & 9 & 6 & 3 \\ 9 & 9 & 3 & 6 \end{bmatrix}$ and $F = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Note that $EF = FE = 3F$ and $F^2 = 0$. Therefore, using an induction argument, it is easy to check that $(A^{-1})^k = 3^{-2k}E^k + k3^{-k-1}F$. Hence, A^{-1} is not eventually nonnegative because the (1,4) and (2,3) entries are always negative.

It is well-known that a Z-matrix $A \in \mathbb{R}^{n \times n}$ is a nonsingular M-matrix if and only if A is positive stable, i.e., the real part of any eigenvalue of A is positive; see, e.g., [2, p.137]. In the following proposition, we prove an analogous result with GZ-matrices and GM-matrices.

Proposition 4.1 *A GZ-matrix $A \in \mathbb{R}^{n \times n}$ is a nonsingular GM-matrix if and only if A is positive stable.*

Proof. Let A be a GZ-matrix in $\mathbb{R}^{n\times n}$ with eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. If A is a nonsingular GM-matrix, then Theorem 4.1 implies that $0 < \lambda_n < Re(\lambda_i)$ for all $\lambda_i \neq \lambda_n$. Thus, $Re(\lambda_i) > 0$ for $i = 1, 2, \ldots, n$. Hence, A is positive stable. Conversely, suppose that A is positive stable, then it follows that 0 is not an eigenvalue of A, which implies that A is nonsingular. Moreover, since A is a GZ-matrix we can decompose A in the following manner $A = sI - B$ where $B \in WPF_n$ and $s \geq 0$. If $s \leq \rho(B)$ then $(s - \rho(B))$ is a nonpositive eigenvalue of A, which contradicts the positive stability of A. Hence, $s > \rho(B)$, which shows that A is a nonsingular GM-matrix. □

Another useful result is the following; see, e.g., [2, p.136].

Theorem 4.3 *A Z-matrix $A \in \mathbb{R}^{n\times n}$ is a nonsingular M-matrix if and only if there is a positive vector x such that Ax is positive.*

In Theorem 4.4 below, we prove an analogous result for pseudo-M-matrices. The results in the following lemma are proved in [31, Theorem 2.6].

Lemma 4.1 *If $B \in \mathbb{R}^{n\times n}$ has a left Perron-Frobenius eigenvector and $x = [x_1 \ \cdots \ x_n]^T$ is any positive vector then either $\frac{\sum_{j=1}^n b_{ij}x_j}{x_i} = \rho(B)$ for all $i \in \{1, 2, \ldots, n\}$ or $min_{i=1}^n \frac{\sum_{j=1}^n b_{ij}x_j}{x_i} \leq \rho(B) \leq max_{i=1}^n \frac{\sum_{j=1}^n b_{ij}x_j}{x_i}$.*

Theorem 4.4 *If $A = sI - B$ where $B \in PF_n$, then the following are equivalent:*

(i) A is a pseudo-M-matrix.

(ii) There is a positive vector x such that Ax is positive.

Proof. Suppose $A = sI - B$ is a pseudo-M-matrix and let x be a right Perron-Frobenius eigenvector of B. Then, $Ax = (sI - B)x = (s - \rho(B))x$ is a positive vector. Conversely, suppose there is a positive vector x such that $Ax = (sI - B)x = sx - Bx$ is positive. Then, $max_{i=1}^n \frac{\sum_{j=1}^n b_{ij}x_j}{x_i} < s$, and by Lemma 4.1, $\rho(B) \leq max_{i=1}^n \frac{\sum_{j=1}^n b_{ij}x_j}{x_i}$. Hence, $\rho(B) < s$ which proves that A is a pseudo-M-matrix. □

We next give a characterization of nonsingular GM-matrices which has the flavor of Theorem 4.3.

Theorem 4.5 *If $A = sI - B$ is a GZ-matrix ($B \in WPF_n$), then the following are equivalent:*

(i) A is a nonsingular GM-matrix.

(ii) There is an orthogonal matrix Q such that Qx and QAx are positive where x is a right Perron-Frobenius eigenvector of B.

(iii) There is an orthogonal matrix Q such that Qy and QA^Ty are positive where y is a left Perron-Frobenius eigenvector of B.

Proof. We prove the equivalence of (i) and (ii) and we omit the proof of the equivalence of (i) and (iii) since it is analogous. Suppose $A = sI - B$ is a nonsingular GM-matrix and let x be a right Perron-Frobenius eigenvector of B. Then, by Lemma 3.7, there is an orthogonal matrix Q such that Qx is positive. Moreover,

$$QAx = Q(sI - B)x = (s - \rho(B))Qx$$

is positive since A is nonsingular having $\rho(B) < s$. Hence, $(i) \Rightarrow (ii)$. Conversely, suppose (ii) is true. Then,

$$QAx = Q(sI - B)x = (s - \rho(B))Qx$$

is positive, and thus, $\rho(B) < s$. □

We end this section with a result on the classes of an EM-matrix.

Proposition 4.2 *Let $A = sI - B$ be an EM-matrix (B eventually nonnegative and $0 < \rho(B) \leq s$). If A is singular, then for every class α of B the following holds:*

1. *$A[\alpha]$ is a singular irreducible EM-matrix if α is basic.*

2. $A[\alpha]$ is a nonsingular irreducible EM-matrix if α is not basic.

Proof. If A is a singular EM-matrix and α is a class of B, then $A[\alpha] = sI - B[\alpha]$, where I is the identity matrix having the appropriate dimension. If α is a basic class of B, then $B[\alpha]$ is an irreducible submatrix of B and $\rho(B[\alpha]) = \rho(B) > 0$. Since the eigenvalues of A are of the form $s - \mu$ where $\mu \in \sigma(B)$ and since A is singular, it follows that $\rho(B) = s$. Hence, $\rho(B[\alpha]) = s$ and $A[\alpha] = sI - B[\alpha]$ must be singular, as well. Moreover, since $B[\alpha]$ is irreducible, it follows that the graph $G(B[\alpha])$ is strongly connected. Note that the graph $G(A[\alpha]) = G(sI - B[\alpha])$ may differ from the graph $G(B[\alpha])$ only in having or missing some loops on some vertices. This means that the graph $G(A[\alpha])$ is also strongly connected because adding or removing loops from vertices of a strongly connected graph does not affect strong connectivity. Hence, $A[\alpha]$ is irreducible. Moreover, if $\kappa = (\alpha_1, \ldots, \alpha_m)$ is an ordered partition of $\{1, 2, \ldots, n\}$ that gives the Frobenius normal form of B (see, e.g., [4]), then B_κ is block triangular and it is permutationally similar to B. Thus, B_κ is eventually nonnegative and so is each of its diagonal blocks. In particular, there is a diagonal block in B_κ which is permutationally similar to $B[\alpha]$ (because α is a class of B). Hence, $B[\alpha]$ is eventually nonnegative, which proves part 1. Similarly, if α is not a basic class of B, then part 2 holds. □

4.3 Splittings and GM-Matrices

Recall that a *splitting* of a matrix $A = (a_{ij})$ is an expression of the form $A = M - N$ where M is a nonsingular matrix. The matrix $M^{-1}N$ is called the *iteration matrix* of the splitting $A = M - N$. If $M = diag(a_{11}, \ldots, a_{nn})$, then we call such a splitting a *Jacobi splitting*. If the (i, j)-entry of M is a_{ij} whenever $i \geq j$ and 0 otherwise, then we call such a splitting a *Gauss-Seidel splitting*. If $\rho(M^{-1}N) < 1$ then we say that the splitting $A = M - N$ is *convergent*; see, e.g., [2], [18], [49]. In this section, we define various splittings of a GM-matrix, give sufficient conditions for convergence, and we illustrate this with examples. We begin by listing some preliminary definitions.

Definition 4.1 *Let $A = M - N$ be a splitting. Then, such a splitting is called*

- *weak (or nonnegative) if $M^{-1}N \geq 0$.*
- *weak-regular if $M^{-1}N \geq 0$ and $M^{-1} \geq 0$ [32].*
- *regular if $M^{-1} \geq 0$ and $N \geq 0$ [49].*
- *M-splitting if M is an M-matrix and $N \geq 0$ [41].*
- *Perron-Frobenius splitting if $M^{-1}N$ possesses the Perron-Frobenius property [31].*

We list now the new splittings introduced in this section. We begin first by defining the splitting having the Perron singular property, which is a splitting for an arbitrary nonsingular matrix. Then we proceed to define the splittings specific to nonsingular GM-matrices.

Definition 4.2 *Let A be nonsingular. We say that the splitting $A = M - N$ has the Perron singular property if $\gamma M + (1-\gamma)N$ is singular for some $\gamma \in \mathbb{R}$, $\gamma \neq 0$ and $M^{-1}N$ has the Perron-Frobenius property.*

Note that a splitting with the Perron singular property is, in particular, a Perron-Frobenius splitting.

Definition 4.3 *Let $A = M - N$ be a splitting of a nonsingular GM-matrix $A = sI - B$ ($B \in WPF_n$ and $\rho(B) < s$). Then, such a splitting is called*

- *a G-regular splitting if M^{-1} and N are in WPF_n.*
- *a GM-splitting if M is a GM-matrix and $N \in WPF_n$.*
- *an overlapping splitting if for a dominant eigenvalue λ of $M^{-1}N$ the vector space $E_\lambda(M^{-1}N) \cap E_{\rho(B)}(B)$ contains a right Perron-Frobenius eigenvector of B.*

- *a commuting bounded splitting if M and N commute and $\rho(M) < s$.*

Remark 4.3 A GM-splitting is a G-regular splitting but not conversely. For example, consider the GM-matrix $A = diag(1,4,4) = sI - B$ where $s = 5$ and $B = diag(4,1,1)$. An example of a G-regular splitting of A is the splitting $A = M - N$ where $M = diag(2,32,-4)$ and $N = diag(1,28,-8)$. Note that M^{-1} is in WPF_n yet, by Theorem 4.1, M is not a GM-matrix. Hence, this G-regular splitting is not a GM-splitting.

Lemma 4.2 *Let $A = M - N$ be a splitting of a nonsingular matrix A. Then, the following are equivalent:*

(i) The splitting is convergent.

(ii) $min\ \{Re(\lambda) \mid \lambda \in \sigma(NA^{-1})\} > -\frac{1}{2}$.

(iii) $min\ \{Re(\lambda) \mid \lambda \in \sigma(A^{-1}N)\} > -\frac{1}{2}$.

Proof. We prove first the equivalence of (i) and (ii). Let $P = M^{-1}NA^{-1}M$. Thus, P and NA^{-1} are similar matrices, and therefore, they have the same eigenvalues with the same multiplicities. Moreover, the following relation between P and $M^{-1}N$ holds:

$$P = M^{-1}NA^{-1}M = M^{-1}N(M-N)^{-1}M = M^{-1}N(I - M^{-1}N)^{-1}.$$

Hence, the eigenvalues of NA^{-1} and $M^{-1}N$ are related as follows: $\mu \in \sigma(M^{-1}N)$ if and only if there is a unique $\lambda \in \sigma(NA^{-1})$ such that $\mu = \frac{\lambda}{1+\lambda}$. The splitting is convergent, i.e., $\rho(M^{-1}N) < 1$ if and only if for all $\mu \in \sigma(M^{-1}N)$, we have $|\mu| < 1$. That is, if for all $\lambda \in \sigma(NA^{-1})$, we have $\left|\frac{\lambda}{1+\lambda}\right| < 1$, or equivalently, $\frac{(Re(\lambda))^2+(Im(\lambda))^2}{(1+Re(\lambda))^2+(Im(\lambda))^2} < 1$, which holds only whenever $2Re(\lambda) + 1 > 0$, or whenever (ii) is true. As for the equivalence of (i) and (iii), it follows similarly by noting the following relation between $A^{-1}N$ and $M^{-1}N$:

$$A^{-1}N = (M-N)^{-1}N = (I - M^{-1}N)^{-1}M^{-1}N. \qquad \square$$

Corollary 4.3 *Let $A = M - N$ be a splitting of a nonsingular matrix A. If $A^{-1}N$ or NA^{-1} is an inverse GM-matrix, then the splitting is convergent.*

Proof. Let P denote $A^{-1}N$ or NA^{-1}. If P is an inverse GM-matrix then, by Corollary 4.1, $Re(\lambda^{-1}) > (\rho(P))^{-1} > 0$ for all $\lambda \in \sigma(P)$, $\lambda \neq \rho(P)$. This implies that $Re(\lambda) = |\lambda|^2 Re(\lambda^{-1}) > -\frac{1}{2}$ for all $\lambda \in \sigma(P)$, $\lambda \neq \rho(P)$. Thus, $Re(\lambda) > -\frac{1}{2}$ for all $\lambda \in \sigma(P)$, which is equivalent to condition *(ii)* of Lemma 4.2 if $P = NA^{-1}$, or equivalent to condition *(iii)* of Lemma 4.2 if $P = A^{-1}N$. Hence, the given splitting is convergent. □

The following lemma is part of Theorem 3.1 of [31].

Lemma 4.3 *Let $A = M - N$ be a Perron-Frobenius splitting of a nonsingular matrix $A \in \mathbb{R}^{n \times n}$. Then, the following are equivalent:*

(i) The splitting $A = M - N$ is convergent.

(ii) $A^{-1}N$ possesses the Perron-Frobenius property.

(iii) $\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)}$.

Corollary 4.4 *Let $A = M - N$ be a splitting of a nonsingular matrix A such that N is nonsingular and $N^{-1}M$ is a nonsingular GM-matrix. Then, the following are equivalent:*

(i) The splitting $A = M - N$ is convergent.

(ii) $A^{-1}N$ possesses the Perron-Frobenius property.

(iii) $\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)}$.

Proof. Since $N^{-1}M$ is a nonsingular GM-matrix, it follows that $(N^{-1}M)^{-1} = M^{-1}N \in \mathrm{WPF}_n$. Hence, $M^{-1}N$ satisfies the Perron-Frobenius property, which implies that the splitting $A = M - N$ is a Perron-Frobenius splitting and the equivalence of the statements in the corollary follows from Lemma 4.3. □

Theorem 4.6 *If $A = M - N$ is a splitting having the Perron singular property, then any of the following conditions is sufficient for convergence:*

Sign Conditions

(A1) $A^{-1}N$ *is eventually positive.*

(A2) $A^{-1}N$ *is eventually nonnegative.*

Spectral Conditions

(B1) $A^{-1}N \in WPF_n$.

(B2) $A^{-1}N$ *has a simple positive and strictly dominant eigenvalue with a positive spectral projector of rank 1.*

Combinatorial Conditions

(C1) For all $1 \leq i, j \leq n$, *the total weight of positive* $A^{-1}N$*-alternating walks from* i *to* j *in* $G(A^{-1}) \cup G(N)$ *eventually majorizes the absolute value of the total weight of negative* $A^{-1}N$*-alternating walks from* i *to* j *of the same length in* $G(A^{-1}) \cup G(N)$.

(C2) For all $1 \leq i, j \leq n$, *the total weight of positive* $A^{-1}N$*-alternating walks from* i *to* j *in* $G(A^{-1}) \cup G(N)$ *eventually majorizes and strictly dominates the absolute value of the total weight of negative* $A^{-1}N$*-alternating walks from* i *to* j *of the same length in* $G(A^{-1}) \cup G(N)$.

(C3) For all $1 \leq i, j \leq n$, *the total weight of positive walks from* i *to* j *in* $G(A^{-1}N)$ *eventually majorizes the absolute value of the total weight of negative walks from* i *to* j *of the same length in* $G(A^{-1}N)$.

(C4) For all $1 \leq i, j \leq n$, *the total weight of positive walks from* i *to* j *in* $G(A^{-1}N)$ *eventually majorizes and strictly dominates the absolute value of the total weight of negative walks from* i *to* j *of the same length in* $G(A^{-1}N)$.

(C5) $\bigcup_{\tau \in Even_k} \prod_{i=1}^{k} G((A^{-1}N)^{\tau(i)}) \succ \bigcup_{\tau \in Odd_k} \prod_{i=1}^{k} G(|(A^{-1}N)^{\tau(i)}|)$

for all $k \geq k_0$ for some $k_0 \geq 1$.

(C6) $A^{-1}N$ has a basic and an initial class α such that $(A^{-1}N)[\alpha]$ has a right Perron-Frobenius eigenvector.

Geometric Conditions

(D1) $\bigcup_{l=k_0}^{\infty} Hull((A^{-1}N)^l) \subset \bigcap_{j=1}^{n} H((A^{-1}N)_{*j})$ *for some $k_0 \geq 0$.*

(D2) $Hull((A^{-1}N)) \subset \bigcap_{l=k_0}^{\infty} \bigcap_{j=1}^{n} H(((A^{-1}N)^l)_{*j})$ *for some $k_0 \geq 0$.*

Proof. We prove first $(A2) \Rightarrow (B1) \Rightarrow$ convergence of the given splitting. Suppose that $A^{-1}N$ is eventually nonnegative. Since $A = M - N$ is a spliting having the *Perron singular property*, it follows that there is a nonzero complex scalar γ such that $\gamma M + (1-\gamma)N$ is singular. Hence, $\gamma A + N$ is singular $\Leftrightarrow det(\gamma A + N) = 0 \Leftrightarrow det(\gamma I + A^{-1}N) = 0 \Leftrightarrow det(-\gamma I - A^{-1}N) = 0$. In other words, $-\gamma$ is a nonzero eigenvalue of $A^{-1}N$. By Lemma 1.1, $A^{-1}N$ and its transpose possess the Perron-Frobenius property, i.e., $A^{-1}N \in WPF_n$. And thus, the given splitting converges by Lemma 4.3. As for the rest of the sufficient conditions, we outline the proof using the following diagram:

$$
\begin{array}{ccccccc}
(A1) & \Rightarrow & (A2) & \Rightarrow & (B1) & \Rightarrow & \text{convergence} \\
\Updownarrow & & \Updownarrow & & \Uparrow & & \\
(B2) & & (C1) & & (C6) & & \\
\Updownarrow & & \Updownarrow & & & & \\
(C2) & & (C3) & & & & \\
\Updownarrow & & \Updownarrow & & & & \\
(C4) & & (C5) & & & & \\
 & & \Updownarrow & & & & \\
 & & (D1) & & & & \\
 & & \Updownarrow & & & & \\
 & & (D2) & & & &
\end{array}
$$

The equivalencies and implications in the above diagram follow from the spectral, combinatorial, and geometric characterizations of eventually positive matrices, eventually nonnegative matrices, and matrices in WPF_n proved in Chapter 2. □

Remark 4.4 Recall that a regular splitting $A = M - N$ of a monotone matrix (i.e., when $A^{-1} \geq 0$) is convergent [49]. Thus, Theorem 4.6 is a genaralization of this situation since we do not require that A^{-1} nor N, nor their product $A^{-1}N$ to be nonnegative.

Example 4.4 Let $A = \begin{bmatrix} 7 & -2 & -3 \\ -3 & 4 & 1 \\ 1 & -2 & 3 \end{bmatrix}$ and consider the splitting $A = M - N$, where $M = \frac{1}{4}\begin{bmatrix} 29 & -6 & -11 \\ -11 & 18 & 5 \\ 5 & -6 & 13 \end{bmatrix}$ and $N = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$. For $\gamma = -\frac{1}{2}$ the matrix $\gamma M + (1-\gamma)N = \frac{1}{4}\begin{bmatrix} -13 & 6 & 7 \\ 7 & -6 & -1 \\ -1 & 6 & -5 \end{bmatrix}$ is singular. Moreover, $M^{-1}N = \frac{1}{12}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ is a positive matrix and thus it possesses the Perron-Frobenius property. Hence, this splitting is a splitting with the Perron singular property. Since $A^{-1}N = \frac{1}{8}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ is a positive matrix (and hence eventually positive), it follows from Theorem 4.6 that this splitting is convergent. In fact, $\rho(M^{-1}N) = \frac{1}{3} < 1$.

Proposition 4.3 *If $A = sI - B$ is a GM-matrix and the splitting $A = M - N$ is an overlapping splitting (for which $E_\lambda(M^{-1}N) \cap E_{\rho(B)}(B)$ contains a right Perron-Frobenius eigenvector of B and $|\lambda| = \rho(M^{-1}N)$), then* *such a splitting is convergent if and only if there is* $\eta = \frac{s-\rho(B)}{1-\lambda} \in \sigma(M)$ *such that* $Re(\eta) > \frac{s-\rho(B)}{2}$.

Proof. Note first that if $A = sI - B = M - N$ is an overlapping splitting then we can pick $v \in E_{\rho(B)}(B) \cap E_{\lambda}(M^{-1}N)$ where v is a right Perron-Frobenius eigenvector of B. And for this vector, we have:

$$\begin{aligned}
& (sI - B)v = Av = (M - N)v = M^{-1}(I - M^{-1}N)v \\
\Leftrightarrow\ & (s - \rho(B))Mv = (I - M^{-1}N)v = (1 - \lambda)v \\
\Leftrightarrow\ & Mv = \frac{(1-\lambda)}{(s - \rho(B))}v \\
\Rightarrow\ & \exists\, \eta \in \sigma(M) \ni \eta = \frac{(1-\lambda)}{(s-\rho(B))} \\
\Leftrightarrow\ & \exists\, \eta \in \sigma(M) \ni \lambda = \frac{\eta - (s - \rho(B))}{\eta}.
\end{aligned}$$

Hence, if $A = M - N$ is an overlapping splitting then there is an eigenvalue $\eta \in \sigma(M)$ such that $\lambda = \frac{\eta-(s-\rho(B))}{\eta}$. Therefore, an overlapping splitting is convergent, i.e., $\rho(M^{-1}N) = |\lambda| < 1$, when $|\eta - (s - \rho(B))| < |\eta|$ for some $\eta \in \sigma(M)$, or equivalently whenever η lies in the right-half plane determined by the perpendicular bisector of the segment on the real axis whose endpoints are 0 and $(s - \rho(B))$, i.e., whenever $Re(\eta) > \frac{s-\rho(B)}{2}$. □

Corollary 4.5 *Let* $A = M - N$ *be an overlapping splitting of a nonsingular GM-matrix* A *and suppose that* $M^{-1}N \in WPF_n$. *If* $\frac{s-\rho(B)}{1-\rho(M^{-1}N)} \in \sigma(M)$ *then* $\rho(M^{-1}N) < 1$, *i.e. the splitting is convergent.*

Example 4.5 Let A be as in Example 4.4. Then, A is a nonsingular GM-matrix. In fact, $A = sI - B$, where $s = 10$, $B = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 6 & -1 \\ -1 & 2 & 7 \end{bmatrix} \in WPF_3$, and $\rho(B) = 8$. An overlapping splitting of the matrix A is $A = M - N$ where $M = \frac{1}{8}\begin{bmatrix} 55 & -18 & -25 \\ -25 & 30 & 7 \\ 7 & -18 & 23 \end{bmatrix}$ and $N = \frac{1}{8}\begin{bmatrix} -1 & -2 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{bmatrix}$. Note that

$M^{-1}N = \frac{2}{3}\begin{bmatrix} -1 & -2 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{bmatrix}$ and that for $\lambda = -\frac{1}{3} \in \sigma(M^{-1}N)$ we have $|\lambda| = \frac{1}{3} = \rho(M^{-1}N)$ and $E_\lambda(M^{-1}N) = E_{\rho(B)}(B) = Span\{[1\ 1\ 1]^T\}$. Hence, this overlapping splitting is convergent. Proposition 4.3 predicts the existence of an eigenvalue η of M such that $\eta = \frac{s-\rho(B)}{1-\lambda} = \frac{10-8}{1-(-1/3)} = \frac{3}{2}$ and $Re(\eta) = \frac{3}{2} > \frac{s-\rho(B)}{2} = 1$. If we look at the spectrum of M we see that $\frac{3}{2} \in \sigma(M) = \{\frac{3}{2}, 6\}$ just as predicted by Proposition 4.3. On the other hand, if $A = M - N$ where $M = \frac{1}{4}\begin{bmatrix} 27 & -10 & -13 \\ -13 & 14 & 3 \\ 3 & -10 & 11 \end{bmatrix}$ and $N = \frac{1}{4}\begin{bmatrix} -1 & -2 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{bmatrix}$ then $M^{-1}N = N = \frac{1}{4}\begin{bmatrix} -1 & -2 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{bmatrix}$ and for $\lambda = -1 \in \sigma(M^{-1}N)$ we have $|\lambda| = 1 = \rho(M^{-1}N)$ and $E_\lambda(M^{-1}N) = E_{\rho(B)}(B) = Span\{[1\ 1\ 1]^T\}$. Hence, the latter splitting is an overlapping splitting but it does not converge. Proposition 4.3 predicts that for all $\eta \in \sigma(M)$ either $\eta \neq \frac{s-\rho(B)}{1-\lambda} = \frac{10-8}{1-(-1)} = 1$ or $Re(\eta) \leq \frac{s-\rho(B)}{2} = 1$, which is true about the spectrum of M since $\sigma(M) = \{1, 6\}$.

Theorem 4.7 *A GM-matrix $A = sI - B$ having a commuting bounded splitting $A = M - N$ induces a splitting of B of the form $B = M' - N'$ where $M' = \frac{1}{\omega}(sI - M)$, $\omega \in \mathbb{R}$ and $\omega \neq 0$. Moreover, if $|\omega| < min\ \left\{\frac{s}{\rho(M')}, \frac{1}{2\rho(M')}\right\}$, then the commuting bounded splitting is convergent.*

Proof. Suppose that the GM-matrix $A = sI - B$ has a commuting bounded splitting $A = M - N$ and let $M' = \frac{1}{\omega}(sI - M)$ and $N' = A - sI + M'$ for some $\omega \in \mathbb{R}$, $\omega \neq 0$. Then, M' is nonsingular because $\rho(M) < s$ and $M = sI - \omega M'$. Moreover, we can write $A = (sI - \omega M') - ((1-\omega)M' - N')$. Note that the iteration matrix of the commuting bounded splitting of A is $M^{-1}N = (sI - \omega M')^{-1}((1-\omega)M' - N')$. Since M and N commute, so do M' and N'. Furthermore, there is a single unitary matrix U that produces the Schur decomposition (see, e.g., [22, p.81]) of both M' and N'. Hence, an eigenvalue of $M^{-1}N$ would have the form $(s-\omega\lambda)^{-1}((1-\omega)\lambda-\mu)$ where $\lambda \in \sigma(M')$ and $\mu \in \sigma(N')$. But, since M' and N' are simultaneously Schur decomposable, it follows that the same unitary matrix that produces the Schur decomposition of M' and N' also produces the Schur decomposition of B and A. Therefore, $\lambda-\mu$ is an eigenvalue of B which does not exceed s in modulus (since A is a GM-matrix). Thus,

$$\left|(s-\omega\lambda)^{-1}((1-\omega)\lambda-\mu)\right| \leq \frac{|\lambda-\mu|+|\omega||\lambda|}{|s-\omega\lambda|} \leq \frac{s+|\omega|\rho(M')}{|s-\omega\lambda|}$$

Moreover, if we choose $|\omega| < \dfrac{s}{\rho(M')}$, then $s-|\omega|\rho(M') > 0$. Hence,

$$\left|(s-\omega\lambda)^{-1}((1-\omega)\lambda-\mu)\right| \leq \frac{s+|\omega|\rho(M')}{|s-\omega\lambda|} \leq \frac{s+|\omega|\rho(M')}{s-|\omega|\rho(M')}.$$

Therefore, if

$$\frac{s+|\omega|\rho(M')}{s-|\omega|\rho(M')} < 1 \tag{4.1}$$

then the splitting $A = M - N$ is convergent. But (4.1) is equivalent to $|\omega| < \frac{1}{2\rho(M')}$. Hence, if $|\omega| < min\left\{\frac{s}{\rho(M')}, \frac{1}{2\rho(M')}\right\}$, then the splitting $A = M - N$ is convergent. □

Example 4.6 Let $A = \frac{1}{40}\begin{bmatrix} 3 & 1 & 2 \\ 1 & 5 & 0 \\ 1 & 1 & 4 \end{bmatrix}$. Then, $A = sI - B$ is a GM-matrix, where $s = 1$, $B = \frac{1}{40}\begin{bmatrix} 37 & -1 & -2 \\ -1 & 35 & 0 \\ -1 & -1 & 36 \end{bmatrix} \in WPF_3$, and $\rho(B) = 0.95$. A commuting bounded splitting of A is $A = M - N$, where $M = \frac{1}{80}\begin{bmatrix} 73 & 2 & 1 \\ 1 & 74 & 1 \\ 1 & 2 & 73 \end{bmatrix}$ and $N = \frac{1}{80}\begin{bmatrix} 67 & 0 & -3 \\ -1 & 64 & 1 \\ -1 & 0 & 65 \end{bmatrix}$. Note that $\rho(M) = 0.95 < 1 = s$ and that

$MN = NM = \frac{1}{1600}\begin{bmatrix} 1222 & 32 & -38 \\ -2 & 1184 & 34 \\ -2 & 32 & 1186 \end{bmatrix}$. Furthermore, let $\omega = 5$ and let $M' = \frac{1}{\omega}(sI - M) = \frac{1}{400}\begin{bmatrix} 7 & -2 & -1 \\ -1 & 6 & -1 \\ -1 & -2 & 7 \end{bmatrix}$. Then, $\rho(M') = 0.02$ making $|\omega| = 5 < min\left\{\frac{s}{\rho(M')}, \frac{1}{2\rho(M')}\right\} = min\left\{\frac{1}{0.02}, \frac{1}{2(0.02)}\right\} = 12.5$. Hence, by Theorem 4.7, the splitting $A = M - N$ is convergent. In fact, $\rho(M^{-1}N) \approx 0.9444 < 1$.

Theorem 4.8 *If $A = M - N$ is a splitting of a GM-matrix A, then any Type I condition (listed below) implies that such a splitting is a G-regular splitting. Moreover, if the splitting $A = M - N$ is a G-regular splitting that satisfies one of Type II conditions (listed below), then any one of Type III conditions (listed below) is sufficient for convergence.*

Type I Conditions

(D1) M^{-1} and N are eventually positive.

(D2) M^{-1} and N are eventually nonnegative with N being nonnilpotent.

(D3) Each of M^{-1} and N has a simple positive and strictly dominant eigenvalue with a positive spectral projector having a rank equal to 1.

(D4) For all $1 \leq i, j \leq n$, the total weight of positive walks from i to j in $G(M^{-1})$ and $G(N)$ eventually majorizes the absolute value of the total weight of the negative walks from i to j of the same length and N is nonnilpotent.

(D5) For all $1 \leq i, j \leq n$, the total weight of positive walks from i to j in $G(M^{-1})$ and $G(N)$ eventually majorizes and strictly dominates the absolute value of the total weight of the negative walks from i to j of the same length.

(D6) *N is nonnilpotent and the following statement is true for $X = M^{-1}$ and for $X = N$: there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,*

$$\bigcup_{\tau \in Even_k} \prod_{i=1}^{k} G(X^{\tau(i)}) \succ \bigcup_{\tau \in Odd_k} \prod_{i=1}^{k} G(|X^{\tau(i)}|).$$

(D7) *The following statement is true for $X = M^{-1}$ and for $X = N$: X has two classes α and β, not necessarily distinct, such that:*

(i) *α is basic, initial, and $X[\alpha]$ has a right Perron-Frobenius eigenvector.*

(ii) *β is basic, final, and $X[\beta]$ has a left Perron-Frobenius eigenvector.*

(D8) *N is nonnilpotent and the following statement is true for $X = M^{-1}$ and for $X = N$:*

$$\bigcup_{l=k_0}^{\infty} Hull(X^l) \subset \bigcap_{j=1}^{n} H(X_{*j}) \text{ for some } k_0 \geq 0.$$

(D9) *N is nonnilpotent and the following statement is true for $X = M^{-1}$ and for $X = N$:*

$$Hull(X) \subset \bigcap_{l=k_0}^{\infty} \bigcap_{j=1}^{n} H((X^l)_{*j}) \text{ for some } k_0 \geq 0.$$

Type II Conditions

(E1) *$M^{-1}N$ is eventually positive.*

(E2) *$M^{-1}N$ is nonnilpotent eventually nonnegative.*

(E3) *$M^{-1}N \in WPF_n$.*

(E4) *$M^{-1}N$ has a simple positive and strictly dominant eigenvalue with a positive spectral projector of rank 1.*

(E5) *For all $1 \leq i, j \leq n$, total weight of positive $M^{-1}N$-alternating walks from i to j in $G(M^{-1}) \cup G(N)$ eventually majorizes the absolute value of the total weight of the negative $M^{-1}N$-alternating walks from i to j of the same length, and $M^{-1}N$ is nonnilpotent.*

(E6) *For all* $1 \leq i, j \leq n$, *total weight of positive* $M^{-1}N$*-alternating walks from* i *to* j *in* $G(M^{-1}) \cup G(N)$ *eventually majorizes and strictly dominates the absolute value of the total weight of the negative* $M^{-1}N$*-alternating walks from* i *to* j *of the same length.*

(E7) *For all* $1 \leq i, j \leq n$, *the total weight of positive walks from* i *to* j *in* $G(M^{-1}N)$ *eventually majorizes the absolute value of the total weight of negative walks from* i *to* j *of the same length in* $G(M^{-1}N)$, *and* $M^{-1}N$ *is nonnilpotent.*

(E8) *For all* $1 \leq i, j \leq n$, *the total weight of positive walks from* i *to* j *in* $G(M^{-1}N)$ *eventually majorizes and strictly dominates the absolute value of the total weight of negative walks from* i *to* j *of the same length in* $G(M^{-1}N)$.

(E9) $\bigcup_{\tau \in Even_k} \prod_{i=1}^{k} G((M^{-1}N)^{\tau(i)}) \succ \bigcup_{\tau \in Odd_k} \prod_{i=1}^{k} G(|(M^{-1}N)^{\tau(i)}|)$

for all $k \geq k_0$ *for some* $k_0 \geq 1$, *and* $M^{-1}N$ *is nonnilpotent.*

(E10) $M^{-1}N$ *has a basic and an initial class* α *such that* $(M^{-1}N)[\alpha]$ *has a right Perron-Frobenius eigenvector.*

(E11) $\bigcup_{l=k_0}^{\infty} Hull((M^{-1}N)^l) \subset \bigcap_{j=1}^{n} H((A^{-1}N)_{*j})$ *for some* $k_0 \geq 0$, *and* $M^{-1}N$ *is nonnilpotent.*

(E12) $Hull((M^{-1}N)) \subset \bigcap_{l=k_0}^{\infty} \bigcap_{j=1}^{n} H(((A^{-1}N)^l)_{*j})$ *for some* $k_0 \geq 0$, *and* $M^{-1}N$ *is nonnilpotent.*

Type III Conditions

(F1) $A^{-1}N$ *is eventually positive.*

(F2) $A^{-1}N$ *is nonnilpotent eventually nonnegative.*

(F3) $A^{-1}N \in WPF_n$.

(F4) *$A^{-1}N$ has a simple positive and strictly dominant eigenvalue with a positive spectral projector of rank 1.*

(F5) *For all $1 \le i, j \le n$, total weight of positive $A^{-1}N$-alternating walks from i to j in $G(A^{-1}) \cup G(N)$ eventually majorizes the absolute value of the total weight of the negative $A^{-1}N$-alternating walks from i to j of the same length, and $A^{-1}N$ is nonnilpotent.*

(F6) *For all $1 \le i, j \le n$, total weight of positive $A^{-1}N$-alternating walks from i to j in $G(A^{-1}) \cup G(N)$ eventually majorizes and strictly dominates the absolute value of the total weight of the negative $A^{-1}N$-alternating walks from i to j of the same length.*

(F7) *For all $1 \le i, j \le n$, the total weight of positive walks from i to j in $G(A^{-1}N)$ eventually majorizes the absolute value of the total weight of negative walks from i to j of the same length in $G(A^{-1}N)$, and $A^{-1}N$ is nonnilpotent.*

(F8) *For all $1 \le i, j \le n$, the total weight of positive walks from i to j in $G(A^{-1}N)$ eventually majorizes and strictly dominates the absolute value of the total weight of negative walks from i to j of the same length in $G(A^{-1}N)$.*

(F9) $\bigcup_{\tau \in Even_k} \prod_{i=1}^{k} G((A^{-1}N)^{\tau(i)}) \succ \bigcup_{\tau \in Odd_k} \prod_{i=1}^{k} G(|(A^{-1}N)^{\tau(i)}|)$

for all $k \ge k_0$ for some $k_0 \ge 1$, and $A^{-1}N$ is nonnilpotent.

(F10) *$A^{-1}N$ has a basic and an initial class α such that $(A^{-1}N)[\alpha]$ has a right Perron-Frobenius eigenvector.*

(F11) $\bigcup_{l=k_0}^{\infty} Hull((A^{-1}N)^l) \subset \bigcap_{j=1}^{n} H((A^{-1}N)_{*j})$ *for some $k_0 \ge 0$, and $A^{-1}N$ is nonnilpotent.*

(F12) $Hull((A^{-1}N)) \subset \bigcap_{l=k_0}^{\infty} \bigcap_{j=1}^{n} H(((A^{-1}N)^l)_{*j})$ *for some $k_0 \ge 0$, and $A^{-1}N$ is nonnilpotent.*

Proof. We prove the theorem for the following Type I, Type II, and Type III conditions, respectively: (D1), (E1), and (F1), and then we outline the rest of the proof. Suppose that the splitting $A = M - N$ satisfies condition (D1). Then, (D1) is true if and only if M^{-1} and N are in $PF_n \subset WPF_n$. Hence, the splitting $A = M - N$ is a G-regular splitting. Moreover, suppose that $A = M - N$ is a G-regular splitting and that (E1) is true. Then, $M^{-1}N \in PF_n$ and thus $M^{-1}N$ has the Perron-Frobenius property. In particular, the given G-regular splitting becomes a Perron-Frobenius splitting. If (F1) is true then $A^{-1}N \in PF_n$ and thus $A^{-1}N$ possesses the Perron-Frobenius property. Hence, by Lemma 4.3, the G-regular splitting converges. With regards to the remaining of the conditions, we use the following diagrams to outline the proofs:

$$\begin{array}{ccccccc}
(D1) & \Rightarrow & (D2) & \Rightarrow & M^{-1}, N \in WPF_n & \Leftrightarrow & A = M - N \text{ is a } G\text{-regular splitting} \\
\Updownarrow & & \Updownarrow & & \Uparrow & & \\
(D3) & & (D4) & & (D7) & & \\
\Updownarrow & & \Updownarrow & & & & \\
(D5) & & (D6) & & & & \\
 & & \Updownarrow & & & & \\
 & & (D8) & & & & \\
 & & \Updownarrow & & & & \\
 & & (D9) & & & &
\end{array}$$

$$
\begin{array}{ccccccc}
 & & & & & & (E10) \\
 & & & & & & \Downarrow \\
(E1) & \Rightarrow & (E2) & \Rightarrow & (E3) & \Rightarrow & M^{-1}N \text{ has the Perron-Frobenius property} \\
\Updownarrow & & \Updownarrow & & & & \Downarrow \\
(E4) & & (E5) & & & & A = M - N \text{ is a Perron-Frobenius} \\
\Updownarrow & & \Updownarrow & & & & \text{splitting as well as a } G\text{-regular splitting} \\
(E6) & & (E7) & & & & \\
\Updownarrow & & \Updownarrow & & & & \\
(E8) & & (E9) & & & & \\
 & & \Updownarrow & & & & \\
 & & (E11) & & & & \\
 & & \Updownarrow & & & & \\
 & & (E12) & & & &
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & & & & & (F10) \\
 & & & & & & \Downarrow \\
(F1) & \Rightarrow & (F2) & \Rightarrow & (F3) & \Rightarrow & A^{-1}N \text{ has the Perron-Frobenius} \\
\Updownarrow & & \Updownarrow & & & & \text{property} \\
(F4) & & (F5) & & & & \Downarrow \\
\Updownarrow & & \Updownarrow & & & & \text{The splitting converges} \\
(F6) & & (F7) & & & & \text{by Lemma 4.3} \\
\Updownarrow & & \Updownarrow & & & & \\
(F8) & & (F9) & & & & \\
 & & \Updownarrow & & & & \\
 & & (F11) & & & & \\
 & & \Updownarrow & & & & \\
 & & (F12) & & & &
\end{array}
$$

All the above implications and equivalencies follow from the combinatorial, spectral, and geometric characterizations of eventually nonnegative matrices, eventually positive matrices, and matrices in WPF_n proved in Chapter 2. □

Example 4.7 Let $A = \begin{bmatrix} 8 & -3 & -4 \\ -4 & 4 & 0 \\ 0 & -3 & 3 \end{bmatrix} = 8I - B = 8I - \begin{bmatrix} 0 & 3 & 4 \\ 4 & 4 & 0 \\ 0 & 3 & 5 \end{bmatrix}$.
Then, A is a nonsingular GM-matrix and $\rho(B) = \frac{1}{2}(7 + \sqrt{73}) \approx 7.7720 < 8$.
Consider the splitting $A = M - N$ where $M = \begin{bmatrix} 7 & -2 & -3 \\ -3 & 4 & 1 \\ 1 & -2 & 3 \end{bmatrix}$ (a nonsingular GM-matrix from the previous examples) and $N = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \in WPF_3$.
Thus, this splitting of A is a GM-splitting, and hence, a G-regular splitting.
Note that $M^{-1}N = \frac{1}{36}\begin{bmatrix} 4 & 12 & 13 \\ 8 & 6 & 17 \\ 16 & 12 & 7 \end{bmatrix}$ is an eventually positive matrix, a Type II condition in Theorem 4.8. Moreover, $A^{-1}N = \frac{1}{12}\begin{bmatrix} 25 & 28 & 33 \\ 28 & 28 & 36 \\ 32 & 32 & 36 \end{bmatrix}$ is an eventually positive matrix, a Type III condition in Theorem 4.8. Hence, Theorem 4.8 predicts the convergence of this G-regular splitting. In fact, $\rho(M^{-1}N) \approx 0.8859 < 1$.

We end this section with few results on the relation between the Jacobi and Gauss-Seidel splittings on one hand and the GM- and G-regular splittings on the other hand.

Lemma 4.4 *If $D = diag(d_1, \ldots, d_n) \in \mathbb{R}^{n \times n}$ then the following are equivalent:*

(i) D is a positive diagonal matrix.

(ii) D is a nonsingular GM-matrix.

Proof. If D is a positive diagonal matrix then so is $D^{-1} = diag(d_1^{-1}, \ldots, d_n^{-1})$. Hence, $D^{-1} \in WPF_n$. Moreover, $min_{i=1}^{n} d_i < d_j = Re(d_j)$ whenever

$d_j \neq min_{i=1}^{n} d_i$. Thus, the minimum modulus eigenvalue of D is strictly less than the real part of any other eigenvalue of D. Theorem 4.1 implies that D is a nonsingular GM-matrix. Conversely, if $D = (d_1, \ldots, d_n)$ is a nonsingular GM-matrix then, by Theorem 4.1, $D^{-1} \in WPF_n$ and $min_{i=1}^{n} |d_i| = \rho(D^{-1}) < Re(d_j) = d_j$ whenever $d_j \neq \rho(D^{-1})$. But, $\rho(D^{-1}) > 0$ because $D^{-1} \in WPF_n$. Therefore, D is a positive diagonal matrix. □

Corollary 4.6 *A GM-spliting $A = M - N$ of a GM-matrix $A = (a_{ij})$ is a Jacobi splitting if and only if $M = diag(a_{11}, \ldots, a_{nn})$ is a positive diagonal matrix and $M - A \in WPF_n$.*

Remark 4.5 If $A = M - N$ is a Gauss-Seidel splitting then such a splitting can not be a G-regular splitting. If $A = M - N$ is a Gauss-Seidel splitting then N is a strictly upper triangular matrix. Hence, N is nilpotent, and thus, $N \notin WPF_n$. Therefore, the Gauss-Seidel splitting $A = M - N$ can neither be a GM-splitting nor a G-regular splitting.

CHAPTER 5

CONCLUSION

We summarize below some of the main results of this dissertation.

1. We gave a complete characterization of the collection of eventually positive matrices, PF_n, in terms of the spectral projector.

2. We gave a characterization of the sub-collection of WPF_n for which the maximum modulus eigenvalue is simple, positive and strictly dominant in terms of the spectral projector.

3. We gave combinatorial characterizations of the collections of eventually nonnegative and eventually positive matrices in terms of walks in the graph and in terms of products and unions of the graphs of the positive and negative parts of a matrix.

4. We gave characterizations of the collections of eventually nonnegative and eventually positive matrices in terms of the hull of a matrix and the half-spaces determined by their columns.

5. We established that all the containments in the following statement are

proper:

$$\begin{aligned} PF_n &= \{\text{Eventually Positive Matrices}\} \\ &\subset \{\text{Nonnilpotent Eventually Nonnegative Matrices}\} \\ &\subset WPF_n \end{aligned}$$

6. We showed that Rothblum's result [35] on the algebraic eigenspace for the spectral radius of a nonnegative matrix carries to eventually nonnegative matrices whose index is 0 or 1.

7. We showed that a matrix is eventually in WPF_n (PF_n) if and only if that matrix is in WPF_n (PF_n, respectively).

8. We characterized all similarity transformations that preserve WPF_n, PF_n, matrices with the Perron-Frobenius property, and matrices with the strong Perron-Frobenius property.

9. We gave an example that illustrates the fact that the collection of symmetric matrices with a Perron-Frobenius eigenpair is not a cone.

10. We proved that the collection of matrices with the Perron-Frobenius property and the collection of matrices with the strong Perron-Frobenius property are path-connected.

11. We proved that the collection of matrices with the strong Perron-Frobenius property and PF_n are simply connected.

12. We established that the closure $\overline{WPF_n} = WPF_n \cup \{$nilpotent matrices with a pair of right and left nonnegative eigenvectors$\}$.

13. We established that every normal matrix in WPF_n is the limit of normal matrices in PF_n.

14. We gave sufficient conditions for the nonnegativity of the right and left singular vectors corresponding to maximum singular value of a normal matrix in WPF_n.

15. We introduced the class of GM-matrices, which is a class of matrices that generalizes the class of M-matrices using the Perron-Frobenius property.

16. We showed how the collections of GM-, EM-, pseudo-M-, and M-matrices are related to one another.

17. We gave a spectral characterization of nonsingular GM-matrices (pseudo-M-matrices) and as a result we determined a condition on the spectrum that must be satisfied by a matrix in WPF_n (PF_n) to be an inverse GM-matrix (the inverse of a pseudo-M-matrix, respectively).

18. We showed by a counter-example that the inverse of a nonsingular EM-matrix does not have to be eventually nonnegative and thus a spectral characterization analogous to that of nonsingular GM- or pseudo-M-matrices does not hold for nonsingular EM-matrices.

19. We proved that the positive stable GZ-matrices are precisely the nonsingluar GM-matrices.

20. We proved that the GZ-matrices of the form $A = sI - B$ where $B \in PF_n$ that map at least one positive vector to a positive vector are precisely the pseudo-M-matrices.

21. We proved that the GZ-matrices $A = sI - B$ where $B \in WPF_n$ for which the right (left) Perron-Frobenius eigenvector of B and its image under A (under A^T, respectively) can be rotated to become positive are precisely the nonsingular GM-matrices.

22. We determined those classes of an eventually nonnegative matrix B in an EM-matrix $A = sI - B$, that result in singular (nonsingular, respectively) irreducible principal submatrices of A.

23. We introduced the following splittings with sufficient conditions for convergence and we illustrated this by examples: the splitting having the

Perron singular property for an arbitrary nonsingular matrix, the G-regular splitting of a GM-matrix, the GM-splitting of a GM-matrix, the overlapping splitting of a GM-matrix, and the commuting bounded splitting of a GM-matrix.

CHAPTER 6

FUTURE WORK

We list below other problems which are related to the work discussed in this dissertation and which are to be considered as projects for future work:

- Characterizing the matrix functions or linear transformations that preserve PF_n.
- Characterizing the matrix functions or linear transformations that preserve WPF_n.
- Characterizing the matrix functions or linear transformations that preserve nonnilpotent eventually nonnegative matrices.
- Characterizing the matrix functions or linear transformations that preserve GM-matrices.
- Proving comparison theorems for the splittings defined in Chapter 4 for GM-matrices.
- Generalizing the characterizations proved for GM-matrices to the analogues of M-matrices defined using cones in Banach spaces, in particular, the cone of positive semidefinite matrices in the space of Hermitian matrices.

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APPENDIX A

PROOFS OF PRELIMINARIES

We present here the postponed proofs of some of the results in sections 2.1.

Proof of Theorem 2.1. Let A be any $n \times n$ complex matrix with d distinct eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_d|$. Then, there exists a matrix $X \in Gl(n, \mathbb{C})$ such that the Jordan canonical form of A is given by

$$J(A) = X^{-1}AX = Box(\lambda_1) \oplus Box(\lambda_2) \oplus \cdots \oplus Box(\lambda_d), \qquad \text{(A.1)}$$

where $Box(\lambda_j)$ is the Jordan box corresponding to λ_j. Note that for all $1 \leq j \leq d$, if $g(j)$ is the number of Jordan blocks in $Box(\lambda_j)$, we have the following identity

$$\begin{aligned} Box(\lambda_j) &= J_{k_{j1}}(\lambda_j) \oplus J_{k_{j2}}(\lambda_j) \oplus \cdots \oplus J_{k_{jg(j)}}(\lambda_j) \\ &= [\lambda_j I_{k_{j1}} + N_{k_{j1}}] \oplus \cdots \oplus [\lambda_j I_{k_{jg(j)}} + N_{k_{jg(j)}}] \\ &= \Big[[\lambda_j I_{k_{j1}}] \oplus \cdots \oplus [\lambda_j I_{k_{jg(j)}}]\Big] + [N_{k_{j1}} \oplus \cdots \oplus N_{k_{jg(j)}}] \\ &= \lambda_j I_{m_j} + \tilde{N}_{m_j} \end{aligned}$$

where $\tilde{N}_{m_j} = N_{k_{j1}} \oplus N_{k_{j2}} \oplus \cdots \oplus N_{k_{jg(j)}}$. In particular, $Box(\lambda_1) = \lambda_1 I_{m_1} + \tilde{N}_{m_1}$, implying that

$$
\begin{aligned}
X^{-1}AX &= Box(\lambda_1) \oplus Box(\lambda_2) \oplus \cdots \oplus Box(\lambda_d) \\
&= [Box(\lambda_1) \oplus O_{n-m_1}] + [O_{m_1} \oplus Box(\lambda_2) \oplus \cdots \oplus Box(\lambda_d)] \\
&= \Big[[\lambda_1 I_{m_1} + \tilde{N}_{m_1}] \oplus O_{n-m_1}\Big] \\
&\quad + [O_{m_1} \oplus Box(\lambda_2) \oplus \cdots \oplus Box(\lambda_d)] \\
&= [\lambda_1 I_{m_1} \oplus O_{n-m_1}] + [\tilde{N}_{m_1} \oplus O_{n-m_1}] \\
&\quad + [O_{m_1} \oplus Box(\lambda_2) \oplus \cdots \oplus Box(\lambda_d)] \\
&= \lambda_1 [I_{m_1} \oplus O_{n-m_1}] + [\tilde{N}_{m_1} \oplus Box(\lambda_2) \oplus \cdots \oplus Box(\lambda_d)].
\end{aligned}
$$

Thus, the matrix A can be written as:

$$
A = \lambda_1 X[I_{m_1} \oplus O_{n-m_1}]X^{-1} + X[\tilde{N}_{m_1} \oplus Box(\lambda_2) \oplus \cdots \oplus Box(\lambda_d)]X^{-1}.
$$

If we define the matrices P and Q as follows:

$$
P := X[I_{m_1} \oplus O_{n-m_1}]X^{-1} \tag{A.2}
$$

$$
Q := X[\tilde{N}_{m_1} \oplus Box(\lambda_2) \oplus \cdots \oplus Box(\lambda_d)]X^{-1} \tag{A.3}
$$

then, it is easy to check that the following three statements are true:

$PQ = QP$.

$\rho(Q) = |\lambda_2| \leq |\lambda_1| = \rho(A)$.

$index_{\lambda_1}(A) = 1 \quad \Leftrightarrow \quad \tilde{N}_{m_1} = O_{m_1} \quad \Leftrightarrow \quad PQ = QP = O$.

This proves (ii), (iii) and (iv). To prove that P is the projection onto $G_{\lambda_1}(A)$ along $\bigoplus_{j=2}^{d} G_{\lambda_j}(A)$, we pick $v \in \mathbb{C}^n$ and we write v as a linear combination of the columns of X. In other words, we write v as the linear combination $v = \sum_{j=1}^{n} \alpha_j \; X_{*j}$ where the α_j's are scalars. This is possible because $X \in Gl(n, \mathbb{C})$. It is important here to note that the first m_1 columns of X form a basis for $G_{\lambda_1}(A)$ and the rest of the columns form a basis for the remaining generalized eigenspaces. Thus,

$$\begin{aligned} Pv &= \sum_{j=1}^{n} \alpha_j \ P \ X_{*j} = \sum_{j=1}^{n} \alpha_j \ P \ Xe_j = \sum_{j=1}^{n} \alpha_j \ X[I_{m_1} \oplus O_{n-m_1}]e_j \\ &= \sum_{j=1}^{m_1} \alpha_j \ Xe_j = \sum_{j=1}^{m_1} \alpha_j \ X_{*j} \ \in \ G_{\lambda_1}(A), \end{aligned}$$

and it is easy to see that $P^2 = P$, which proves (i). □

Proof of Lemma 2.1. Let $(i \ \ k)$ denote the permutation of the set $\{1, 2, 3, \ldots, s\}$ that exchanges i and k while keeping all other elements of $\{1, 2, 3, \ldots, s\}$ fixed. We call $(i \ \ k)$ a transposition of the set $\{1, 2, 3, \ldots, s\}$. Define the permutation σ of the set $\{1, 2, 3, \ldots, s\}$ in terms of transpositions as follows:

$$\sigma = (1 \quad s) \circ (2 \quad s-1) \circ (3 \quad s-2) \circ \cdots \circ (\left\lfloor \frac{s}{2} \right\rfloor \quad s - \left\lfloor \frac{s}{2} \right\rfloor + 1),$$

where $\lfloor \frac{s}{2} \rfloor$ is the integer part of $\frac{s}{2}$. If $\{e_1, e_2, \ldots, e_s\}$ denotes the standard basis of $\mathbb{C}^s$, then we define the matrix R_{js} as follows:

$$R_{js}e_i = e_{\sigma(i)} \qquad i = 1, 2, 3, \ldots, s.$$

Obviously, R_{js} is a permutation matrix because it is a rearrangement of the columns of the identity matrix. Also, R_{js} is orthogonal, i.e., $R_{js}^T = R_{js}^{-1}$, because the columns of R_{js} form an orthonormal basis. Moreover, since the transpositions that appear in σ are disjoint, it follows that σ^2 is the identity permutation of the set $\{1, 2, 3, \ldots, s\}$, which implies that $(R_{js})^2 = I_s$. This proves (i). As for (ii), it follows immediately by evaluating the matrices on both sides of the equality at the vectors of the standard basis $\{e_1, e_2, e_3, \ldots, e_s\}$. □

Proof of Corollary 2.1. Let $g(j)$ denote the number of Jordan blocks in $Box(\lambda_j)$ and let k_{ji} denote the size of the ith Jordan block in $Box(\lambda_j)$. Let $R_j := R_{jk_{j1}} \oplus R_{jk_{j2}} \oplus \cdots \oplus R_{jk_{jg(j)}}$, where R_{js} for $s = k_{j1}, \ldots, k_{jg(j)}$ is the matrix defined in Lemma 2.1. Then, R_j is another permutation matrix such

that:
(*i*) $R_j = R_j^{-1} = R_j^T$, and
(*ii*) $[Box(\lambda_j)]^T = R_j\ Box(\lambda_j)\ R_j$. □

Proof of Lemma 2.2. First, note that $Box(\lambda_2) \oplus \cdots \oplus Box(\lambda_d)$ is a Jordan matrix of size $n - m_1$. By Corollary 2.2, there is a permutation matrix R of size $n - m_1$ such that
(*i*) $R = R^{-1} = R^T$, and
(*ii*) $[Box(\lambda_2) \oplus \cdots \oplus Box(\lambda_d)]^T = R\ [Box(\lambda_2) \oplus \cdots \oplus Box(\lambda_d)]\ R$
Second, note that since $index_{\lambda_1}(A) = 1$, the Jordan canonical form of A is given by $J(A) = [\lambda_1 I_{m_1}] \oplus Box(\lambda_2) \oplus \cdots \oplus Box(\lambda_d)$, and therefore,

$$\begin{aligned}
[J(A)]^T &= [[\lambda_1 I_{m_1}] \oplus Box(\lambda_2) \oplus \cdots \oplus Box(\lambda_d)]^T \\
&= [\lambda_1 I_{m_1}] \oplus [R\ [Box(\lambda_2) \oplus \cdots \oplus Box(\lambda_d)]\ R] \\
&= [I_{m_1} \oplus R]\ [[\lambda_1 I_{m_1}] \oplus Box(\lambda_2) \oplus \cdots \oplus Box(\lambda_d)]\ [I_{m_1} \oplus R] \\
&= [I_{m_1} \oplus R]\ J(A)\ [I_{m_1} \oplus R]
\end{aligned}$$

Note also here that $[I_{m_1} \oplus R] = [I_{m_1} \oplus R]^{-1} = [I_{m_1} \oplus R]^T$. Moreover, since $A = X\ J(A)\ X^{-1}$, it follows that

$$A^T = [X^{-1}]^T\ [J(A)]^T\ X^T = [X^{-1}]^T\ [I_{m_1} \oplus R]\ J(A)\ [I_{m_1} \oplus R]\ X^T.$$

If we let S denote the matrix $[X^{-1}]^T\ [I_{m_1} \oplus R]$, then

$$A^T = S\ J(A)\ S^{-1}.$$

Thus, S is the similarity matrix giving the Jordan canonical form of A^T. Hence, the first m_1 columns of S form a basis for $G_{\lambda_1}(A^T)$, the generalized eigenspace of λ_1 for A^T. But, if we look closely at these columns of S we see that

$$\begin{aligned}
Se_i &= [X^{-1}]^T\ [I_{m_1} \oplus R]e_i = [X^{-1}]^T e_i \quad \text{for} \quad i = 1, \ldots, m_1 \\
&= [e_i^T X^{-1}]^T \quad \text{for} \quad i = 1, \ldots, m_1. \qquad \square
\end{aligned}$$