Comprehensive Examination in Geometry & Topology
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Part I. Solve three of the following problems.

I.1 Give an example (with proof) of the following:

a) A connected cell complex whose fundamental group is finite and nonabelian.

b) A connected cell complex that is not homotopy equivalent to a closed oriented manifold of any dimension.

c) A connected smooth manifold M whose tangent bundle is nontrivial, i.e. not isomorphic to $M \times \mathbb{R}^n$ for $n = \dim(M)$.

I.2 Show the equations

$$x^2 - y^2 - z^2 + w^2 = 2z$$

$$xy - zw = w$$

define a submanifold $M \subset \mathbb{R}^4$. Find the dimension of M and its tangent space at the origin.

I.3 Let M be a compact, connected, orientable 3–manifold with boundary, such that ∂M is a surface of genus g. Recall that Poincaré–Lefschetz duality provides an isomorphism $H_2(M, \partial M) \cong H^1(M)$.

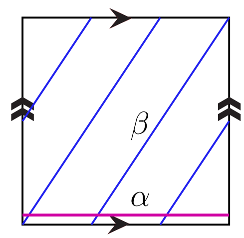
a) Write down the long exact sequence of the pair $(M, \partial M)$. Identify all homology groups that can be computed from the information given. Identify any homomorphisms that are necessarily the 0 homomorphism.

b) Assuming $g \geq 1$, prove that $H_1(M) \neq 0$.

I.4 Let M be a compact smooth manifold. Prove that there is a smooth embedding $f\colon M \to \mathbb{R}^k$, for some k depending on M.

Part II. Solve two of the following problems.

II.1 Let T be the torus, obtained as the quotient of a square with opposite sides identified. Let X be the 2–complex obtained from T by attaching a 2–cell to the curve α and a 2–cell to the curve β.



a) Compute the homology groups of X.

b) Compute $\pi_1(X)$.

c) Sketch the preimage of T, α, and β in the universal cover $\widetilde{X}$.

II.2 Consider the following 2–form on $\mathbb{R}^3 \setminus \{0\}$:

$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

You may assume that ω is closed.

a) Prove that ω is not exact. *Hint:* consider the restriction to S^2.

b) Use ω to prove the following claim. **Claim:** there does not exist a smooth function $s\colon \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{0\}$, which is the identity outside a ball of radius $1/2$ about 0.

II.3 Let A be a path connected, closed subcomplex of a path connected cell complex X and let $\iota\colon A \to X$ be the inclusion map. Let $p\colon \widetilde{X} \to X$ be the universal cover.

a) Show that $\iota_*\colon \pi_1(A, a) \to \pi_1(X, a)$ is surjective if and only if $p^{-1}(A) \subset \widetilde{X}$ is path connected.

b) Show that $\iota_*\colon \pi_1(A, a) \to \pi_1(X, a)$ is injective if and only if each path component of the preimage $p^{-1}(A)$ is simply connected.