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POLYHEDRAL SUMS AND THETA SERIES

A Dissertation
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in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

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David DeSario
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ABSTRACT

POLYHEDRAL SUMS AND THETA SERIES

David DeSario
DOCTOR OF PHILOSOPHY

Temple University, August, 2007

Professor Sinai Robins, Chair

The Ehrhart polynomial of a d-dimensional integral polytope $\mathcal{P}$ is a polynomial of degree d whose evaluation at the positive integer r gives the discrete volume of $r\mathcal{P}$, i.e. the number of integer points contained in the r-fold dilation of $\mathcal{P}$. This counting function was first studied by Ehrhart [23], who proved that it is always a polynomial. He also showed that the Ehrhart polynomial encodes the continuous volume of $\mathcal{P}$ by showing that its leading coefficient is in fact vol$\mathcal{P}$. In discrete geometry, there is often an intriguing interplay between a discrete property of an object and its continuous counterpart. In this dissertation, we use polyhedral sums and theta series to study both discrete and continuous volumes of polytopes.

In chapter 1 we extend the methods of Diaz and Robins [21] to obtain computable formulas for the Ehrhart quasi-polynomial of simple rational polytopes. In chapter 2 we study solid angles of polyhedra by analyzing the asymptotics of polyhedral theta series. Solid angles are the generalizations of two-dimensional angles to higher dimensions and they can be interpreted as the volume of spherical polytopes. We also define new solid angles with respect to l^p-norm and find computable formulas for l^1-solid angles. In chapter 3 we use Fourier methods to generalize the solid angle theory in *Computing the Continuous Discretely. Integer-Point Enumeration in Polyhedra* by Beck and Robins [7] by extending several results to include real polytopes.

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To Elaine + ϵ

with all my love

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CHAPTER 1

The Ehrhart Polynomial of a Rational Simple Polytope, Using Fourier Methods

1.1 Introduction

We begin by defining some key terms that will be used throughout this dissertation. A **convex polytope** is the convex hull of finitely many points in $\mathbb{R}^d$. Therefore, given a finite set of points $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset \mathbb{R}^d$, the polytope $\mathcal{P}$ is the smallest convex set containing those points: that is

$$\mathcal{P} = \{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n : \text{all } \lambda_k \geq 0 \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.$$

This definition is called the **vertex description** of $\mathcal{P}$ and we note that a polytope is a closed subset of $\mathbb{R}^d$. Every polytope is the bounded intersection of finitely many half-spaces and hyperplanes, and therefore also has a **hyperplane description**. A hyperplane $H = \{\mathbf{x} \in \mathbb{R}^d | \, \mathbf{a} \cdot \mathbf{x} = b\}$ is called a **supporting hyperplane** of $\mathcal{P}$ if $\mathcal{P}$ lies entirely on one side of H. A **face** of $\mathcal{P}$ is a set of the form $\mathcal{P} \cap H$, where H is a supporting hyperplane. The $(d-1)$-dimensional faces are called **facets**, the 1-dimensional faces are called **edges**, and the 0-dimensional faces are called **vertices**.

If a polytope $\mathcal{P}$ is of dimension d, we call it a d-polytope. If each vertex of a d-polytope $\mathcal{P}$ lies precisely on d edges of $\mathcal{P}$, we call $\mathcal{P}$ **simple**. $\mathcal{P}$ is called **integral (rational)** if all of its vertices have integer (rational) coordinates.

A **convex cone** $\mathcal{K} \subseteq \mathbb{R}^d$ is the intersection of finitely many half-spaces of the form $\{\mathbf{x} \in \mathbb{R}^d | \, \mathbf{a} \cdot \mathbf{x} \leq b\}$ whose corresponding hyperplanes $\{\mathbf{x} \in \mathbb{R}^d | \, \mathbf{a} \cdot \mathbf{x} = b\}$ meet in at least one point. A cone is called **pointed** if the defining hyperplanes meet in exactly one point.

Now that our terminology is set, we can discuss the content of this chapter. In this chapter, we obtain computable formulas for the integer point enumerator of any simple rational polytope $\mathcal{P}$, for any *rational* dilate of $\mathcal{P}$, and for any *real translate* of $\mathcal{P}$. The r^{th} dilate of $\mathcal{P}$ is by definition $r\mathcal{P} := \{rx | x \in \mathcal{P}\}$. The integer point enumerator that we study here is

$$\#\left\{\mathbb{Z}^d \cap \{r\mathcal{P} - T\}\right\},$$

the number of integer points inside the polytope obtained by dilating $\mathcal{P}$ by any rational dilation $r \in \mathbb{Q}$ and translating the dilate by any real vector $T \in \mathbb{R}^d$. To fix notation, we let

$$L_{\mathcal{P}}(r, T, y) = \sum_{l \in \mathbb{Z}^d} 1_{r\mathcal{P}}(l + T) e^{2\pi \langle l, y \rangle}, \tag{1.1}$$

a generalization of the integer point enumerator, called the *integer point transform* of $\mathcal{P}$ (see [7]), and defined for any rational number r, any real vector $T \in \mathbb{R}^d$, and any complex vector $y \in \mathbb{C}^d$.

We note that $L_{\mathcal{P}}(r, T, 0) = \sum_{l \in \mathbb{Z}^d} 1_{r\mathcal{P}}(l + T) = \#\left\{\mathbb{Z}^d \cap \{r\mathcal{P} - T\}\right\}$, the integer point enumerator of a translated rational polytope, and that specializing further to $T = 0$ gives us the classical integer point enumerator $L_P(r) := \sum_{l \in \mathbb{Z}^d} 1_{r\mathcal{P}}(l)$ studied by many authors ([14],[16],[21]) and known as an Ehrhart quasi-polynomial in $r \in \mathbb{Z}$.

Our use of the word *computable* refers to computing in polynomial time, and in fixed dimension, certain universal functions defined over any convex rational polytope that we call polyhedral Dedekind sums (see equation (1.12) below). Thus, this chapter focuses on the structure of the polyhedral Dedekind sums, and how we may take linear combinations of them to obtain the integer point transform $L_{\mathcal{P}}(r,T,y)$ and integer point enumerator $\sum_{l\in\mathbb{Z}^d} 1_{r\mathcal{P}}(l)$ of any convex rational polytope.

We will use Fourier methods, and in particular the Poisson summation formula, extending the methods of Diaz and Robins [21] to rewrite $L_{\mathcal{P}}(r,T,y)$ as a finite linear combination of the polyhedral Dedekind sums $S(v,u,y,T)$ over the vertices of $\mathcal{P}$. One difference from the paper [21] is that the method therein was based on first coning over the polytope, whereas here we deal with the polytope directly. Although the analysis here is more involved than in [21], the resulting formulas are much more general. In particular, we obtain computable formulas for any *rational* dilate $r\mathcal{P}$, in contrast with previously known formulas that hold only for integer values of the dilation parameter r.

1.2 Poisson summation, and statements of the results

The main theorem, stated at the end of this section, is retrieved using Poisson summation, Lipschitz summation, and various facts from Harmonic analysis. Poisson summation states that if f is a "sufficiently nice" function (for example, a function which is L^1 and continuous, and has a Fourier transform which is also L^1 and continuous), then

$$\sum_{l\in\mathbb{Z}^d} f(l) = \sum_{m\in\mathbb{Z}^d} \hat{f}(m). \tag{1.2}$$

Since the indicator function of a polytope is discontinuous on $\mathbb{R}^d$, we must first smooth the function $1_{r\mathcal{P}}$ by convolution before we can apply the Poisson summation formula. We wish to apply Poisson summation to the function

$$f(x) = 1_{r\mathcal{P}}(x+T)e^{-2\pi\langle x,y\rangle}\text{, for fixed } T\in\mathbb{R}^d \text{ and } y\in\mathbb{C}^d, \tag{1.3}$$

but to avoid infinities in the computations, we first smooth f with the approximate identity function

$$\phi_\epsilon(x) = |\epsilon|^{-d} \frac{f(\frac{x}{\epsilon})}{\int_{\mathbb{R}^d} f(x)dx},$$

and then use the following lemma:

Lemma 1.2.1.

$$\lim_{\epsilon \to 0} (f * \phi_\epsilon)(l) = f(l), \textit{ for all } l \in \mathbb{R}^d.$$

Proof.

$$\begin{aligned}
\lim_{\epsilon \to 0} (f * \phi_\epsilon)(l) &= \lim_{\epsilon \to 0} (\phi_\epsilon * f)(l) \\
&= \lim_{\epsilon \to 0} \int_{\mathbb{R}^d} \phi_\epsilon(x) f(l-x)dx \\
&= \lim_{\epsilon \to 0} \frac{|\epsilon|^{-d}}{\int_{\mathbb{R}^d} f(x)dx} \int_{\mathbb{R}^d} f\left(\frac{x}{\epsilon}\right) f(l-x)dx. \qquad (1.4)
\end{aligned}$$

Making the substitution $x = \epsilon z$ in equation (1.4) gives us

$$\begin{aligned}
\lim_{\epsilon \to 0} (f * \phi_\epsilon)(l) &= \lim_{\epsilon \to 0} \frac{1}{\int_{\mathbb{R}^d} f(x)dx} \int_{\mathbb{R}^d} f(z) f(l - \epsilon z)dz \\
&= \lim_{\epsilon \to 0} \frac{1}{\int_{\mathbb{R}^d} f(x)dx} \int_{\mathbb{R}^d} f(z) 1_{r\mathcal{P}}(l - \epsilon z + T) e^{-2\pi \langle l - \epsilon z, y \rangle} dz \\
&= \frac{1}{\int_{\mathbb{R}^d} f(x)dx} \int_{\mathbb{R}^d} \lim_{\epsilon \to 0} f(z) 1_{r\mathcal{P}}(l - \epsilon z + T) e^{-2\pi \langle l - \epsilon z, y \rangle} dz \\
&= \frac{1}{\int_{\mathbb{R}^d} f(x)dx} \int_{\mathbb{R}^d} f(z) f(l) dz \\
&= f(l).
\end{aligned}$$

We note that f has compact support due to the indicator function of $r\mathcal{P}$ in (1.3), the definition of f. Therefore, the integral above converges and we can bring the limit inside the integral sign by Lebesgue's Dominated Convergence Theorem. ■

We will use various facts from Harmonic Analysis, which can be found in the Appendix, such as the fact that for any simple rational convex polytope

$\mathcal{P}$, the Fourier-Laplace transform of its indicator function is given by

$$\hat{1}_{r\mathcal{P}}(m) = \frac{1}{(-2\pi i)^d} \sum_{\mathbf{v} \text{ a vertex of } r\mathcal{P}} \frac{\exp(2\pi i \langle \mathbf{v}, m\rangle) |\det A_{\mathbf{v}}|}{\prod_{k=1}^d \langle \mathbf{w}_k(\mathbf{v}), m\rangle}, \tag{1.5}$$

where $\mathbf{w}_1(\mathbf{v}), \ldots, \mathbf{w}_d(\mathbf{v})$ are the 1 dimensional edges of the vertex tangent cone $K_{\mathbf{v}} := \{\lambda \mathbf{v} + (1-\lambda)x : x \in r\mathcal{P}, \lambda \in \mathbb{R}_{\geq 0}\}$. Throughout the chapter, $A_{\mathbf{v}}$ is the matrix whose k^{th} column is the edge vector $\mathbf{w}_k(\mathbf{v})$.

Let $\epsilon > 0$. Then we have

$$\sum_{l \in \mathbb{Z}^d} 1_{r\mathcal{P}}(l+T) e^{-2\pi \langle l, y\rangle}$$

$$= \lim_{\epsilon \to 0} \sum_{l \in \mathbb{Z}^d} (f * \phi_\epsilon)(l) \tag{1.6}$$

$$= \lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}^d} \widehat{(f * \phi_\epsilon)}(m) \tag{1.7}$$

$$= \lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}^d} \hat{f}(m) \cdot \hat{\phi}_\epsilon(m) \tag{1.8}$$

$$= \lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}^d} \hat{1}_{r\mathcal{P}}(m+iy) e^{-2\pi i \langle T, m+iy\rangle} \cdot \hat{\phi}_\epsilon(m) \tag{1.9}$$

$$= \lim_{\epsilon \to 0} \frac{(-1)^d}{(2\pi i)^d} \sum_{\substack{\mathbf{v} \text{ a} \\ \text{vertex} \\ \text{of } r\mathcal{P}}} \sum_{m \in \mathbb{Z}^d} \frac{\exp(2\pi i \langle \mathbf{v}, m+iy\rangle) |\det A_{\mathbf{v}}|}{\prod_{k=1}^d \langle \mathbf{w}_k(\mathbf{v}), m+iy\rangle} \cdot e^{-2\pi i \langle T, m+iy\rangle} \hat{\phi}_\epsilon(m) \tag{1.10}$$

$$= \lim_{\epsilon \to 0} \frac{(-1)^d}{(2\pi i)^d} \sum_{\substack{\mathbf{v} \text{ a vertex} \\ \text{of } r\mathcal{P}}} |\det A_{\mathbf{v}}| e^{2\pi i \langle \mathbf{v} - T, iy\rangle} \sum_{m \in \mathbb{Z}^d} \frac{\exp(2\pi i \langle \mathbf{v} - T, m\rangle)}{\prod_{k=1}^d \langle \mathbf{w}_k(\mathbf{v}), m+iy\rangle} \cdot \hat{\phi}_\epsilon(m),$$

where we used Lemma 1.2.1 in equation (1.6) and Poisson summation in equation (1.7). We also used Facts 2, 3 and 4 from the Appendix in equations (1.8), (1.9), and (1.10) respectively. We note that the infinite lattice sum is now absolutely convergent due to the presence of the damping function $\hat{\phi}_\epsilon$. To evaluate this inner sum, we first let

$$n_k = \langle \mathbf{w}_k(\mathbf{v}), m\rangle, \text{ and } z_k = \langle \mathbf{w}_k(\mathbf{v}), iy\rangle.$$

Thus, $A_{\mathbf{v}}^{\mathrm{t}} \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} = \begin{pmatrix} n_1 \\ \vdots \\ n_d \end{pmatrix}$ and hence $m = A_{\mathbf{v}}^{-\mathrm{t}} n$ and $iy = A_{\mathbf{v}}^{-\mathrm{t}} z$. Similarly,

$\langle \mathbf{w}_k(\mathbf{u}), m\rangle$ is the k^{th} element of $A_{\mathbf{u}}^{\mathrm{t}} m = A_{\mathbf{u}}^{\mathrm{t}} A_{\mathbf{v}}^{-\mathrm{t}} n$. Also, we will use Facts 4 and 5 to get

$$\begin{aligned}\hat{\phi}_\epsilon(m) &= |\epsilon|^{-d}\frac{\hat{f}(\frac{m}{\epsilon})}{\int_{\mathbb{R}^d} f(x)dx} = \frac{|\epsilon|^{-d}|\epsilon|^d \hat{f}(\epsilon m)}{\int_{\mathbb{R}^d} f(x)dx} = \frac{\hat{f}(\epsilon m)}{\int_{\mathbb{R}^d} f(x)dx}\\ &= \frac{(-2\pi i)^{-d}}{\int_{\mathbb{R}^d} f(x)dx}\sum_{\substack{\mathbf{u}\text{ a vertex}\\ \text{of } r\mathcal{P}}}\frac{\exp(2\pi i\langle \mathbf{u}, \epsilon m + iy\rangle)|\det A_{\mathbf{u}}|}{\prod_{k=1}^d \langle \mathbf{w}_k(\mathbf{u}), \epsilon m + iy\rangle}\cdot e^{-2\pi i\langle T,\, \epsilon m + iy\rangle}.\end{aligned}$$

Then we have

$$\begin{aligned}&\sum_{l\in\mathbb{Z}^d} 1_{r\mathcal{P}}(l+T)e^{-2\pi\langle l,\, y\rangle}\\ &= \lim_{\epsilon\to 0}\sum_{m\in\mathbb{Z}^d}\hat{f}(m)\cdot\hat{\phi}_\epsilon(m)\\ &= \lim_{\epsilon\to 0}\frac{(-2\pi i)^{-2d}}{\int_{\mathbb{R}^d} f(x)dx}\sum_{\substack{\mathbf{v},\,\mathbf{u}\text{ vertices}\\ \text{of } r\mathcal{P}}}|\det A_{\mathbf{v}}||\det A_{\mathbf{u}}|e^{2\pi i\langle \mathbf{v}+\mathbf{u}-2T,\, iy\rangle}\cdot S,\end{aligned}$$

where S is the infinite lattice sum

$$S := \sum_{m\in\mathbb{Z}^d}\frac{\exp(2\pi i(\langle \mathbf{v}-T, m\rangle + \langle \mathbf{u}-T, \epsilon m\rangle))}{\prod_{k=1}^d\langle \mathbf{w}_k(\mathbf{v}), m+iy\rangle\langle \mathbf{w}_k(\mathbf{u}), \epsilon m + iy\rangle}. \tag{1.11}$$

By the results of Section 1.3 below, we have

$$\begin{aligned}\lim_{\epsilon\to 0} S = \frac{(2\pi i)^d}{|\det A_{\mathbf{v}}|}\sum_{g\in\mathbb{Z}^d/A_{\mathbf{v}}\mathbb{Z}^d}\prod_{k=1}^d\frac{1}{\langle C_k(\mathbf{u},\mathbf{v}), z\rangle}\cdot\prod_{k\notin R(\mathbf{u},\mathbf{v})}\frac{e^{-2\pi i\{\alpha_k+\beta_k\}z_k}}{1-e^{-2\pi i z_k}}\cdot\\ \prod_{k\in R(\mathbf{u},\mathbf{v})}\left(e^{2\pi i|\alpha_k|\langle C_k(\mathbf{u},\mathbf{v}),\, z\rangle/C_{kk}(\mathbf{u},\mathbf{v})} + \frac{1}{1-e^{-2\pi i z_k}}\right),\end{aligned}$$

where $A_{\mathbf{u}}^{\mathrm{t}}A_{\mathbf{v}}^{-\mathrm{t}} = [C_{ij}(\mathbf{u},\mathbf{v})]$, $C_k(\mathbf{u},\mathbf{v})$ is the k^{th} row of $[C_{ij}(\mathbf{u},\mathbf{v})]$, the α_k's and the β_k's are defined by:

$$\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_d\end{pmatrix} = A_{\mathbf{v}}^{-1}(\mathbf{v}-T),\quad \begin{pmatrix}\beta_1\\ \vdots\\ \beta_d\end{pmatrix} = A_{\mathbf{v}}^{-1}(g),$$

and $R(\mathbf{u},\mathbf{v}) = \{\, k \mid 1\le k\le d,\ C_{kk}(\mathbf{u},\mathbf{v})\neq 0,\ \{\alpha_k+\beta_k\} = 0 \text{ and } \alpha_k < 0\}$.

We will omit the notation $(\mathbf{u},\mathbf{v})$ whenever the dependance on $\mathbf{u}$ and $\mathbf{v}$ is clear from context. Recall that $z = iA_{\mathbf{v}}^{\mathrm{t}}y$. Therefore, we have $z_k = i\,(A_{\mathbf{v}}^{\mathrm{t}}y)_k$, the k^{th} element of the vector $iA_{\mathbf{v}}^{\mathrm{t}}y$. From the definition of $A_{\mathbf{v}}$, we see that $i\,(A_{\mathbf{v}}^{\mathrm{t}}y)_k = i\langle \mathbf{w}_k(\mathbf{v}), y\rangle$. Putting this all together we have the following:

Theorem 1.2.1. *Let $\mathcal{P}$ be a simple rational convex polytope, $T \in \mathbb{R}^d$, and $y \in \mathbb{C}^d$. Then*

$$\sum_{l\in \mathbb{Z}^d} 1_{r\mathcal{P}}(l+T)e^{-2\pi\langle l,\, y\rangle} \int_{\mathbb{R}^d} 1_{r\mathcal{P}}(x+T)e^{-2\pi\langle x,\, y\rangle}dx$$
$$= \frac{1}{(-2\pi)^d} \sum_{\substack{\mathbf{v},\,\mathbf{u}\ \textit{vertices}\\ \textit{of } r\mathcal{P}}} \frac{|\det A_{\mathbf{u}}|\; e^{-2\pi\langle \mathbf{v}+\mathbf{u}-2T,\, y\rangle}}{\prod_{k=1}^{d}\langle C_k, A_{\mathbf{v}}^{t}y\rangle} \cdot S(\mathbf{v},\mathbf{u},y,T),$$

where

$$S(\mathbf{v},\mathbf{u},y,T) = \sum_{g\in\mathbb{Z}^d/A_{\mathbf{v}}\mathbb{Z}^d}\ \prod_{k\notin R} \frac{e^{2\pi\{\alpha_k+\beta_k\}\langle \mathbf{w}(\mathbf{v}),\, y\rangle}}{1-e^{2\pi\langle \mathbf{w}(\mathbf{v}),\, y\rangle}} \prod_{k\in R}\left(e^{-2\pi|\alpha_k|\langle C_k,\, A_{\mathbf{v}}^{t}y\rangle/C_{kk}} + \frac{1}{1-e^{2\pi\langle \mathbf{w}(\mathbf{v}),\, y\rangle}}\right) \tag{1.12}$$

is the polyhedral Dedekind sum.

We note that the singular set in Theorem 1.2.1 is contained in a countably infinite union of hyperplanes, defined by

$$\Omega = \bigcup_{\substack{\mathbf{u},\mathbf{v}\text{ vertices of } r\mathcal{P}\\ k=1,\ldots,d}} \left\{y\in\mathbb{C}^d \,|\, \langle A_{\mathbf{v}}C_k(\mathbf{u},\mathbf{v}), y\rangle = 0\right\} \cup \left\{y\in\mathbb{C}^d \,|\, \langle \mathbf{w}_k(\mathbf{v}), y\rangle \in i\mathbb{Z}\right\}.$$

1.3 Polyhedral Dedekind Sums

In this section, we use Lipschitz summation formulas and some basic facts from Harmonic Analysis to evaluate the limiting value, as ϵ approaches zero, of our infinite lattice sum

$$S = \sum_{m\in\mathbb{Z}^d} \frac{\exp(2\pi i(\langle \mathbf{v}-T, m\rangle + \langle \mathbf{u}-T, \epsilon m\rangle))}{\prod_{k=1}^{d}\langle \mathbf{w}_k(\mathbf{v}), m+iy\rangle\langle \mathbf{w}_k(\mathbf{u}), \epsilon m+iy\rangle}. \tag{1.13}$$

This evaluation is given in the following theorem.

Theorem 1.3.1.

$$\lim_{\epsilon\to 0} S = \frac{(2\pi i)^d}{|\det A_{\mathbf{v}}|} \sum_{g\in\mathbb{Z}^d/A_{\mathbf{v}}\mathbb{Z}^d} \prod_{k=1}^{d} \frac{1}{\langle C_k(\mathbf{u},\mathbf{v}), z\rangle} \cdot \prod_{k\notin R(\mathbf{u},\mathbf{v})} \frac{e^{-2\pi i\{\alpha_k+\beta_k\}z_k}}{1-e^{-2\pi i z_k}} \cdot$$

$$\prod_{k\in R(\mathbf{u},\mathbf{v})} \left(e^{2\pi i|\alpha_k|\langle C_k(\mathbf{u},\mathbf{v}),\, z\rangle/C_{kk}(\mathbf{u},\mathbf{v})} + \frac{1}{1-e^{-2\pi i z_k}} \right).$$

where $A_{\mathbf{u}}^t A_{\mathbf{v}}^{-t} = [C_{ij}(\mathbf{u},\mathbf{v})]$, $C_k(\mathbf{u},\mathbf{v})$ *is the* k^{th} *row of* $[C_{ij}(\mathbf{u},\mathbf{v})]$, *the* α_k*'s and the* β_k*'s are defined by*

$$A_{\mathbf{v}}^{-1}(\mathbf{v}-T) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} \quad \text{and } A_{\mathbf{v}}^{-1}(g) = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_d \end{pmatrix}$$

respectively, and where

$$R(\mathbf{u},\mathbf{v}) = \{\, k \mid 1 \le k \le d,\ C_{kk}(\mathbf{u},\mathbf{v}) \ne 0,\ \{\alpha_k+\beta_k\} = 0 \text{ and } \alpha_k < 0\}.$$

We will need the following technical lemma, which is a variation of Lipschitz summation, in the proof of Theorem 1.3.1.

Lemma 1.3.1. *Let* $\tau_1 \in H$, $\tau_2 \in \mathbb{C}\backslash\mathbb{R}$ *and* $x \in \mathbb{R}$. *Then*

$$\sum_{m\in\mathbb{Z}} \frac{e^{2\pi i m x}}{(m+\tau_1)(m+\tau_2)} = \frac{2\pi i}{\tau_1-\tau_2}\left[\frac{e^{-2\pi i\{x\}\tau_2}}{1-e^{-2\pi i\tau_2}} - \frac{e^{-2\pi i\{x\}\tau_1}}{1-e^{-2\pi i\tau_1}}\right].$$

Proof. We will use the following Lipschitz summation formulas:
For $\mathrm{Re}(s) > 0$, $\alpha \in \mathbb{R}$, and $\tau \in H$, the complex upper half plane:

$$\sum_{m\in\mathbb{Z}} \frac{e^{2\pi i m\alpha}}{(\tau+m)^s} = \frac{(-2\pi i)^s}{\Gamma(s)} \sum_{n=1}^{\infty} (n-\{\alpha\})^{s-1} e^{2\pi i\tau(n-\{\alpha\})}. \tag{1.14}$$

For $\mathrm{Re}(s) > 0$, $\alpha \in \mathbb{R}$, and $\tau \in -H$, the complex lower half plane:

$$\sum_{m\in\mathbb{Z}} \frac{e^{2\pi i m(1-\alpha)}}{(\tau+m)^s} = \frac{(2\pi i)^s}{\Gamma(s)} \sum_{n=1}^{\infty} (n-\{\alpha\})^{s-1} e^{-2\pi i\tau(n-\{\alpha\})}. \tag{1.15}$$

Note: Formula (1.15) is obtained by putting $-\tau$ into (1.14) and changing the variable m to $-m$.

By Facts 8 and 9 and formula (1.14), we have for τ_1, $\tau_2 \in H$, and $x \in \mathbb{R}$:

$$\sum_{m\in\mathbb{Z}} \frac{e^{2\pi i m x}}{(\tau_1+m)^{s_1}(\tau_2+m)^{s_2}} = \int_0^1 \left(\sum_{n\in\mathbb{Z}} \frac{e^{2\pi i n(x-t)}}{(\tau_1+n)^{s_1}}\right)\left(\sum_{k\in\mathbb{Z}} \frac{e^{2\pi i k t}}{(\tau_2+k)^{s_2}}\right) dt$$

$$= \frac{(-2\pi i)^{s_1+s_2}}{\Gamma(s_1)\Gamma(s_2)} \int_0^1 \left(\sum_{n=1}^{\infty} (n-\{x-t\})^{s_1-1} e^{2\pi i \tau_1 (n-\{x-t\})}\right)\left(\sum_{k=1}^{\infty} (k-t)^{s_2-1} e^{2\pi i \tau_2 (k-t)}\right) dt$$

(Letting $s_1 = s_2 = 1$)

$$= (2\pi i)^2 \sum_{n,\,k=1}^{\infty} e^{2\pi i(\tau_1 n + \tau_2 k)} \int_0^1 e^{-2\pi i(\tau_1\{x-t\}+\tau_2 t)} dt$$

$$= (2\pi i)^2 \frac{e^{2\pi i \tau_1}}{1-e^{2\pi i \tau_1}} \cdot \frac{e^{2\pi i \tau_2}}{1-e^{2\pi i \tau_2}} \int_0^1 e^{-2\pi i(\tau_1\{x-t\}+\tau_2 t)} dt$$

$$= (2\pi i)^2 \frac{e^{2\pi i \tau_1}}{1-e^{2\pi i \tau_1}} \cdot \frac{e^{2\pi i \tau_2}}{1-e^{2\pi i \tau_2}} \left[\int_0^{\{x\}} e^{-2\pi i(\tau_1(\{x\}-t)+\tau_2 t)} dt + \int_{\{x\}}^1 e^{-2\pi i(\tau_1(\{x\}-t+1)+\tau_2 t)} dt\right]$$

$$= (2\pi i)^2 \frac{e^{2\pi i \tau_1}}{1-e^{2\pi i \tau_1}} \cdot \frac{e^{2\pi i \tau_2}}{1-e^{2\pi i \tau_2}} \left[e^{-2\pi i \tau_1\{x\}} \int_0^{\{x\}} e^{-2\pi i t(\tau_2-\tau_1)} dt + e^{-2\pi i \tau_1(\{x\}+1)} \int_{\{x\}}^1 e^{-2\pi i t(\tau_2-\tau_1)} dt\right]$$

$$= (2\pi i)^2 \frac{e^{2\pi i \tau_1}}{1-e^{2\pi i \tau_1}} \cdot \frac{e^{2\pi i \tau_2}}{1-e^{2\pi i \tau_2}} \left[\frac{e^{-2\pi i \tau_1\{x\}}}{-2\pi i(\tau_2-\tau_1)} \left(\left(1-e^{-2\pi i \tau_1}\right) e^{-2\pi i\{x\}(\tau_2-\tau_1)} + e^{-2\pi i \tau_2} - 1\right)\right]$$

$$= \frac{2\pi i}{\tau_1-\tau_2} \cdot \frac{1}{e^{-2\pi i \tau_1}-1} \cdot \frac{1}{e^{-2\pi i \tau_2}-1} \left[\left(1-e^{-2\pi i \tau_1}\right) e^{-2\pi i\{x\}\tau_2} - \left(1-e^{-2\pi i \tau_2}\right) e^{-2\pi i\{x\}\tau_1}\right]$$

$$= \frac{2\pi i}{\tau_1-\tau_2} \left[\frac{e^{-2\pi i\{x\}\tau_2}}{1-e^{-2\pi i \tau_2}} - \frac{e^{-2\pi i\{x\}\tau_1}}{1-e^{-2\pi i \tau_1}}\right].$$

To finish the proof of the Lemma 1.3.1, we now let $\tau_1 \in H$, $\tau_2 \in -H$, and $x \in \mathbb{R}$ and use formula (1.15) to get:

$$\sum_{m\in\mathbb{Z}} \frac{e^{2\pi i m x}}{(\tau_1+m)^{s_1}(\tau_2+m)^{s_2}} = \int_0^1 \left(\sum_{n\in\mathbb{Z}} \frac{e^{2\pi i n(x-t)}}{(\tau_1+n)^{s_1}}\right)\left(\sum_{k\in\mathbb{Z}} \frac{e^{2\pi i k t}}{(\tau_2+k)^{s_2}}\right) dt$$

$$= \frac{(-2\pi i)^{s_1}(2\pi i)^{s_2}}{\Gamma(s_1)\Gamma(s_2)} \int_0^1 \left(\sum_{n=1}^{\infty} (n-\{x-t\})^{s_1-1} e^{2\pi i \tau_1 (n-\{x-t\})}\right) \cdot$$

$$\left(\sum_{k=1}^{\infty} (k-\{1-t\})^{s_2-1} e^{-2\pi i \tau_2 (k-\{1-t\})}\right) dt.$$

Letting $s_1 = s_2 = 1$, we obtain

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \frac{e^{2\pi imx}}{(m+\tau_1)(m+\tau_2)} &= -(2\pi i)^2 \int_0^1 \left(\sum_{n=1}^{\infty} e^{2\pi i\tau_1(n-\{x-t\})}\right)\left(\sum_{k=1}^{\infty} e^{-2\pi i\tau_2(k+t-1)}\right) dt \\
&= -(2\pi i)^2 \sum_{\substack{n=1\\k=0}}^{\infty} e^{2\pi i(\tau_1 n-\tau_2 k)} \int_0^1 e^{-2\pi i(\tau_1\{x-t\}+\tau_2 t)} dt \\
&= -(2\pi i)^2 \frac{e^{2\pi i\tau_1}}{1-e^{2\pi i\tau_1}} \cdot \frac{1}{1-e^{-2\pi i\tau_2}} \int_0^1 e^{-2\pi i(\tau_1\{x-t\}+\tau_2 t)} dt \\
&= (2\pi i)^2 \frac{e^{2\pi i\tau_1}}{1-e^{2\pi i\tau_1}} \cdot \frac{e^{2\pi i\tau_2}}{1-e^{2\pi i\tau_2}} \int_0^1 e^{-2\pi i(\tau_1\{x-t\}+\tau_2 t)} dt \\
&= \frac{2\pi i}{\tau_1-\tau_2}\left[\frac{e^{-2\pi i\{x\}\tau_2}}{1-e^{-2\pi i\tau_2}} - \frac{e^{-2\pi i\{x\}\tau_1}}{1-e^{-2\pi i\tau_1}}\right], \text{ as above.}
\end{aligned}$$

■

Proof of Theorem (1.3.1). We first note that the notation $(\mathbf{u}, \mathbf{v})$ is suppressed throughout the proof since $\mathbf{u}$ and $\mathbf{v}$ remain fixed. To evaluate the sum

$$S = \sum_{m\in\mathbb{Z}^d} \frac{\exp(2\pi i(\langle \mathbf{v}-T, m\rangle + \langle \mathbf{u}-T, \epsilon m\rangle))}{\prod_{k=1}^d \langle \mathbf{w}_k(\mathbf{v}), m+iy\rangle\langle \mathbf{w}_k(\mathbf{u}), \epsilon m+iy\rangle},$$

we first diagonalize the linear forms in the denominator by letting $n_k = \langle \mathbf{w}_k(\mathbf{v}), m\rangle$ and $z_k = \langle \mathbf{w}_k(\mathbf{v}), iy\rangle$. Thus $A_{\mathbf{v}}^{\mathrm{t}} \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} = \begin{pmatrix} n_1 \\ \vdots \\ n_d \end{pmatrix}$ and hence $m = A_{\mathbf{v}}^{-\mathrm{t}} n$ and $iy = A_{\mathbf{v}}^{-\mathrm{t}} z$. Similarly, $\langle \mathbf{w}_k(\mathbf{u}), m\rangle$ is the k^{th} element of the vector $A_{\mathbf{u}}^{\mathrm{t}} m = A_{\mathbf{u}}^{\mathrm{t}} A_{\mathbf{v}}^{-\mathrm{t}} n$. Hence,

$$\begin{aligned}
S &= \sum_{m\in\mathbb{Z}^d} \frac{\exp(2\pi i(\langle \mathbf{v}-T, m\rangle + \langle \mathbf{u}-T, \epsilon m\rangle))}{\prod_{k=1}^d \langle \mathbf{w}_k(\mathbf{v}), m+iy\rangle\langle \mathbf{w}_k(\mathbf{u}), \epsilon m+iy\rangle} \\
&= \sum_{m\in\mathbb{Z}^d} \frac{\exp(2\pi i(\langle \mathbf{v}-T, m\rangle + \langle \mathbf{u}-T, \epsilon m\rangle))}{\prod_{k=1}^d (\langle \mathbf{w}_k(\mathbf{v}), m\rangle + \langle \mathbf{w}_k(\mathbf{v}), iy\rangle)(\langle \mathbf{w}_k(\mathbf{u}), m\rangle\epsilon + \langle \mathbf{w}_k(\mathbf{u}), iy\rangle)} \\
&= \sum_{m\in\mathbb{Z}^d} \frac{\exp(2\pi i(\langle \mathbf{v}-T, m\rangle + \langle \mathbf{u}-T, \epsilon m\rangle))}{\prod_{k=1}^d (n_k+z_k)(\langle \mathbf{w}_k(\mathbf{u}), m\rangle\epsilon + \langle \mathbf{w}_k(\mathbf{u}), iy\rangle)} \\
&= \sum_{m\in\mathbb{Z}^d} \frac{\exp(2\pi i(\langle \mathbf{v}-T, m\rangle + \langle \mathbf{u}-T, \epsilon m\rangle))}{\prod_{k=1}^d (n_k+z_k)(A_{\mathbf{u}}^{\mathrm{t}} A_{\mathbf{v}}^{-\mathrm{t}}(n\epsilon+z))_k},
\end{aligned}$$

where $(A_{\mathrm{u}}^{\mathrm{t}}A_{\mathrm{v}}^{-\mathrm{t}}(n\epsilon+z))_k$ is the k^{th} element of the vector $A_{\mathrm{u}}^{\mathrm{t}}A_{\mathrm{v}}^{-\mathrm{t}}(n\epsilon+z)$. Note that the coefficient of ϵ is a rational linear combination of $n_1,\ldots,n_d$, and as m runs through $\mathbb{Z}^d$, n runs through the lattice $\mathbb{L} := A_{\mathrm{v}}^{\mathrm{T}} \circ \mathbb{Z}^d$. With this change of variable, our lattice sum over $m \in \mathbb{Z}^d$ is transformed into a lattice sum over $n \in \mathbb{L}$. We now have

$$\begin{aligned} S &= \sum_{n\in\mathbb{L}} \frac{\exp(2\pi i(\langle \mathbf{v}-T, A_{\mathrm{v}}^{-\mathrm{t}}n\rangle + \langle \mathbf{u}-T, \epsilon A_{\mathrm{v}}^{-\mathrm{t}}n\rangle))}{\prod_{k=1}^{d}(n_k+z_k)(A_{\mathrm{u}}^{\mathrm{t}}A_{\mathrm{v}}^{-\mathrm{t}}(n\epsilon+z))_k} \\ &= \sum_{n\in\mathbb{L}} F(n) \\ &= \frac{1}{|\det A_{\mathrm{v}}|} \sum_{g\in\mathbb{Z}^d/A_{\mathrm{v}}\mathbb{Z}^d} \sum_{l\in\mathbb{Z}^d} F(l) e^{2\pi i\langle A_{\mathrm{v}}^{-1}g,\, l\rangle}, \end{aligned}$$

using the orthogonality relations of characters on the finite abelian group $\mathbb{Z}^d/A_{\mathrm{v}}\mathbb{Z}^d$ (Fact 6 in the appendix), where F is defined by the penultimate equality above. Therefore

$$\begin{aligned} S &= \frac{1}{|\det A_{\mathrm{v}}|} \sum_{g\in\mathbb{Z}^d/A_{\mathrm{v}}\mathbb{Z}^d} \sum_{l\in\mathbb{Z}^d} \frac{\exp(2\pi i(\langle \mathbf{v}-T, A_{\mathrm{v}}^{-\mathrm{t}}l\rangle + \langle \mathbf{u}-T, \epsilon A_{\mathrm{v}}^{-\mathrm{t}}l\rangle + \langle A_{\mathrm{v}}^{-1}g, l\rangle))}{\prod_{k=1}^{d}(l_k+z_k)(A_{\mathrm{u}}^{\mathrm{t}}A_{\mathrm{v}}^{-\mathrm{t}}(l\epsilon+z))_k} \\ &= \frac{1}{|\det A_{\mathrm{v}}|} \sum_{g\in\mathbb{Z}^d/A_{\mathrm{v}}\mathbb{Z}^d} \sum_{l\in\mathbb{Z}^d} \frac{\exp(2\pi i(\langle A_{\mathrm{v}}^{-1}(\mathbf{v}-T+g), l\rangle + \langle A_{\mathrm{v}}^{-1}(\mathbf{u}-T), \epsilon l\rangle))}{\prod_{k=1}^{d}(l_k+z_k)(A_{\mathrm{u}}^{\mathrm{t}}A_{\mathrm{v}}^{-\mathrm{t}}(l\epsilon+z))_k} \\ &= \frac{1}{|\det A_{\mathrm{v}}|} \sum_{g\in\mathbb{Z}^d/A_{\mathrm{v}}\mathbb{Z}^d} \sum_{l\in\mathbb{Z}^d} \prod_{k=1}^{d} \frac{\exp(2\pi i((\alpha_k+\beta_k)l_k + \alpha_k l_k \epsilon))}{(l_k+z_k)(A_{\mathrm{u}}^{\mathrm{t}}A_{\mathrm{v}}^{-\mathrm{t}}(l\epsilon+z))_k} \\ &= \frac{1}{|\det A_{\mathrm{v}}|} \sum_{g\in\mathbb{Z}^d/A_{\mathrm{v}}\mathbb{Z}^d} \prod_{k=1}^{d} \sum_{l_k\in\mathbb{Z}} \frac{\exp(2\pi i((1+\epsilon)\alpha_k+\beta_k)l_k)}{(l_k+z_k)(A_{\mathrm{u}}^{\mathrm{t}}A_{\mathrm{v}}^{-\mathrm{t}}(l\epsilon+z))_k}, \end{aligned}$$

where the α_k's and the β_k's are defined by $A_{\mathrm{v}}^{-1}(\mathbf{v}-T) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix}$ and $A_{\mathrm{v}}^{-1}(g) = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_d \end{pmatrix}$ respectively. We note that the order of the product and sum can be switched due to the absolute convergence of the sum. To streamline the notation, we let $A_{\mathbf{u}}^{t}A_{\mathbf{v}}^{-t} = [C_{ij}]$. Then the k^{th} element of $A_{\mathbf{u}}^{t}A_{\mathbf{v}}^{-t}(l\epsilon + z)$ can be written as

$$C_{k1}(l_1\epsilon + z_1) + C_{k2}(l_2\epsilon + z_2) + \cdots + C_{kd}(l_d\epsilon + z_d).$$

We now investigate the cases when $C_{kk} = 0$ and $C_{kk} \neq 0$ for $1 \leq k \leq d$.

Case 1: $C_{kk} = 0$.

Then

$$\begin{aligned}
&\sum_{l_k \in \mathbb{Z}} \frac{\exp(2\pi i((1+\epsilon)\alpha_k + \beta_k)l_k)}{(l_k + z_k)(A_{\mathbf{u}}^{t}A_{\mathbf{v}}^{-t}(l\epsilon + z))_k} \\
&= \sum_{l_k \in \mathbb{Z}} \frac{\exp(2\pi i((1+\epsilon)\alpha_k + \beta_k)l_k)}{(l_k + z_k)(C_{k1}(l_1\epsilon + z_1) + \cdots + C_{kk}\widehat{(l_k\epsilon + z_k)} + \cdots + C_{kd}(l_d\epsilon + z_d))} \\
&= \frac{1}{(C_{k1}(l_1\epsilon + z_1) + \cdots + C_{kk}\widehat{(l_k\epsilon + z_k)} + \cdots + C_{kd}(l_d\epsilon + z_d))} \sum_{l_k \in \mathbb{Z}} \frac{e^{2\pi i((1+\epsilon)\alpha_k + \beta_k)l_k}}{l_k + z_k},
\end{aligned}$$

where the $\widehat{\ }$ means that the corresponding term is missing. Now we can apply Lipschitz summation (1.14) to write this sum as

$$\begin{aligned}
&\frac{-2\pi i}{(C_{k1}(l_1\epsilon + z_1) + \cdots + C_{kk}\widehat{(l_k\epsilon + z_k)} + \cdots + C_{kd}(l_d\epsilon + z_d))} \sum_{n=1}^{\infty} e^{2\pi i z_k(n - \{(1+\epsilon)\alpha_k + \beta_k\})} \\
&= \frac{-2\pi i\, e^{-2\pi i z_k\{(1+\epsilon)\alpha_k + \beta_k\}}}{(C_{k1}(l_1\epsilon + z_1) + \cdots + C_{kk}\widehat{(l_k\epsilon + z_k)} + \cdots + C_{kd}(l_d\epsilon + z_d))} \cdot \frac{e^{2\pi i z_k}}{1 - e^{2\pi i z_k}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\lim_{\epsilon \to 0} \sum_{l_k \in \mathbb{Z}} \frac{\exp(2\pi i((1+\epsilon)\alpha_k + \beta_k)l_k)}{(l_k + z_k)(A_{\mathbf{u}}^{t}A_{\mathbf{v}}^{-t}(l\epsilon + z))_k} \\
&= \lim_{\epsilon \to 0} \frac{-2\pi i\, e^{-2\pi i z_k\{(1+\epsilon)\alpha_k + \beta_k\}}}{(C_{k1}(l_1\epsilon + z_1) + \cdots + C_{kk}\widehat{(l_k\epsilon + z_k)} + \cdots + C_{kd}(l_d\epsilon + z_d))} \cdot \frac{e^{2\pi i z_k}}{1 - e^{2\pi i z_k}} \\
&= \frac{-2\pi i\, e^{-2\pi i z_k\{\alpha_k + \beta_k\}}}{C_{k1}z_1 + \cdots + C_{kd}z_d} \cdot \frac{e^{2\pi i z_k}}{1 - e^{2\pi i z_k}} \\
&= \frac{1}{\langle C_k, z\rangle} \cdot \frac{2\pi i e^{-2\pi i\{\alpha_k + \beta_k\}z_k}}{1 - e^{-2\pi i z_k}}.
\end{aligned}$$

Case 2: $C_{kk} \neq 0$.

Since $C_{kk} \neq 0$, the k^{th} element of $A_{\text{u}}^{\text{t}} A_{\text{v}}^{-\text{t}}(l\epsilon + z)$ can be written as

$$\begin{aligned} & C_{k1}(l_1\epsilon + z_1) + C_{k2}(l_2\epsilon + z_2) + \cdots + C_{kd}(l_d\epsilon + z_d) \\ = \ & \epsilon C_{k1} l_1 + \epsilon C_{k2} l_2 + \cdots + \epsilon C_{kd} l_d + C_{k1} z_1 + C_{k2} z_2 + \cdots + C_{kd} z_d \\ = \ & \epsilon C_{kk} \left(l_k + \frac{C_{k1} l_1 + C_{k2} l_2 + \ldots \widehat{C_{kk} l_k} + \cdots + C_{kd} l_d}{C_{kk}} + \frac{C_{k1} z_1 + C_{k2} z_2 + \cdots + C_{kd} z_d}{\epsilon C_{kk}} \right) \\ = \ & \epsilon C_{kk}(l_k + \tau), \end{aligned}$$

$$\text{for } \tau = \frac{C_{k1} l_1 + \cdots + \widehat{C_{kk} l_k} + \cdots + C_{kd} l_d}{C_{kk}} + \frac{C_{k1} z_1 + \cdots + C_{kd} z_d}{\epsilon C_{kk}}, \tag{1.16}$$

where the $\widehat{\ }$ means that the corresponding term is missing. Thus we have

$$\begin{aligned} & \sum_{l_k \in \mathbb{Z}} \frac{\exp(2\pi i((1+\epsilon)\alpha_k + \beta_k) l_k)}{(l_k + z_k)(A_{\text{u}}^{\text{t}} A_{\text{v}}^{-\text{t}}(l\epsilon + z))_k} \\ = \ & \frac{1}{\epsilon C_{kk}} \sum_{l_k \in \mathbb{Z}} \frac{\exp(2\pi i((1+\epsilon)\alpha_k + \beta_k) l_k)}{(l_k + z_k)(l_k + \tau)}. \end{aligned}$$

Now we use Lemma 1.3.1 with $x = (1+\epsilon)\alpha_k + \beta_k$, $\tau_1 = z_k$, and $\tau_2 = \tau$ to get:

$$\begin{aligned} & \lim_{\epsilon \to 0} \frac{1}{\epsilon C_{kk}} \sum_{l_k \in \mathbb{Z}} \frac{\exp(2\pi i((1+\epsilon)\alpha_k + \beta_k) l_k)}{(l_k + z_k)(l_k + \tau)} \\ = \ & \lim_{\epsilon \to 0} \frac{1}{\epsilon C_{kk}} \sum_{l_k \in \mathbb{Z}} \frac{\exp(2\pi i x l_k)}{(l_k + \tau_1)(l_k + \tau_2)} \\ = \ & \lim_{\epsilon \to 0} \frac{1}{\epsilon C_{kk}} \cdot \frac{2\pi i}{\tau_1 - \tau_2} \left[\frac{e^{-2\pi i \{x\} \tau_2}}{1 - e^{-2\pi i \tau_2}} - \frac{e^{-2\pi i \{x\} \tau_1}}{1 - e^{-2\pi i \tau_1}} \right] \\ = \ & \lim_{\epsilon \to 0} \frac{1}{\epsilon C_{kk}} \cdot \frac{2\pi i}{z_k - \tau} \left[\frac{e^{-2\pi i \{(1+\epsilon)\alpha_k + \beta_k\} \tau}}{1 - e^{-2\pi i \tau}} - \frac{e^{-2\pi i \{(1+\epsilon)\alpha_k + \beta_k\} z_k}}{1 - e^{-2\pi i z_k}} \right] \\ = \ & \frac{-2\pi i}{C_{k1} z_1 + C_{k2} z_2 + \cdots + C_{kd} z_d} \left[\lim_{\epsilon \to 0} \frac{e^{-2\pi i \{(1+\epsilon)\alpha_k + \beta_k\} \tau}}{1 - e^{-2\pi i \tau}} - \frac{e^{-2\pi i \{\alpha_k + \beta_k\} z_k}}{1 - e^{-2\pi i z_k}} \right], \end{aligned}$$

using definition (1.16) of τ. To finish Case 2, we must evaluate the remaining nontrivial limit. We may assume without loss of generality that $\tau \in H$.

Then $2\pi i\tau \to 0$ as $\epsilon \to 0$. Thus if $\{\alpha_k + \beta_k\} \neq 0$, we have

$$\lim_{\epsilon\to 0} \frac{e^{-2\pi i\{(1+\epsilon)\alpha_k+\beta_k\}\tau}}{1-e^{-2\pi i\tau}} = \lim_{\epsilon\to 0} \frac{e^{2\pi i(1-\{(1+\epsilon)\alpha_k+\beta_k\})\tau}}{e^{2\pi i\tau}-1} = 0.$$

Therefore, if $C_{kk} \neq 0$ and $\{\alpha_k + \beta_k\} \neq 0$, then

$$\lim_{\epsilon\to 0} \sum_{l_k\in\mathbb{Z}} \frac{\exp(2\pi i((1+\epsilon)\alpha_k+\beta_k)l_k)}{(l_k+z_k)(A_{\mathbf{u}}^{\mathrm{t}}A_{\mathbf{v}}^{-\mathrm{t}}(l\epsilon+z))_k} = \frac{2\pi i}{\langle C_k,\, z\rangle} \cdot \frac{e^{-2\pi i\{\alpha_k+\beta_k\}z_k}}{1-e^{-2\pi i z_k}}.$$

We now assume that $\{\alpha_k + \beta_k\} = 0$ and hence

$$\{(1+\epsilon)\alpha_k+\beta_k\} = \{\epsilon\alpha_k+\alpha_k+\beta_k\} = \begin{cases} \epsilon\alpha_k & \text{if } \alpha_k \geq 0 \\ 1-\epsilon|\alpha_k| & \text{if } \alpha_k < 0 \end{cases}.$$

We recall that

$$\tau = \frac{C_{k1}l_1 + \cdots + \widehat{C_{kk}l_k} + \cdots + C_{kd}l_d}{C_{kk}} + \frac{\langle C_k, z_k\rangle}{\epsilon C_{kk}}$$

and for simplicity we write $\tau = A + \frac{B}{\epsilon}$, where $\mathrm{Im}(B) = \mathrm{Im}\left(\frac{\langle C_k,\, z_k\rangle}{C_{kk}}\right) > 0$. Then

$$\begin{aligned}
\lim_{\epsilon\to 0} \frac{e^{-2\pi i\{(1+\epsilon)\alpha_k+\beta_k\}\tau}}{1-e^{-2\pi i\tau}} &= \begin{cases} \lim_{\epsilon\to 0} \frac{e^{-2\pi i\epsilon\alpha_k\tau}}{1-e^{-2\pi i\tau}} & \text{if } \alpha_k \geq 0 \\ \lim_{\epsilon\to 0} \frac{e^{-2\pi i(1-\epsilon|\alpha_k|)\tau}}{1-e^{-2\pi i\tau}} & \text{if } \alpha_k < 0 \end{cases} \\
&= \begin{cases} \lim_{\epsilon\to 0} \frac{e^{-2\pi i\alpha_k(\epsilon A+B)}}{1-e^{-2\pi i\tau}} & \text{if } \alpha_k \geq 0 \\ \lim_{\epsilon\to 0} \frac{e^{-2\pi i\tau}}{1-e^{-2\pi i\tau}} \cdot e^{2\pi i|\alpha_k|(\epsilon A+B)} & \text{if } \alpha_k < 0 \end{cases} \\
&= \begin{cases} 0 & \text{if } \alpha_k \geq 0 \\ -e^{2\pi i|\alpha_k|B} & \text{if } \alpha_k < 0 \end{cases} \\
&= \begin{cases} 0 & \text{if } \alpha_k \geq 0 \\ -e^{2\pi i|\alpha_k|\langle C_k,\, z\rangle/C_{kk}} & \text{if } \alpha_k < 0 \end{cases}.
\end{aligned}$$

To summarize cases 1 and 2, we let

$$R = \{\, k \mid 1 \leq k \leq d,\ C_{kk} \neq 0, \{\alpha_k+\beta_k\} = 0 \text{ and } \alpha_k < 0\}.$$

Therefore

$$\lim_{\epsilon\to 0}\sum_{l_k\in\mathbb{Z}}\frac{\exp(2\pi i((1+\epsilon)\alpha_k+\beta_k)l_k)}{(l_k+z_k)(A_{\mathrm{u}}^{\mathrm{t}}A_{\mathrm{v}}^{-\mathrm{t}}(l\epsilon+z))_k}$$

$$= \begin{cases} \frac{2\pi i}{\langle C_k,\, z\rangle}\cdot\frac{e^{-2\pi i\{\alpha_k+\beta_k\}z_k}}{1-e^{-2\pi i z_k}} & \text{if } \; k\notin R \\ \frac{2\pi i}{\langle C_k,\, z\rangle}\left(e^{2\pi i|\alpha_k|\langle C_k,\, z\rangle/C_{kk}}+\frac{1}{1-e^{-2\pi i z_k}}\right) & \text{if } \; k\in R. \end{cases}$$

Putting this all together, we finally obtain the following limit:

$$\begin{aligned}
\lim_{\epsilon\to 0} S &= \lim_{\epsilon\to 0}\frac{1}{|\det A_{\mathrm{v}}|}\sum_{g\in\mathbb{Z}^d/A_{\mathrm{v}}\mathbb{Z}^d}\prod_{k=1}^{d}\sum_{l_k\in\mathbb{Z}}\frac{\exp(2\pi i((1+\epsilon)\alpha_k+\beta_k)l_k)}{(l_k+z_k)(A_{\mathrm{u}}^{\mathrm{t}}A_{\mathrm{v}}^{-\mathrm{t}}(l\epsilon+z))_k} \\
&= \frac{1}{|\det A_{\mathrm{v}}|}\sum_{g\in\mathbb{Z}^d/A_{\mathrm{v}}\mathbb{Z}^d}\left(\prod_{k\notin R}\frac{2\pi i}{\langle C_k,z\rangle}\cdot\frac{e^{-2\pi i\{\alpha_k+\beta_k\}z_k}}{1-e^{-2\pi i z_k}}\right)\cdot \\
&\quad \left(\prod_{k\in R}\frac{2\pi i}{\langle C_k,z\rangle}\cdot\left(e^{2\pi i|\alpha_k|\langle C_k,\, z\rangle/C_{kk}}+\frac{1}{1-e^{-2\pi i z_k}}\right)\right) \\
&= \frac{(2\pi i)^d}{|\det A_{\mathrm{v}}|}\sum_{g\in\mathbb{Z}^d/A_{\mathrm{v}}\mathbb{Z}^d}\prod_{k=1}^{d}\frac{1}{\langle C_k,z\rangle}\cdot\prod_{k\notin R}\frac{e^{-2\pi i\{\alpha_k+\beta_k\}z_k}}{1-e^{-2\pi i z_k}}\cdot \\
&\quad \prod_{k\in R}\left(e^{2\pi i|\alpha_k|\langle C_k,\, z\rangle/C_{kk}}+\frac{1}{1-e^{-2\pi i z_k}}\right).
\end{aligned}$$

■

1.4 The Ehrhart Quasi-polynomial of a Translated Simple Rational Polytope

In this section, as a corollary to Theorem 1.2.1, we find a formula for the Ehrhart quasi-polynomial of a translated simple rational polytope $\mathcal{P}+T$ given by

$$\sum_{l\in\mathbb{Z}^d}1_{r\mathcal{P}}(l+T).$$

Recall that for fixed $T \in \mathbb{R}^d$ and $y \in \mathbb{C}^d$, the left-hand side of Theorem 1.2.1 gives an explicit formula for the integer point transform

$$L_P(r, T, y) = \sum_{l \in \mathbb{Z}^d} 1_{r\mathcal{P}}(l + T) e^{-2\pi \langle l,\, y \rangle}.$$

We would like to set $y = 0$ to get the integer point enumerator that we are seeking. Unfortunately, $y = 0$ is a singularity of the right hand side of Theorem 1.2.1. Fortunately, we can find our way around this singularity, which is the content of this section. We are free to choose any complex value for y as long as it is not a singularity of the right-hand side of Theorem 1.2.1.

There are an infinite number of such choices since the set of these singularities, Ω, is collected in a countably infinite union of hyperplanes. We recall that the singular set in Theorem 1.2.1 is

$$\Omega = \bigcup_{\substack{\mathbf{u},\mathbf{v} \text{ vertices of } r\mathcal{P} \\ k=1,\ldots,d}} \{y \in \mathbb{C}^d | \langle A_{\mathbf{v}} C_k(\mathbf{u}, \mathbf{v}), y \rangle = 0\} \cup \{y \in \mathbb{C}^d | \langle \mathbf{w}_k(\mathbf{v}), y \rangle \in i\mathbb{Z}\}.$$

In particular, we choose $y = iv/N$ for some $v = (v_1, v_2, \ldots, v_d) \in \mathbb{Z}^d$ such that $\gcd(v_1, v_2, \ldots, v_d) = 1$ and $N \in \mathbb{Z}_{>0}$ such that $y \notin \Omega$. Now consider the following set of discrete hyperplanes:

$$\mathbb{L}_j := \{l \in \mathbb{Z}^d | \, \langle l, v \rangle \equiv j \,(\mathrm{mod}\, N)\}, \quad j = 0, \ldots, N-1.$$

It is clear that the integer lattice is stratified by these N discrete hyperplanes; that is, $\mathbb{Z}^d = \bigcup_{j=0}^{N-1} \mathbb{L}_j$. By making the substitution for $y = iv/N$, we have

$$\sum_{l \in \mathbb{Z}^d} 1_{r\mathcal{P}}(l + T) e^{-2\pi \langle l,\, y \rangle} = \sum_{l \in \mathbb{Z}^d} 1_{r\mathcal{P}}(l + T) e^{\frac{2\pi i}{N} \langle l,\, v \rangle} \qquad (1.17)$$

$$= \sum_{j=0}^{N-1} \sum_{l \in \mathbb{L}_j} 1_{r\mathcal{P}}(l + T) e^{\frac{2\pi i j}{N}}. \qquad (1.18)$$

Making the substitutions $y = ivk/N$ for $k = 1, 2, 3, \ldots, N-1$ and adding up all the corresponding sums (1.17) we get

$$\begin{aligned}\sum_{k=1}^{N-1}\sum_{l\in\mathbb{Z}^d} 1_{r\mathcal{P}}(l+T)e^{\frac{2\pi ik}{N}\langle l,v\rangle} &= \sum_{k=1}^{N-1}\sum_{j=0}^{N-1}\sum_{l\in\mathbb{L}_j} 1_{r\mathcal{P}}(l+T)e^{2\pi i(\frac{jk}{N})}\\ &= \sum_{j=0}^{N-1}\sum_{l\in\mathbb{L}_j} 1_{r\mathcal{P}}(l+T)\sum_{k=1}^{N-1} e^{2\pi i(\frac{jk}{N})}.\end{aligned}$$

By the orthogonality relation of the roots of unity, we have

$$\sum_{k=1}^{N-1} e^{2\pi i(\frac{jk}{N})} = \begin{cases} N-1 & \text{if } j=0\\ -1 & \text{if } j\neq 0\end{cases}.$$

Therefore

$$\begin{aligned}\sum_{k=1}^{N-1}\sum_{l\in\mathbb{Z}^d} 1_{r\mathcal{P}}(l+T)e^{\frac{2\pi ik}{N}\langle l,v\rangle} &= (N-1)\sum_{l\in\mathbb{L}_0} 1_{r\mathcal{P}}(l+T) - \sum_{j=1}^{N-1}\sum_{l\in\mathbb{L}_j} 1_{r\mathcal{P}}(l+T)\\ &= N\sum_{l\in\mathbb{L}_0} 1_{r\mathcal{P}}(l+T) - \sum_{j=0}^{N-1}\sum_{l\in\mathbb{L}_j} 1_{r\mathcal{P}}(l+T)\\ &= N\sum_{l\in\mathbb{L}_0} 1_{r\mathcal{P}}(l+T) - \sum_{l\in\mathbb{Z}^d} 1_{r\mathcal{P}}(l+T).\end{aligned}$$

Since $L_{\mathcal{P}}(r,T,0) = \sum_{l\in\mathbb{Z}^d} 1_{r\mathcal{P}}(l+T)$, we retrieve from the last equation a formula for the Ehrhart quasi-polynomial of a translated simple rational polytope $\mathcal{P}-T$:

Corollary 1.4.1.

$$L_{\mathcal{P}}(r,T,0) = N\sum_{l\in\mathbb{L}_0} 1_{r\mathcal{P}}(l+T) - \sum_{k=1}^{N-1}\sum_{l\in\mathbb{Z}^d} 1_{r\mathcal{P}}(l+T)e^{\frac{2\pi ik}{N}\langle l,v\rangle}.$$

We note that the right-hand side of this formula is computable by using Theorem 1.2.1. Indeed, the double sum is just $N-1$ evaluations of theorem 1.2.1, where y ranges over ivk/N for $k=1,2,3,\ldots,N-1$. To evaluate the sum $\sum_{l\in\mathbb{L}_0} 1_{r\mathcal{P}}(l+T)$, we note that $\mathbb{L}_0$ is a sublattice of $\mathbb{Z}^d$, say $\mathbb{L}_0 = M^t\mathbb{Z}^d$ for some $M\in GL_d(\mathbb{Z})$. Then we can use Fact 6 from the Appendix to get

$$\sum_{l\in\mathbb{L}_0} 1_{r\mathcal{P}}(l+T) = \frac{1}{|\det M|}\sum_{g\in\mathbb{Z}^d/M\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} 1_{r\mathcal{P}}(l+T)e^{2\pi i\langle M^{-1}g,l\rangle},$$

which can now be computed by theorem 1.2.1 with the additional assumption that for all $g\in\mathbb{Z}^d/M\mathbb{Z}^d$, $iM^{-1}g$ is not in the singular set Ω.

CHAPTER 2

Asymptotics of Polyhedral Theta Series and Solid Angles

2.1 Introduction

Theta series are important tools in many areas of mathematics, including elliptic functions, modular forms, algebraic and analytic number theory, and discrete geometry, to name only a few. For instance, the classical

theta function, defined by $$\theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi t n^2}, \quad \text{for } t > 0, \tag{2.1}$$

satisfies the following transformation law $\theta(t) = t^{-\frac{1}{2}}\theta\left(\frac{1}{t}\right)$ of Jacobi [29] and was used by Riemann [40] to prove the functional equation for the Riemann zeta function by means of the integral

$$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s) = \frac{1}{2}\int_0^\infty [\theta(t) - 1]t^{s/2}\frac{dt}{t}.$$

In this chapter, we define polyhedral theta series and investigate their connection with solid angles, the generalization of two-dimensional angles to any dimension. We will show that polyhedral theta series are useful tools for studying solid angles by analyzing the asymptotics of such series. Overall, this chapter serves as an introduction to the tools and methods used to study solid angles and it represents a starting point for future study on asymptotics of polyhedral theta series. As a warmup, in the following section we present an application of a one-dimensional theta series, namely the Twisted Landsberg-Schaar Identity.

2.2 Twisted Landsberg-Schaar Identity

The Poisson summation formula states that if f is sufficiently nice, then

$$\sum_{l \in \mathbb{Z}^d} f(l) = \sum_{m \in \mathbb{Z}^d} \hat{f}(m). \tag{2.2}$$

One application of this formula is in the proof of Jacobi's transformation law for the classical theta function. This transformation law is in turn used to prove the following identity of Landsberg and Schaar [33]: For positive relatively prime integers p and q,

$$\frac{1}{\sqrt{p}} \sum_{n=0}^{p-1} e^{2\pi i n^2 q/p} = \frac{e^{\pi i/4}}{\sqrt{2q}} \sum_{n=0}^{2q-1} e^{-\pi i n^2 p/2q}.$$

Formula (2.2) can be "twisted" to include Dirichlet characters, but we first recall the definition of a **Dirichlet character** modulo an integer N.

Definition 2.2.1. A Dirichlet character modulo N is a function $\chi : \mathbb{Z} \to \mathbb{C}$, which is periodic with period N, such that $\chi(nm) = \chi(n)\chi(m)$ and

$$|\chi(n)| = \begin{cases} 1 & \text{if } (n, N) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

A Dirichlet character χ mod N is called *primitive* if χ is not a character mod M for any divisor M of N.

The following is the twisted Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} \chi(n) f(n) = \frac{\chi(-1) g(\chi)}{N} \sum_{n=-\infty}^{\infty} \overline{\chi(n)} \hat{f}\left(\frac{n}{N}\right), \tag{2.3}$$

where χ is a primitive character mod N and $g(\chi)$ is the Gauss sum defined by the formula

$$g(\chi) = \sum_{n \bmod N} \chi(n) e^{2\pi i n/N}.$$

(For a proof of (2.3), see [15]). As an application of (2.3), we let χ be a nontrivial primitive character mod N such that $\chi(-1) = 1$, (so, χ is even). Then for $f(x) = e^{-\pi t x^2}$, $t > 0$, we have

$$\sum_{n=-\infty}^{\infty} \chi(n) e^{-\pi t n^2} = \frac{g(\chi)}{N\sqrt{t}} \sum_{n=-\infty}^{\infty} \overline{\chi(n)} e^{-\pi n^2/N^2 t}. \tag{2.4}$$

If we let $\tau = it$, for $t > 0$, we get

$$\sum_{n=-\infty}^{\infty} \chi(n) e^{\pi i \tau n^2} = \frac{g(\chi)}{N\sqrt{-i\tau}} \sum_{n=-\infty}^{\infty} \overline{\chi(n)} e^{-\pi i n^2/N^2 \tau}. \tag{2.5}$$

By analytic continuation, (2.5) is true for all $\tau \in H$, the upper half complex plane. We will now use (2.5) to prove the following result:

Theorem 2.2.1 (Twisted Landsberg-Schaar Identity). *Let χ be a nontrivial even primitive Dirichlet character* mod N *and let $q \in \mathbb{Z}_{>0}$ such that $(q, N) = 1$, then*

$$\sum_{j=0}^{N-1} \chi(j) e^{2\pi i q j^2/N} = \frac{g(\chi)\sqrt{2q}}{\sqrt{-iN}} \sum_{j=0}^{4qN-1} \overline{\chi(j)} e^{-2\pi i j^2/4qN}.$$

Proof. We start with the left-hand side of equation (2.5) and $\tau = it + 2q/N$:

$$\sum_{n=-\infty}^{\infty} \chi(n) e^{-\pi t n^2 + 2\pi i q n^2/N} = \sum_{j=0}^{N-1} \chi(j) e^{2\pi i q j^2/N} \sum_{n \equiv j \bmod N} e^{-\pi t n^2}. \tag{2.6}$$

We used the periodicity of $\chi(n)$ and $e^{2\pi i q n^2/N}$ as n varies over $\mathbb{Z}$. We will need the following claim:

Claim:

$$\lim_{t \to 0} \sqrt{t} \sum_{n \equiv j \bmod N} e^{-\pi t n^2} = \frac{1}{N}.$$

Proof of Claim.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx \\ &= \lim_{\Delta x\to 0}\sum_{n=-\infty}^{\infty} \Delta x f(x_n) \\ &= \lim_{t\to 0}\sum_{n=-\infty}^{\infty} \sqrt{t}Ne^{-\pi t(Nn+j)^2} \\ &= \lim_{t\to 0}\sqrt{t}N\sum_{n\equiv j \bmod N} e^{-\pi t n^2}, \end{aligned}$$

where we used $f(x)=e^{-\pi x^2}$, $x_n=\sqrt{t}(Nn+j)$ and hence $\Delta x=\sqrt{t}N$. ∎

This claim, along with (2.6), shows that

$$\sqrt{t}\sum_{n=-\infty}^{\infty}\chi(n)e^{-\pi t n^2+2\pi i q n^2/N} \to \frac{1}{N}\sum_{j=0}^{N-1}\chi(j)e^{2\pi i q j^2/N},\ \text{as } t\to 0. \tag{2.7}$$

Now we play a similar game with the right-hand side of equation (2.5). Again, we let $\tau = it+2q/N$ and get

$$\begin{aligned} &\frac{g(\chi)}{\sqrt{N^2t-2iqN}}\sum_{n=-\infty}^{\infty}\overline{\chi(n)}e^{-\pi i n^2/N^2t-2iqN} \\ &= \frac{g(\chi)}{\sqrt{N^2t-2iqN}}\sum_{n=-\infty}^{\infty}\overline{\chi(n)}\exp\left(\frac{-\pi t n^2}{N^2t^2+4q^2}-\frac{2\pi i q n^2}{N^3t^2+4q^2N^2}\right) &(2.8) \\ &= \frac{g(\chi)}{\sqrt{N^2t-2iqN}}\sum_{n=-\infty}^{\infty}\overline{\chi(n)}e^{-\pi t n^2/4q^2}e^{-2\pi i n^2/4qN+O(t^4)},\ \text{as}\, t\to 0, &(2.9) \\ &= \frac{g(\chi)}{\sqrt{N^2t-2iqN}}\sum_{j=0}^{4qN-1}\overline{\chi(j)}e^{-2\pi i j^2/4qN}\sum_{n\equiv j \bmod 4qN} e^{-\pi t n^2/4q^2+O(t^4)},\ \text{as}\, t\to 0. &(2.10) \end{aligned}$$

In (2.8) we wrote the exponent as $a+bi$ where $a,b\in\mathbb{R}$ and in (2.9) we used the Taylor series representations of both a and b centered at 0. In (2.10) we used the periodicity of $\overline{\chi(n)}$ and $e^{-2\pi i n^2/4qN}$ as n varies over $\mathbb{Z}$. We note that $\overline{\chi}$ has period N, and thus is also periodic with period $4qN$. By the above claim, it is easy to see that

$$\lim_{t\to 0}\sqrt{t}\sum_{n\equiv j \bmod 4qN} e^{-\pi t n^2/4q^2} = \frac{2q}{N}.$$

This limit, along with (2.10), shows that as $t \to 0$

$$\frac{g(\chi)\sqrt{t}}{\sqrt{N^2t - 2iqN}} \sum_{n=-\infty}^{\infty} \overline{\chi(n)} e^{-\pi i n^2/N^2 t - 2iqN} \to \frac{g(\chi)\sqrt{2q}}{N\sqrt{-iN}} \sum_{j=0}^{4qN-1} \overline{\chi(j)} e^{-2\pi i j^2/4qN}. \tag{2.11}$$

By equating the asymptotic equalities (2.7) and (2.11), we obtain the desired result. ■

2.3 Solid Angles

As we stated earlier, the motivation for this chapter is the study of solid angles via appropriately defined theta functions. Suppose $\mathcal{P} \subset \mathbb{R}^d$ is a convex d-polytope. Then the **solid angle $\omega_{\mathcal{P}}(\mathbf{x})$ of a point $\mathbf{x}$** (with respect to $\mathcal{P}$) equals the proportion of a small ball centered at $\mathbf{x}$ that is contained in $\mathcal{P}$. Thus, for all positive ϵ sufficiently small,

$$\omega_{\mathcal{P}}(\mathbf{x}) = \frac{\operatorname{vol}(B_\epsilon(\mathbf{x}) \cap \mathcal{P})}{\operatorname{vol} B_\epsilon(\mathbf{x})},$$

where $B_\epsilon(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{y}\|_2 < \epsilon\}$. This definition holds for any polytope $\mathcal{P}$ or pointed cone $\mathcal{K}$.

Before discussing solid angles further, we first define several terms from the language of polyhedra. A **pointed cone** $\mathcal{K} \subseteq \mathbb{R}^d$ is a set of the form

$$\mathcal{K} = \{\mathbf{v} + \lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \cdots + \lambda_m\mathbf{w}_m | \, \lambda_1, \lambda_2, \ldots, \lambda_m \geq 0\},$$

where $\mathbf{v}, \mathbf{w}_1, \ldots, \mathbf{w}_m \in \mathbb{R}^d$ are such that there exists a hyperplane $H = \{\mathbf{x} \in \mathbb{R}^d | \, \mathbf{a} \cdot \mathbf{x} = b\}$ for which $H \cap \mathcal{K} = \mathbf{v}$. The vector $\mathbf{v}$ is called the **apex** of $\mathcal{K}$, and the $\mathbf{w}_k$'s are the **generators** (or edges) of $\mathcal{K}$. The pointed cone is **rational** if $\mathbf{v}, \mathbf{w}_1, \ldots, \mathbf{w}_m \in \mathbb{Q}^d$. We say that H is a **supporting hyperplane** of $\mathcal{K}$ if $\mathcal{K}$ lies entirely on one side of H. A **face** of a cone $\mathcal{K}$ is a set of the form $\mathcal{K} \cap H$, where H is a supporting hyperplane of $\mathcal{K}$. If $\mathcal{K}$ is of dimension d, then we call it a d-cone. The d-cone $\mathcal{K}$ is called **simple** if $\mathcal{K}$ has exactly d linearly independent generators.

With the language now set, we have the following alternate definition of the solid angle of a cone:

Definition 2.3.1. Given a cone $\mathcal{K} \subseteq \mathbb{R}^d$ with apex at the origin,

$$\omega_{\mathcal{K}}(0) := \int_{\mathcal{K}} e^{-\pi\|x\|^2} dx.$$

We note that one can replace the integrand $e^{-\pi\|x\|^2}$ with any radially symmetric function $f(\|x\|)$ in definition 2.3.1, as long as one divides by the total mass, $\int_{\mathbb{R}^d} f(\|x\|)\, dx$. The benefit of using the Gaussian function in our definition is that the total mass equals 1. We now define the tool that will help us analyze the solid angle of a cone.

Definition 2.3.2. Given a cone $\mathcal{K} \subseteq \mathbb{R}^d$, we define the following **conic theta function** for $t > 0$:

$$\theta_{\mathcal{K}}(t) := \sum_{m \in \mathcal{K} \cap \mathbb{Z}^d} e^{-\pi t\|m\|^2}. \tag{2.12}$$

The connection between $\theta_{\mathcal{K}}(t)$ and $\omega_{\mathcal{K}}(0)$ becomes apparent when we discretize the integral definition 2.3.1 as a Riemann sum and obtain the asymptotic result:

Theorem 2.3.1. *Given a cone $\mathcal{K} \subseteq \mathbb{R}^d$ with its apex at the origin,*

$$\theta_{\mathcal{K}}(t) \sim \frac{\omega_{\mathcal{K}}(0)}{t^{d/2}}, \quad \text{as } t \to 0^+.$$

Proof. Let $f(x) = e^{-\pi\|x\|^2}$. Then we have that

$$\begin{aligned}\omega_{\mathcal{K}}(0) &:= \int_{\mathcal{K} \subseteq \mathbb{R}^d} e^{-\pi\|x\|^2} dx \\ &= \lim_{\Delta x \to 0^+} \sum_{n \in \mathcal{K} \cap \mathbb{Z}^d} (\Delta x)^d f(\Delta x \cdot n).\end{aligned}$$

If we let $\triangle x = t^{1/2}$, it follows that

$$\begin{aligned}\omega_{\mathcal{K}}(0) &= \lim_{t\to 0^+} \sum_{n\in\mathcal{K}\cap\mathbb{Z}^d} t^{d/2} f(t^{1/2}n) \\ &= \lim_{t\to 0^+} \sum_{n\in\mathcal{K}\cap\mathbb{Z}^d} t^{d/2} e^{-\pi\|t^{1/2}n\|^2} \\ &= \lim_{t\to 0^+} \sum_{n\in\mathcal{K}\cap\mathbb{Z}^d} t^{d/2} e^{-\pi t\|n\|^2} \\ &= \lim_{t\to 0^+} t^{d/2}\theta_{\mathcal{K}}(t).\end{aligned}$$

Therefore, we have the desired result:

$$\lim_{t\to 0^+} \frac{t^{d/2}\theta_{\mathcal{K}}(t)}{\omega_{\mathcal{K}}(0)} = 1.$$

■

2.4 Asymptotics of the Conic Theta Functions

We have just shown that the conic theta function $\theta_{\mathcal{K}}(t) \sim \omega_{\mathcal{K}}(0)/t^{d/2}$, as $t \to 0^+$. In this section, we study finer asymptotics of $\theta_{\mathcal{K}}(t)$ using Euler-Maclaurin summation and the transformation law of the classical theta function. In particular, we will use the geometry of the positive orthant, $\mathcal{O}$, to study the behavior of $\theta_{\mathcal{O}}(t)$ as $t \to 0^+$.

The classical Euler-Maclaurin summation formula for a function f having $2m$ continuous derivatives on the interval $[1, \infty)$ can be written in the form

$$\begin{aligned}f(1) + \cdots + f(n) &= \int_1^n f(x)dx + C + \frac{1}{2}f(n) + \frac{B_2}{2!}f'(n) + \frac{B_4}{4!}f'''(n) + \\ &\quad + \cdots + \frac{B_{2m}}{(2m)!}f^{(2m-1)}(n) - \int_1^n f^{(2m)}(x)\frac{B_{2m}(x-\lfloor x\rfloor)}{(2m)!}\,dx.\end{aligned} \tag{2.13}$$

The B_k's are the Bernoulli numbers and they have the generating function

$$\frac{z}{e^z - 1} = \sum_{k\geq 0} \frac{B_k}{k!} z^k \qquad (|z| < 2\pi).$$

The $B_k(x)$'s are the Bernoulli polynomials and they are defined through the generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k\geq 0} \frac{B_k(x)}{k!} z^k \qquad (|z| < 2\pi).$$

The symbol $\lfloor x \rfloor$ means the greatest integer $\leq x$. The number C is a constant independent of n given by

$$C = \frac{1}{2} f(1) - \frac{B_2}{2!} f'(1) - \cdots - \frac{B_{2m}}{(2m)!} f^{(2m-1)}(1).$$

It is known that [26]

$$B_{2m}(x - \lfloor x \rfloor) = 2(2m)!(2\pi)^{-2m}(-1)^{m+1} \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k^{2m}} \qquad \text{for } m = 1, 2, 3, \ldots.$$

Therefore, we have the following:

$$|B_{2m}(x - \lfloor x \rfloor)| \leq 2(2m)!(2\pi)^{-2m} \sum_{k=1}^{\infty} \frac{1}{k^{2m}} = |B_{2m}(0)| = |B_{2m}|. \qquad (2.14)$$

This bound is useful in estimating the last integral in (2.13).

For fixed $\alpha > 0$ and $n \in \mathbb{Z}_{>0}$, we consider the sum

$$S_N(\alpha, n) = \sum_{k=-N}^{N} e^{-\alpha k^2/n}.$$

Using the Euler-Maclaurin summation formula (2.13) with $f(x) = e^{-\alpha x^2/n}$, we have

$$\begin{aligned} S_N(\alpha, n) &= \int_N^N f(x)dx + \frac{1}{2}(f(N) + f(-N)) + \frac{B_2}{2!}(f'(N) - f'(-N)) + \\ &+ \cdots + \frac{B_{2m}}{(2m)!}\left(f^{(2m-1)}(N) - f^{(2m-1)}(-N)\right) - R_m, \end{aligned}$$

where

$$R_m = \int_{-N}^{N} f^{(2m)}(x) \frac{B_{2m}(x - \lfloor x \rfloor)}{(2m)!} dx.$$

We now take the limit as $N \to \infty$ to get

$$\begin{aligned} S(\alpha, n) &:= \sum_{k=-\infty}^{\infty} e^{-\alpha k^2/n} \\ &= \int_{-\infty}^{\infty} f(x)dx - \int_{-\infty}^{\infty} f^{(2m)}(x)\frac{B_{2m}(x - \lfloor x \rfloor)}{(2m)!}\,dx \\ &= \sqrt{\frac{\pi n}{\alpha}} - \int_{-\infty}^{\infty} f^{(2m)}(x)\frac{B_{2m}(x - \lfloor x \rfloor)}{(2m)!}\,dx, \end{aligned} \tag{2.15}$$

since $f^{(k)}(\pm N) \to 0$ for $k = 0, 1, 2, \ldots$. We denote the integral in (2.15) by R and notice that R does not depend on m, since $-R = S(\alpha, n) - \sqrt{\pi n/\alpha}$. We can use (2.14) to estimate R:

$$\begin{aligned} |R| &= \left| \int_{-\infty}^{\infty} f^{(2m)}(x)\frac{B_{2m}(x - \lfloor x \rfloor)}{(2m)!}dx \right| \\ &\le \int_{-\infty}^{\infty} \left| f^{(2m)}(x)\frac{B_{2m}(x - \lfloor x \rfloor)}{(2m)!} \right| dx \\ &\le \left| \frac{B_{2m}}{(2m)!} \right| \int_{-\infty}^{\infty} \left| f^{(2m)}(x) \right| dx. \end{aligned}$$

In [20], de Bruijn uses Hermite polynomials and Stirling's formula to show that $-R = O\left(ne^{-\pi^2 n/\alpha}\right)$, as $n \to \infty$. Alternatively, we realize that $S(\alpha, n) = \theta(\alpha/\pi n)$, where $\theta(t)$ is the classical theta function

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi t n^2}, \quad \text{for } t > 0.$$

We know that $\theta(t)$ satisfies the following transformation law

$$\theta(t) = \frac{1}{\sqrt{t}}\,\theta\left(\frac{1}{t}\right). \tag{2.16}$$

Therefore

$$\begin{aligned} S(\alpha, n) &= \sum_{k=-\infty}^{\infty} e^{-\alpha k^2/n} \\ &= \sqrt{\frac{\pi n}{\alpha}} \sum_{k=-\infty}^{\infty} e^{-k^2\pi^2 n/\alpha} \\ &= \sqrt{\frac{\pi n}{\alpha}} + 2\sqrt{\frac{\pi n}{\alpha}} \sum_{k=1}^{\infty} e^{-k^2\pi^2 n/\alpha}, \end{aligned}$$

and hence we obtain the improved result:

$$-R = 2\sqrt{\frac{\pi n}{\alpha}}\sum_{k=1}^{\infty} e^{-k^2\pi^2 n/\alpha} = 2\sqrt{\frac{\pi n}{\alpha}}e^{-\pi^2 n/\alpha} + O\left(\sqrt{n}e^{-4\pi^2 n/\alpha}\right), \quad \text{as } n \to \infty.$$

Thus, we have shown that

$$\sum_{k=-\infty}^{\infty} e^{-\alpha k^2/n} = \sqrt{\frac{\pi n}{\alpha}} + 2\sqrt{\frac{\pi n}{\alpha}}e^{-\pi^2 n/\alpha} + O\left(\sqrt{n}e^{-4\pi^2 n/\alpha}\right), \quad \text{as } n \to \infty. \tag{2.17}$$

Let us express this result in terms of $\theta(t)$. By making the substitution $\alpha/n = \pi t$, equation (2.17) becomes

$$\theta(t) := \sum_{k=-\infty}^{\infty} e^{-\pi t k^2} = \frac{1}{\sqrt{t}} - \frac{2}{\sqrt{t}}e^{-\pi/t} + O\left(\frac{1}{\sqrt{t}}e^{-4\pi/t}\right), \quad \text{as } t \to 0^+. \tag{2.18}$$

Equation (2.18) is the key result that we will use to find asymptotics of the conic theta function $\theta_{\mathcal{O}}(t)$ defined over the positive orthant. But first, we will need to define a weighted conic theta function over a cone $\mathcal{K}$. The **weighted characteristic function** $\mathbf{C}_{\mathcal{K}}(x)$ is the function on $\mathbb{R}^d$ that takes the value 0 if $x \notin \mathcal{K}$, the value 1 if $x \in \mathcal{K}^\circ$, the interior of $\mathcal{K}$, and the value $1/2^k$ on the relative interior of a face of $\mathcal{K}$ of codimension k. So for example, for the cone $[a, \infty) \subset \mathbb{R}$, the function $\mathbf{C}_{[a,\infty)}(x)$ assigns the value 1 to $x > a$, 0 to points $x < a$, and $1/2$ to a. Then the **weighted conic theta function over** $\mathcal{K}$, $\Theta_{\mathcal{K}}(t)$, is defined as

$$\Theta_{\mathcal{K}}(t) := \sum_{m \in \mathcal{K}\cap\mathbb{Z}^d} \mathbf{C}_{\mathcal{K}}(m)e^{-\pi t\|m\|^2} \quad \text{for } t > 0.$$

The positive orthant in $\mathbb{R}^d$ is the set

$$\mathcal{O} = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \,|\, x_i \geq 0, \ \forall i\}.$$

Clearly, $\mathcal{O}$ is a pointed cone and we want to study the asymptotics of $\Theta_{\mathcal{O}}(t)$ as $t \to 0$. As we shall see in the next result, the geometry of the positive orthant is particularly nice for studying conic theta functions because we can reduce the analysis to the one-dimensional classical theta function.

Theorem 2.4.1. *The weighted conic theta function defined over the positive orthant in* $\mathbb{R}^d$, $\Theta_{\mathcal{O}}(t)$, *is a product of classical theta functions,* $\theta(t)$. *In fact,*

$$\Theta_{\mathcal{O}}(t) = \frac{\theta^d(t)}{2^d}, \quad \textit{for } t > 0. \tag{2.19}$$

Proof.

$$\begin{aligned}
\frac{\theta^d(t)}{2^d} &= \frac{1}{2^d}\left(\sum_{m=-\infty}^{\infty} e^{-\pi t m^2}\right)^d \\
&= \frac{1}{2^d}\left(1 + 2\sum_{m=1}^{\infty} e^{-\pi t m^2}\right)^d \\
&= \left(\frac{1}{2} + \sum_{m=1}^{\infty} e^{-\pi t m^2}\right)^d \\
&= \left(\sum_{m=1}^{\infty} e^{-\pi t m^2}\right)^d + \binom{d}{1}\frac{1}{2}\left(\sum_{m=1}^{\infty} e^{-\pi t m^2}\right)^{d-1} + \cdots + \binom{d}{d-1}\frac{1}{2^{d-1}}\sum_{m=1}^{\infty} e^{-\pi t m^2} + \frac{1}{2^d} \\
&= \underbrace{\sum_{m_1,\ldots,m_d=1}^{\infty} e^{-\pi t(m_1^2+\cdots+m_d^2)}}_{\text{interior of } \mathcal{O}} + \underbrace{\sum_{\substack{m_i=1\\ i\neq d}}^{\infty} \frac{1}{2} e^{-\pi t(m_1^2+\cdots+m_{d-1}^2)} + \cdots + \sum_{\substack{m_i=1\\ i\neq 1}}^{\infty} \frac{1}{2} e^{-\pi t(m_2^2+\cdots+m_d^2)}}_{(d-1)-\text{ dimensional facets of } \mathcal{O}} \\
&\quad + \cdots + \underbrace{\sum_{m_1=1}^{\infty} \frac{1}{2^{d-1}} e^{-\pi t m_1^2} + \cdots + \sum_{m_d=1}^{\infty} \frac{1}{2^{d-1}} e^{-\pi t m_d^2}}_{1-\text{ dimensional edges of } \mathcal{O}} + \underbrace{\frac{1}{2^d}}_{\text{apex of } \mathcal{O}} \\
&= \sum_{m\in\mathcal{O}\cap\mathbb{Z}^d} C_{\mathcal{O}}(m) e^{-\pi t\|m\|^2} \\
&= \Theta_{\mathcal{O}}(t).
\end{aligned}$$

∎

From equation (2.19), we get essentially for free a transformation law for $\Theta_{\mathcal{O}}$.

Corollary 2.4.2. *The weighted conic theta function defined over the positive orthant in* $\mathbb{R}^d$, $\Theta_{\mathcal{O}}(t)$, *satisfies the following transformation law:*

$$\Theta_{\mathcal{O}}(t) = \frac{1}{t^{d/2}}\Theta_{\mathcal{O}}\left(\frac{1}{t}\right), \quad \textit{for } t > 0.$$

Proof. This follows from Theorem (2.4.1) and the transformation law for $\theta(t)$ in equation (2.1).

$$\Theta_{\mathcal{O}}(t) = \frac{\theta^d(t)}{2^d} = \frac{\left(\frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right)\right)^d}{2^d} = \frac{1}{t^{d/2}} \cdot \frac{\theta^d\left(\frac{1}{t}\right)}{2^d} = \frac{1}{t^{d/2}} \cdot \Theta_{\mathcal{O}}\left(\frac{1}{t}\right).$$

■

We can now use the asymptotics of the classical theta function in equation (2.18) to study the asymptotics of $\Theta_{\mathcal{O}}$. But first we will need the following lemma.

Lemma 2.4.1.

$$\frac{e^{-4\pi/t}}{\sqrt{t}} = O\left(\frac{1-2e^{-\pi/t}}{\sqrt{t}}\right), \quad \text{as } t \to 0^+.$$

Proof. It suffices to prove that

$$\lim_{t\to 0^+} \frac{\frac{e^{-4\pi/t}}{\sqrt{t}}}{\frac{1-2e^{-\pi/t}}{\sqrt{t}}} = \lim_{t\to 0^+} \frac{e^{-4\pi/t}}{1-2e^{-\pi/t}} = 0.$$

By L'Hôpital's rule, we have

$$\lim_{t\to 0^+} \frac{e^{-4\pi/t}}{1-2e^{-\pi/t}} = \lim_{t\to 0^+} \frac{-2e^{-4\pi/t}}{e^{-\pi/t}} = \lim_{t\to 0^+} -2e^{-3\pi/t} = 0.$$

■

Now we can state the asymptotics as $t \to 0^+$ of the weighted conic theta function defined over $\mathcal{O}$, the positive orthant in $\mathbb{R}^d$.

Corollary 2.4.3. *As* $t \to 0^+$,

$$\Theta_{\mathcal{O}}(t) := \sum_{m\in\mathcal{O}\cap\mathbb{Z}^d} C_{\mathcal{O}}(m)e^{-\pi t\|m\|^2} = \left(\frac{1-2e^{-\pi/t}}{2\sqrt{t}}\right)^d + O\left(\frac{e^{-4\pi/t}}{\sqrt{t}} \cdot \left(\frac{1-2e^{-\pi/t}}{\sqrt{t}}\right)^{d-1}\right).$$

Proof. From theorem 2.4.1 and equation (2.18), as $t \to 0^+$, we have that

$$\Theta_{\mathcal{O}}(t) = \frac{\theta^d(t)}{2^d} = \left(\frac{1-2e^{-\pi/t}}{2\sqrt{t}} + O\left(\frac{e^{-4\pi/t}}{\sqrt{t}}\right)\right)^d.$$

If we expand this expression, we get the term

$$\left(\frac{1-2e^{-\pi/t}}{2\sqrt{t}}\right)^d,$$

plus error terms involving products of

$$E_1(t) := \frac{1-2e^{-\pi/t}}{2\sqrt{t}} \text{ and } E_2(t) := \frac{e^{-4\pi/t}}{\sqrt{t}}.$$

By lemma (2.4.1), $E_1(t)$ dominates $E_2(t)$, and hence the largest error term is

$$O\left(\frac{e^{-4\pi/t}}{\sqrt{t}} \cdot \left(\frac{1-2e^{-\pi/t}}{\sqrt{t}}\right)^{d-1}\right) \quad \text{as } t \to 0^+.$$

■

The following lemma relates the asymptotics of the weighted conic theta function, $\Theta_{\mathcal{O}}(t)$, to the non-weighted version, $\theta_{\mathcal{O}}(t)$.

Lemma 2.4.2.
Given a cone $\mathcal{K} \subseteq \mathbb{R}^d$, $\quad \Theta_{\mathcal{K}}(t) \sim \theta_{\mathcal{K}}(t), \quad$ *as* $t \to 0^+$.

Proof. We will use the same proof as in theorem 2.3.1 to show that $\Theta_{\mathcal{K}}(t) \sim \omega_{\mathcal{K}}(0)t^{-d/2}$, as $t \to 0^+$. Then we will have

$$\lim_{t\to 0}\frac{\Theta_{\mathcal{K}}(t)}{\theta_{\mathcal{K}}(t)} = \lim_{t\to 0}\frac{\Theta_{\mathcal{K}}(t)\omega_{\mathcal{K}}(0)t^{-d/2}}{\theta_{\mathcal{K}}(t)\omega_{\mathcal{K}}(0)t^{-d/2}} = 1.$$

We begin with

$$\begin{aligned}\omega_{\mathcal{K}}(0) &:= \int_{\mathcal{K}\subseteq\mathbb{R}^d} e^{-\pi\|x\|^2}dx \\ &= \int_{\mathcal{K}\subseteq\mathbb{R}^d} \mathbf{C}_{\mathcal{K}}(x)e^{-\pi\|x\|^2}dx,\end{aligned}$$

since $\mathbf{C}_{\mathcal{K}}(x)$ only affects the boundary of $\mathcal{K}$ and the integral over the lower-dimensional $\partial\mathcal{K}$ equals 0. Then we let $f(x) = \mathbf{C}_{\mathcal{K}}(x)e^{-\pi\|x\|^2}$ and write the following Riemann sum

$$\omega_{\mathcal{K}}(0) = \lim_{\triangle x\to 0+} \sum_{n\in\mathcal{K}\cap\mathbb{Z}^d} (\triangle x)^d f(\triangle x\cdot n). \tag{2.20}$$

If we let $\triangle x = t^{1/2}$, it follows that

$$\begin{aligned}\omega_{\mathcal{K}}(0) &= \lim_{t\to 0+} \sum_{n\in\mathcal{K}\cap\mathbb{Z}^d} t^{d/2} f(t^{1/2}n) \\ &= \lim_{t\to 0+} \sum_{n\in\mathcal{K}\cap\mathbb{Z}^d} t^{d/2}\mathbf{C}_{\mathcal{K}}(t^{1/2}n)e^{-\pi\|t^{1/2}n\|^2} \\ &= \lim_{t\to 0+} t^{d/2} \sum_{n\in\mathcal{K}\cap\mathbb{Z}^d} \mathbf{C}_{\mathcal{K}}(n)e^{-\pi t\|n\|^2} \\ &= \lim_{t\to 0+} t^{d/2}\Theta_{\mathcal{K}}(t).\end{aligned} \tag{2.21}$$

In (2.21), we used the fact that for $t > 0$, if n is on a face of $\mathcal{K}$ with codimension k then so is $t^{1/2}n$. Therefore $\lim_{t\to 0^+} \frac{\Theta_{\mathcal{K}}(t)}{\omega_{\mathcal{K}}(0)t^{-d/2}} = 1$ and the lemma follows. ■

Corollary 2.4.4. *As* $t \to 0^+$,

$$\theta_{\mathcal{O}}(t) := \sum_{m\in\mathcal{O}\cap\mathbb{Z}^d} e^{-\pi t\|m\|^2} = \left(\frac{1-2e^{-\pi/t}}{2\sqrt{t}}\right)^d + O\left(\frac{e^{-4\pi/t}}{\sqrt{t}}\cdot\left(\frac{1-2e^{-\pi/t}}{\sqrt{t}}\right)^{d-1}\right). \tag{2.22}$$

Proof. This follows directly from lemma 2.4.2 and corollary 2.4.3. ■

In dimension d, there are 2^d orthants and hence the solid angle of the positive orthant is $\omega_{\mathcal{O}} = 1/2^d$. We point out that the leading term in (2.22) is

$$\frac{1}{2^d t^{d/2}} = \frac{\omega_{\mathcal{O}}}{t^{d/2}},$$

which is consistent with theorem 2.3.1.

The goal for future study will be to use corollary 2.4.4 to study solid angles of any rational simple cone $\mathcal{K}$, where $\mathcal{K} = M\mathcal{O}$, for some $d\times d$ matrix M. When we apply the transformation M to the standard orthant, it results in conic theta functions with matrices in the exponent of the Gaussian. This greatly increases the difficulty of studying $\omega_{\mathcal{K}}$ through the asymptotics of such theta functions. In the following sections, we generalize the notion of a solid angle in order to find a more computable measure of volume.

2.5 l^p-Solid Angles

The solid angles that we have studied thus far have measured the volume of the intersection of a polytope with a small ball. Without saying so, we assumed that the ball was defined with respect to the l^2-norm. We now extend our notion of a solid angle by considering balls with respect to l^p-norm for $p \geq 1$.

Given $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, the l^p-norm of $\mathbf{x}$ is defined by

$$\|\mathbf{x}\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_d|^p)^{1/p}, \quad \text{for } p \geq 1.$$

The ball with respect to l^p-norm of radius ϵ centered at $\mathbf{x}$ is the set

$$B_{p,\epsilon}(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^d : \quad \|\mathbf{x} - \mathbf{y}\|_p < \epsilon\}.$$

For any convex d-cone $\mathcal{K} \subset \mathbb{R}^d$, we define the generalized **l^p-solid angle** of a point $\mathbf{x}$, denoted by $\omega_{p,\mathcal{K}}(\mathbf{x})$, to be the proportion of a small l^p-ball centered at $\mathbf{x}$ that is contained in $\mathcal{K}$. That is

$$\omega_{p,\mathcal{K}}(\mathbf{x}) = \frac{\operatorname{vol}(B_{p,\epsilon}(\mathbf{x}) \cap \mathcal{K})}{\operatorname{vol} B_{p,\epsilon}(\mathbf{x})},$$

for all positive ϵ sufficiently small. We note that the l^2-norm is the usual norm associated with $\mathbb{R}^d$ and hence the usual solid angle is the l^2-solid angle. We also have the following integral definition:

Definition 2.5.1. Given a cone $\mathcal{K} \subseteq \mathbb{R}^d$ with apex at the origin,

$$\begin{aligned}\omega_{p,\mathcal{K}}(0) &:= \int_{\mathcal{K}} e^{-c\|x\|_p^p} dx \\ &= \int_{\mathcal{K}} e^{-c(|x_1|^p + |x_2|^p + \cdots + |x_d|^p)} dx,\end{aligned}$$

where c is a constant dependent on p such that if $\mathcal{K} = \mathbb{R}^d$, then $\omega_{p,\mathcal{K}}(0) = 1$.

We will now solve for the constant c in the following lemma:

Lemma 2.5.1. *The following identity holds for all $p \geq 1$:*

$$\int_{\mathbb{R}^d} e^{-c\|x\|_p^p} dx = 1, \quad \textit{for} \ \ c = \left(2\Gamma\left(\frac{1}{p}+1\right)\right)^p,$$

where $\Gamma(s)$ is the Gamma function defined for $s \in \mathbb{C}$ with $Re(s) > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt.$$

Proof. First, we note that

$$\begin{aligned}1 &= \int_{\mathbb{R}^d} e^{-c\|x\|_p^p} dx \\ &= \int_{\mathbb{R}^d} e^{-c(|x_1|^p + \cdots + |x_d|^p)} dx \\ &= \left(\int_{\mathbb{R}} e^{-c|x|^p} dx\right)^d.\end{aligned}$$

Thus we have reduced the identity to the one dimensional integral:

$$1 = \int_{\mathbb{R}} e^{-c|x|^p} dx.$$

Making the substitution $c^{1/p}x = y$, we have

$$\begin{aligned}1 &= \frac{1}{c^{1/p}} \int_{\mathbb{R}} e^{-|y|^p} dy \\ &= \frac{2}{c^{1/p}} \int_0^\infty e^{-y^p} dy \\ &= \frac{2}{c^{1/p}} \cdot \frac{1}{p} \int_0^\infty e^{-t} t^{\frac{1}{p}-1} dt &(2.23)\\ &= \frac{2}{c^{1/p}} \cdot \frac{1}{p} \Gamma\left(\frac{1}{p}\right) \\ &= \frac{2}{c^{1/p}} \cdot \Gamma\left(\frac{1}{p}+1\right), &(2.24)\end{aligned}$$

where we made the substitution $t = y^p$ in equation (2.23) and we used the well known relation $\Gamma(s+1) = s\Gamma(s)$ in equation (2.24). Thus the identity holds for

$$c^{1/p} = 2\Gamma\left(\frac{1}{p} + 1\right).$$

■

Now that we have nailed down the constant c for each $p \geq 1$, we extend to all $\mathbf{x} \in \mathbb{R}^d$ the integral definition 2.5.1 of a solid angle:

Definition 2.5.2. Let $\epsilon > 0$. Then for $\mathbf{x} \in \mathbb{R}^d$ and $p \geq 1$, the ***l^p*-solid angle** of $\mathbf{x}$ with respect to the cone $\mathcal{K}$ is given by

$$\omega_{p,\mathcal{K}}(\mathbf{x}) := \lim_{\epsilon \to 0} \frac{1}{\epsilon^{d/p}} \int_{\mathcal{K}} e^{\frac{-c}{\epsilon}\|t-\mathbf{x}\|_p^p} dt. \tag{2.25}$$

This definition of $\omega_{p,\mathcal{K}}(\mathbf{x})$ is more analytic in nature, as opposed to geometric, and it opens the door to Harmonic Analysis techniques that will be used to study solid angles in the next chapter. As a preview, we analyze (2.25) further.

For $\epsilon > 0$, $p \geq 1$, and $t \in \mathbb{R}^d$ we define

$$\phi_\epsilon(t) := \frac{1}{\epsilon^{d/p}} e^{\frac{-c}{\epsilon}\|t\|_p^p}. \tag{2.26}$$

Notice that $\phi_\epsilon(-t) = \phi_\epsilon(t)$, by the properties of the l^p-norm. Then equation (2.25) becomes

$$\begin{aligned}
\omega_{p,\mathcal{K}}(\mathbf{x}) &= \lim_{\epsilon \to 0} \int_{\mathcal{K}} \phi_\epsilon(t - \mathbf{x}) dt \\
&= \lim_{\epsilon \to 0} \int_{\mathcal{K}} \phi_\epsilon(\mathbf{x} - t) dt \\
&= \lim_{\epsilon \to 0} \int_{\mathbb{R}^d} 1_{\mathcal{K}}(t)\, \phi_\epsilon(\mathbf{x} - t) dt \\
&= \lim_{\epsilon \to 0} \left(1_{\mathcal{K}} * \phi_\epsilon\right)(\mathbf{x}).
\end{aligned}$$

The last equality follows from the definition of the convolution. This fact will be used a great deal in the next chapter, so we highlight it here:

Fact 1.

$$\omega_{p,\mathcal{K}}(\mathbf{x}) = \lim_{\epsilon \to 0} \left(1_{\mathcal{K}} * \phi_\epsilon\right)(\mathbf{x}), \quad \text{for all} \ \ \mathbf{x} \in \mathbb{R}^d.$$

We now take a moment to comment on the properties of ϕ_ϵ, which will also be used later. The Schwartz space ,$\mathscr{S}$, is the vector space of infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{C}$ which are bounded, smooth (i.e., all partial derivatives exist and are continuous), and rapidly decreasing (i.e., $|x|^N f(x)$ approaches zero as $|x| \to \infty$ for any N). The Fourier transform $\hat{f} : \mathbb{R}^d \to \mathbb{C}$ is defined as

$$\hat{f}(y) = \int_{\mathbb{R}^d} e^{2\pi i \langle x, y \rangle} f(x) dx.$$

It is known [32] that if $f \in \mathscr{S}$, then $\hat{f} \in \mathscr{S}$. For example, when $p = 2$, $\phi_\epsilon \in \mathscr{S}$ and hence $\hat{\phi}_\epsilon \in \mathscr{S}$ where

$$\phi_\epsilon(t) = \epsilon^{\frac{-d}{2}} \, e^{-\frac{\pi}{\epsilon}\|t\|_2^2} \quad \text{and} \quad \hat{\phi}_\epsilon(t) = \epsilon^{\frac{1-d}{2}} \, e^{-\pi\epsilon\|t\|_2^2}.$$

The fact that $\hat{\phi}_\epsilon$ is rapidly decreasing will be used to show absolute convergence of series in the next chapter.

2.6 l^p-Conic Theta Functions

In order to study l^p-solid angles, we define the following ***l^p*-conic theta functions** for any $p \geq 1$.

Definition 2.6.1. Given a cone $\mathcal{K} \subseteq \mathbb{R}^d$, the l^p-conic theta function for $t > 0$ and $p \geq 1$ is given by:

$$\theta_{p,\mathcal{K}}(t) \ := \sum_{m \in \mathcal{K} \cap \mathbb{Z}^d} e^{-ct\|m\|_p^p}, \quad \text{where} \ \ c = \left(2\Gamma\left(\frac{1}{p} + 1\right)\right)^p.$$

Let us pause to examine a couple of these new objects before we state and prove a key result on the asymptotics of l^p-solid angles.

Example 2.6.1. When $p = 1$, $c = 2\Gamma(2) = 2$ and we get the following l^1-conic theta function over $\mathcal{K}$:

$$\theta_{1,\mathcal{K}}(t) = \sum_{m \in \mathcal{K} \cap \mathbb{Z}^d} e^{-2t|m|}. \tag{2.27}$$

For $p = 2$, we have $c = (2\Gamma(3/2))^2 = 4(\sqrt{\pi}/2)^2 = \pi$ and hence the l^2-conic theta function over $\mathcal{K}$,

$$\theta_{2,\mathcal{K}}(t) = \sum_{m \in \mathcal{K} \cap \mathbb{Z}^d} e^{-\pi t \|m\|^2}, \tag{2.28}$$

is just the d-dimensional classical theta function defined over $\mathcal{K}$.

■

Solid angles are constant along the relative interior of the apex of any cone $\mathcal{K}$. Therefore, we define the l^p**-solid angle of a cone** $\mathcal{K}$ as $\omega_{p,\mathcal{K}} := \omega_{p,\mathcal{K}}(\mathbf{x})$, for any point $\mathbf{x}$ in the relative interior of the apex of $\mathcal{K}$.

Theorem 2.6.1. *Given a cone $\mathcal{K} \subseteq \mathbb{R}^d$ with its apex at the origin,*

$$\theta_{p,\mathcal{K}}(t) \sim \frac{\omega_{p,\mathcal{K}}}{t^{d/p}}, \quad as\ t \to 0^+.$$

Proof. Let $f(x) = e^{-c\|x\|_p^p}$, where $c = \left(2\Gamma\left(\frac{1}{p}+1\right)\right)^p$. Then we have that

$$\begin{aligned}
\omega_{p,\mathcal{K}} &:= \omega_{p,\mathcal{K}}(0) \\
&= \int_K e^{-c\|x\|_p^p} dx \\
&= \lim_{\triangle x \to 0^+} \sum_{n \in K \cap \mathbb{Z}^d} (\triangle x)^d f(\triangle x \cdot n).
\end{aligned}$$

If we let $\triangle x = t^{1/p}$, it follows that

$$\begin{aligned}\omega_{p,\mathcal{K}} &= \lim_{t\to 0^+} \sum_{n\in\mathcal{K}\cap\mathbb{Z}^d} t^{d/p} f(t^{1/p}n) \\ &= \lim_{t\to 0^+} \sum_{n\in\mathcal{K}\cap\mathbb{Z}^d} t^{d/p} e^{-c\|t^{1/p}n\|_p^p} \\ &= \lim_{t\to 0^+} \sum_{n\in\mathcal{K}\cap\mathbb{Z}^d} t^{d/p} e^{-ct\|n\|_p^p} \\ &= \lim_{t\to 0^+} t^{d/p}\theta_{p,\mathcal{K}}(t).\end{aligned}$$

Therefore, we have the desired result:

$$\lim_{t\to 0^+} \frac{t^{d/2}\theta_{p,\mathcal{K}}(t)}{\omega_{p,K}} = 1.$$

■

2.7 l^1-Solid Angles

Our motivation for studying generalized solid angles is the fact that l^1-solid angles are much easier to compute than l^2-solid angles, as the next two theorems illustrate.

Theorem 2.7.1. *The l^1-solid angle of a d-dimensional simple pointed cone $\mathcal{K} \subset \mathbb{R}^d$ contained in any one orthant with its apex at the origin is given by*

$$\omega_{1,\mathcal{K}}(0) = \frac{|\det \mathcal{K}|}{2^d \prod_{i=1}^d \|\mathbf{w}_i\|_1}, \tag{2.29}$$

where $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d$ are the d edges of $\mathcal{K}$ and $\det \mathcal{K}$ is the determinant of the matrix whose i^{th} column is the edge vector $\mathbf{w}_i$.

Proof. Without loss of generality, we assume $\mathcal{K}$ is contained in the positive orthant of $\mathbb{R}^d$. We know that $\omega_{1,\mathcal{K}}(0)$ is defined as

$$\omega_{1,\mathcal{K}}(0) = \frac{\operatorname{vol}(B_{1,1}(0)\cap\mathcal{K})}{\operatorname{vol} B_{1,1}(0)},$$

where $B_{1,1}(0)$ is the unit ball with respect to the l^1-norm, centered at the origin. It turns out that the closure of this ball, $\overline{B_{1,1}(0)}$, is a well studied polytope called the **cross-polytope**, which is defined by

$$\diamond := \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : |x_1| + |x_2| + \cdots + |x_d| \leq 1\}.$$

Therefore

$$\omega_{1,\mathcal{K}}(0) = \frac{\mathrm{vol}(\diamond \cap \mathcal{K})}{\mathrm{vol}\,\diamond}.$$

It is known that $\mathrm{vol}\diamond = 2^d/d!$. Thus, it remains to show that

$$\mathrm{vol}(\diamond \cap \mathcal{K}) = \frac{|\det \mathcal{K}|}{d! \prod_{i=1}^{d} \|\mathbf{w}_i\|_1}.$$

Since $\mathcal{K}$ is simple, by definition it has exactly d edges, which we assume are contained in the positive orthant. Therefore, the intersection of $\mathcal{K}$ with $\diamond$ forms a d-dimensional simplex with one vertex at the origin and d vertices on the boundary of $\diamond$, corresponding to the d edge vectors of $\mathcal{K}$.

It is known that the volume of a d-dimensional simplex Δ with vertices $0, \mathbf{v}_1, \ldots, \mathbf{v}_d$ is given by

$$\mathrm{vol}\Delta = \frac{1}{d!}\left|\det\begin{pmatrix} \vdots & \vdots & & \vdots \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_d \\ \vdots & \vdots & & \vdots \end{pmatrix}\right|. \tag{2.30}$$

Let $\mathbf{v}_i$ for $i = 1, \ldots, d$, be the non-zero vertex of $\diamond \cap \mathcal{K}$ in the direction of the edge vector $\mathbf{w}_i$. Since $\mathbf{v}_i$ is on the boundary of $\diamond$, we must have $\|\mathbf{v}_i\|_1 = 1$ for each i. Also, v_i is a scalar multiple of $\mathbf{w}_i$. Hence, we must have $v_i = \frac{\mathbf{w}_i}{\|\mathbf{w}_i\|_1}$, for each i, because

$$\left\|\frac{\mathbf{w}_i}{\|\mathbf{w}_i\|_1}\right\|_1 = \frac{\|\mathbf{w}_i\|_1}{\|\mathbf{w}_i\|_1} = 1.$$

Therefore,

$$\begin{aligned}
\mathrm{vol}(\diamond \cap \mathcal{K}) &= \frac{1}{d!}\left|\det\begin{pmatrix} \vdots & \vdots & & \vdots \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_d \\ \vdots & \vdots & & \vdots \end{pmatrix}\right| \\
&= \frac{\left|\det\begin{pmatrix} \vdots & \vdots & & \vdots \\ \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_d \\ \vdots & \vdots & & \vdots \end{pmatrix}\right|}{d!\prod_{i=1}^{d}\|\mathbf{w}_i\|_1} \\
&= \frac{|\det \mathcal{K}|}{2^d\prod_{i=1}^{d}\|\mathbf{w}_i\|_1}.
\end{aligned}$$

■

Before we start the next theorem, we pause to clarify what we mean by a "polynomial time algorithm." Let $\mathcal{C}$ be the class of all rational polytopes in $\mathbb{R}^d$ with d fixed. An algorithm on $\mathcal{C}$ is called a "**polynomial time algorithm in fixed dimension**" if there exits a fixed polynomial f such that the algorithm only takes $f(t_1, \ldots, t_m)$ time, where the t_i's are the $\log_2 |c_i|$ of the coordinates c_i of the vertices of $\mathcal{P}$, for any $\mathcal{P} \in \mathcal{C}$. The polynomial f is known as a polynomial in the input size of $\mathcal{P}$.

Theorem 2.7.2. *Let us fix d. Given any pointed rational d-cone $\mathcal{K} \subset \mathbb{R}^d$, there exists a polynomial time algorithm in fixed dimension which computes the l^1-solid angle at the apex of $\mathcal{K}$.*

Proof. We can assume that the apex $\mathbf{v}$ of $\mathcal{K}$ is at the origin, otherwise we can calculate the l^1-solid angle of the translated cone $\mathcal{K} - \mathbf{v}$. Theorem 2.7.1 gives the formula for simple cones contained in one orthant. We know that l^p-solid angles are additive, i.e. if $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$ where $\dim(\mathcal{K}_1 \cap \mathcal{K}_2) < d$, then $\omega_{p,\mathcal{K}}(x) = \omega_{p,\mathcal{K}_1}(x) + \omega_{p,\mathcal{K}_2}(x)$ for all $x \in \mathbb{R}^d$. Therefore, we wish to decompose $\mathcal{K}$ efficiently into cones, the l^1-solid angles of which are computable using theorem 2.7.1.

As is well-known, any pointed cone can be triangulated into simple cones using no new generators [7], and in fact every rational cone can be triangulated into unimodular cones (simple cones whose edges form a basis of $\mathbb{Z}^d$) [24]. But triangulations are not enough to ensure polynomial time computability. We will use the following theorem of Barvinok [6] which says we can break up $\mathcal{K}$ as a signed decomposition of unimodular cones in polynomial time:

Theorem (Barvinok). *For a fixed dimension d, there exists a polynomial time algorithm, which, given a rational polyhedral cone $\mathcal{K} \subset \mathbb{R}^d$, computes unimodular cones $\mathcal{K}_i$ for $i \in I$ and numbers $\epsilon_i \in \{-1, 1\}$ such that*

$$1_{\mathcal{K}} = \sum_{i \in I} \epsilon_i 1_{\mathcal{K}_i},$$

where $1_{\mathcal{K}}$ is the indicator function of $\mathcal{K}$. In particular, the number $|I|$ of cones in the decomposition is bounded by a polynomial in the input size of $\mathcal{K}$.

Therefore, we assume that $\mathcal{K}$ has such a signed decomposition. Then by the additivity of l^p-solid angles, we have

$$\omega_{1,\mathcal{K}}(0) = \sum_{i \in I} \epsilon_i \omega_{1,\mathcal{K}_i}(0). \tag{2.31}$$

According to [4], computing the edge vectors of each $\mathcal{K}_i$ can be done in polynomial time. Thus, the l^1-solid angle at the vertex of $\mathcal{K}$ can be computed in polynomial time using theorem 2.7.1 and equation (2.31), as long as the unimodular cones in the decomposition are all contained in one orthant.

Alternatively, we can use the fact that $\mathcal{K} \cap \diamond$ is a polytope. Polytopes are simply the bounded intersection of a finite number of half-spaces of the form $\{\mathbf{x} \in \mathbb{R}^d : a_1x_1 + a_2x_2 + \cdots + a_dx_d \leq b\}$. The boundaries of these half-spaces are called hyperplanes. A hyperplane is called **rational** if it is of the form $\{\mathbf{x} \in \mathbb{R}^d : a_1x_1 + a_2x_2 + \cdots + a_dx_d = b\}$ for some $a_1, a_2, \ldots, a_d, b \in \mathbb{Z}$. A polytope is called **rational** if all of its defining hyperplanes are rational.

Since $\diamond$ is a rational polytope and we assume that $\mathcal{K}$ is a rational cone, all of the defining hyperplanes of their intersection $\mathcal{K} \cap \diamond$ will be rational.

Thus, $\mathcal{K} \cap \diamond$ is a rational polytope. In [4], Barvinok gives an algorithm that implies polynomial computability of the Ehrhart quasi-polynomial of a rational polytope when the dimension is fixed. The Ehrhart quasi-polynomial of $\mathcal{P}$ is an expression of the form $L_{\mathcal{P}}(r) = c_d(r)r^d + \cdots + c_1(r)r + c_0(r)$, where $c_d, \ldots, c_0$ are periodic functions in r and $L_{\mathcal{P}}(r) = \#\{r\mathcal{P} \cap \mathbb{Z}^d\}$, the discrete volume of the r^{th} dilate of $\mathcal{P}$. It is known that the leading coefficient, c_d, equals the continuous volume of $\mathcal{P}$. Therefore, the volume of $\mathcal{K} \cap \diamond$ is computable in polynomial time and so is $\omega_{1,\mathcal{K}}(0) = \frac{\text{vol}(\mathcal{K} \cap \diamond)}{\text{vol} \diamond}$.

■

CHAPTER 3

Generalized Solid Angle Theory for Real Polytopes

3.1 Introduction

We have seen in the previous chapter that solid angles are a generalization of two-dimensional angles to higher dimensions. I. G. Macdonald initiated the systematic study of solid-angle sums in integral polytopes. Recently, there has been a resurgence of activity on solid angles and we now have a theory of solid angles that parallels the theory of integer-point enumeration known as Ehrhart theory. This solid-angle theory for rational polytopes, including results from the 1971 paper [34] of Macdonald, can be found in Chapter 11 of [7].

In this chapter, we extend many theorems of solid-angle theory for rational polytopes from [7] to results involving generalized solid angles and *real* polytopes. The proofs we give here rely on Harmonic Analysis and therefore do not resemble the proofs in [7], which are combinatorial in nature. Furthermore, it is the power of Harmonic Analysis that allows us to use generalized solid angles and to extend our results to real polytopes. We also note that solid-angle theory for real polytopes is still in its infancy, primarily due to the considerable increase in difficulty associated with the study of polyhedra with irrational vertices. Theorems for irrational polytopes are hard to come by, even in Ehrhart theory, and thus the significant contribution of this chapter to solid-angle theory is the extension of several fundamental theorems to *real* polytopes. We also note that in this chapter the word cone always refers to a pointed cone.

3.2 Generalized Solid-Angle Generating Functions

The **integer-point transform** of a polytope $\mathcal{P} \in \mathbb{R}^d$, given by

$$\sigma_{\mathcal{P}}(\mathbf{z}) := \sum_{m \in \mathcal{P} \cap \mathbb{Z}^d} \mathbf{z}^m, \tag{3.1}$$

is a multivariate generating function that lists all integer points in $\mathcal{P}$ as a formal sum of monomials. This special format encodes information about the integer points in a way that allows us to use both algebraic and analytic techniques to study the discrete geometry of polyhedra. By analogy, we form the **solid-angle generating function** for a polytope $\mathcal{P}$

$$\alpha_{\mathcal{P}}(\mathbf{z}) := \sum_{m \in \mathcal{P} \cap \mathbb{Z}^d} \omega_{\mathcal{P}}(m)\mathbf{z}^m. \tag{3.2}$$

In order to employ the methods of Harmonic Analysis, we often need to consider functions of a complex variable. For this reason, we redefine $\alpha_{\mathcal{P}}$ using the substitution $z_k = e^{2\pi i s_k}$ for each $k = 1, \ldots, d$, so $\mathbf{z}^m = e^{2\pi i \langle s, m \rangle}$ and we obtain

$$\alpha_{\mathcal{P}}(s) := \sum_{m \in \mathcal{P} \cap \mathbb{Z}^d} \omega_{\mathcal{P}}(m) e^{2\pi i \langle s, m \rangle}, \quad \text{for } s \in \mathbb{C}^d. \tag{3.3}$$

This substitution will prove essential when we use the Poisson summation formula in our proofs. Using this technique will introduce sums of Fourier-Laplace transforms defined over polyhedra and the complex variable will ensure convergence of such sums. We note that while defined similarly, the Fourier-Laplace transform is defined for the complex variable $s \in \mathbb{C}^d$, while the Fourier transform is only defined on $\mathbb{R}^d$.

We wish to point out that $\alpha_{\mathcal{P}}(s)$ is a finite sum for any polytope $\mathcal{P} \subset \mathbb{R}^d$ and for all $s \in \mathbb{C}^d$ because the $\omega_{\mathcal{P}}(m) = 0$ for all $m \notin \mathcal{P}$. Therefore, convergence is not an issue when dealing with polytopes. However, when we consider the solid-angle generating function for a pointed cone $\mathcal{K}$, this is not the case. To discuss the convergence of $\alpha_{\mathcal{K}}(s)$, we need to define K^*, the polar cone associated with K. The polar cone K^* is defined by

$$K^* = \{x \in \mathbb{R}^d \,:\, \langle x,\, y\rangle < 0, \forall\, y \in \mathcal{K}\}.$$

Thus, $\alpha_{\mathcal{K}}(s)$ converges if $s \in \mathbb{C}^d$ such that $-\mathrm{Im}(s) \in \mathcal{K}^*$, because

$$\begin{aligned} & -\mathrm{Im}(s) \in \mathcal{K}^* \\ \Rightarrow\;& \langle -\mathrm{Im}(s),\, m\rangle < 0, \quad \forall\, m \in \mathcal{K} \cap \mathbb{Z}^d \\ \Leftrightarrow\;& \left|e^{2\pi\langle -\mathrm{Im}(s),\, m\rangle}\right| < 1, \quad \forall\, m \in \mathcal{K} \cap \mathbb{Z}^d \\ \Leftrightarrow\;& \left|e^{2\pi i\langle s,\, m\rangle}\right| < 1, \quad \forall\, m \in \mathcal{K} \cap \mathbb{Z}^d. \end{aligned}$$

We now further extend our definition of $\alpha_{\mathcal{P}}$, by allowing $\omega_{\mathcal{P}}(m)$ to be the generalized l^p-solid angle measure defined in (2.25) and which we restate here:

$$\omega_{\mathcal{P}}(\mathbf{x}) := \lim_{\epsilon \to 0} \frac{1}{\epsilon^{d/p}} \int_{\mathcal{P}} e^{\frac{-c}{\epsilon}\|t-\mathbf{x}\|_p^p} dt. \tag{3.4}$$

We will use Fact 1 from the previous chapter which states that:

$$\omega_{\mathcal{P}}(m) = \lim_{\epsilon \to 0} \left(1_{\mathcal{P}} * \phi_\epsilon\right)(m), \tag{3.5}$$

for an appropriate choice of ϕ_ϵ with $\phi_\epsilon(-x) = \phi_\epsilon(x)$. In fact, we will use $\phi_\epsilon(s) = \epsilon^{-\frac{d}{2}} e^{\frac{-\pi}{\epsilon}\langle s,\, s\rangle}$, since we will need $\hat{\phi}_\epsilon$ to be rapidly decreasing. We could use $\phi_\epsilon(t) = \frac{1}{\epsilon^{d/p}} e^{\frac{-c}{\epsilon}\|t\|_p^p}$ for any $p \geq 1$ and $c = \left(2\Gamma\left(\frac{1}{p}+1\right)\right)^p$ as long as $\hat{\phi}_\epsilon$ decreases rapidly enough to ensure absolute convergence in the series that will follow.

We will now show that the solid-angle generating function $\alpha_{\mathcal{K}}(s)$ obeys the following reciprocity relation:

Theorem 3.2.1. *Suppose $\mathcal{K}$ is a simple d-cone in $\mathbb{R}^d$ with vertex at the origin and $s \in \mathbb{C}^d$. Then*

$$\alpha_{\mathcal{K}}(-s) = (-1)^d \alpha_{\mathcal{K}}(s). \tag{3.6}$$

Proof. For $j = 1, \ldots, d$, let $\mathbf{w}_j$ be a generator of the simple cone $\mathcal{K}$. By abuse of notation, we denote the determinant of the matrix whose j^{th} column is the edge vector $\mathbf{w}_j$ by $\det \mathcal{K}$. Then

$$\begin{aligned}
\alpha_{\mathcal{K}}(-s) &= \lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}^d} (1_{\mathcal{K}} * \phi_\epsilon)(m) e^{2\pi i \langle -s, m \rangle} && (3.7)\\
&= \lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}^d} \widehat{(1_{\mathcal{K}} * \phi_\epsilon)}(m - s) && (3.8)\\
&= \lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}^d} \hat{1}_{\mathcal{K}}(m - s) \hat{\phi}_\epsilon(m - s) && (3.9)\\
&= \lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}^d} \frac{(-2\pi i)^{-d} |\det \mathcal{K}|}{\prod_{j=1}^d \langle \mathbf{w}_j, m - s \rangle} \hat{\phi}_\epsilon(m - s) && (3.10)
\end{aligned}$$

The last equality uses the formula for $\hat{1}_{\mathcal{K}}$, which is exercise 10.4 in [7]. We used Poisson summation in the second equality, which is valid because the convolution of $1_{\mathcal{K}}$ with ϕ_ϵ is an integrable and continuous function whenever ϕ_ϵ is integrable and continuous.

Now we will use the fact that the lattice sum is invariant under the substitution $m = -n$. Thus, we have

$$\begin{aligned}
\alpha_{\mathcal{K}}(-s) &= \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d} \frac{(-2\pi i)^{-d} |\det \mathcal{K}|}{\prod_{j=1}^d \langle \mathbf{w}_j, -n - s \rangle} \hat{\phi}_\epsilon(-n - s) && (3.11)\\
&= (-1)^d \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d} \frac{(-2\pi i)^{-d} |\det \mathcal{K}|}{\prod_{j=1}^d \langle \mathbf{w}_j, n + s \rangle} \hat{\phi}_\epsilon(n + s) && (3.12)\\
&= (-1)^d \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d} \hat{1}_{\mathcal{K}}(n + s) \hat{\phi}_\epsilon(n + s) && (3.13)\\
&= (-1)^d \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d} \widehat{(1_{\mathcal{K}} * \phi_\epsilon)}(n + s) && (3.14)\\
&= (-1)^d \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d} (1_{\mathcal{K}} * \phi_\epsilon)(n) e^{2\pi i \langle s, n \rangle} && (3.15)\\
&= (-1)^d \alpha_{\mathcal{K}}(s). && (3.16)
\end{aligned}$$

In (3.12), we used the fact that, for all complex vectors $z \in \mathbb{C}^d$, $\hat{\phi}_\epsilon(-z) = \hat{\phi}_\epsilon(z)$. This last remark holds because

$$\begin{aligned}
\hat{\phi}_\epsilon(-z) &= \int_{\mathbb{R}^d} e^{2\pi i\langle -z,x\rangle}\phi_\epsilon(x)dx \\
&= \int_{\mathbb{R}^d} e^{2\pi i\langle z,-x\rangle}\phi_\epsilon(x)dx \\
&= \int_{\mathbb{R}^d} e^{2\pi i\langle z,u\rangle}\phi_\epsilon(-u)du \\
&= \int_{\mathbb{R}^d} e^{2\pi i\langle z,u\rangle}\phi_\epsilon(u)du \\
&= \hat{\phi}_\epsilon(z).
\end{aligned}$$

■

We now generalize the previous theorem to any real d-cone.

Theorem 3.2.2. *Suppose $\mathcal{K}$ is a d-cone with its vertex at the origin, $\mathbf{v} \in \mathbb{R}^d$, and $s \in \mathbb{C}^d$. Then the solid-angle generating function $\alpha_{\mathbf{v}+\mathcal{K}}(s)$ of the d-cone $\mathbf{v}+\mathcal{K}$ satisfies*

$$\alpha_{\mathbf{v}+\mathcal{K}}(-s) = (-1)^d \alpha_{-v+\mathcal{K}}(s). \tag{3.17}$$

Proof. Since solid angles are additive, it suffices to prove this theorem for simple cones. Therefore, let $\mathbf{w}_j$ for $j = 1, \ldots, d$ be the generators of the simple cone $\mathcal{K}$. Then the cone $\mathbf{v}+\mathcal{K}$ has generators $\mathbf{v}+\mathbf{w}_j$ and we have

$$\begin{aligned}
\alpha_{\mathbf{v}+\mathcal{K}}(-s) &= \lim_{\epsilon\to 0} \sum_{m\in\mathbb{Z}^d} \left(1_{\mathbf{v}+\mathcal{K}} * \phi_\epsilon\right)(m) e^{2\pi i\langle -s,m\rangle} \\
&= \lim_{\epsilon\to 0} \sum_{m\in\mathbb{Z}^d} (\widehat{1_{\mathbf{v}+\mathcal{K}} * \phi_\epsilon})(m-s) \qquad (3.18) \\
&= \lim_{\epsilon\to 0} \sum_{m\in\mathbb{Z}^d} \hat{1}_{\mathbf{v}+\mathcal{K}}(m-s)\hat{\phi}_\epsilon(m-s).
\end{aligned}$$

We used Poisson summation in the (3.18) above and we note that the formula for the Fourier-Laplace transform of the shifted cone $\mathbf{v}+\mathcal{K}$ is obtained from that of $\mathcal{K}$, since $\hat{1}_{\mathbf{v}+\mathcal{K}} = \hat{1}_{\mathcal{K}} \cdot e^{2\pi i\langle \mathbf{v},\cdot\rangle}$. Thus

$$\begin{aligned}
\alpha_{\mathbf{v}+\mathcal{K}}(-s) &= \lim_{\epsilon\to 0}\sum_{m\in\mathbb{Z}^d}\frac{(-2\pi i)^{-d}|\det\mathcal{K}|\; e^{2\pi i\langle \mathbf{v},\, m-s\rangle}}{\prod_{j=1}^{d}\langle \mathbf{w}_j, m-s\rangle}\,\hat{\phi}_\epsilon(m-s)\\
&= \lim_{\epsilon\to 0}\sum_{n\in\mathbb{Z}^d}\frac{(-2\pi i)^{-d}|\det\mathcal{K}|\; e^{2\pi i\langle \mathbf{v},\, -n-s\rangle}}{\prod_{j=1}^{d}\langle \mathbf{w}_j, -n-s\rangle}\,\hat{\phi}_\epsilon(-n-s)\\
&= (-1)^d\lim_{\epsilon\to 0}\sum_{n\in\mathbb{Z}^d}\frac{(-2\pi i)^{-d}|\det\mathcal{K}|\; e^{2\pi i\langle -\mathbf{v},\, n+s\rangle}}{\prod_{j=1}^{d}\langle \mathbf{w}_j, n+s\rangle}\,\hat{\phi}_\epsilon(n+s)\\
&= (-1)^d\lim_{\epsilon\to 0}\sum_{n\in\mathbb{Z}^d}\hat{1}_{-\mathbf{v}+\mathcal{K}}(n+s)\hat{\phi}_\epsilon(n+s)\\
&= (-1)^d\lim_{\epsilon\to 0}\sum_{n\in\mathbb{Z}^d}(\widehat{1_{-\mathbf{v}+\mathcal{K}}*\phi_\epsilon})(n+s)\\
&= (-1)^d\lim_{\epsilon\to 0}\sum_{n\in\mathbb{Z}^d}\left(1_{-\mathbf{v}+\mathcal{K}}*\phi_\epsilon\right)(n)e^{2\pi i\langle s,n\rangle}\\
&= (-1)^d\alpha_{-\mathbf{v}+\mathcal{K}}(s).
\end{aligned}$$

We again used the fact that the lattice sum is invariant under the substitution $m=-n$ and that $\hat{\phi}_\epsilon(-z)=\hat{\phi}_\epsilon(z)$, for all $z\in\mathbb{C}^d$. ∎

We now state and prove the analogue of Brion's theorem in terms of generalized solid angles.

Theorem 3.2.3. *Suppose $\mathcal{P}$ is any convex d-polytope. Then we have the following identity of meromorphic functions for $s\in\mathbb{C}^d$:*

$$\alpha_{\mathcal{P}}(s)=\sum_{\substack{\mathbf{v}\text{ a vertex}\\ \text{of }\mathcal{P}}}\alpha_{\mathcal{K}_{\mathbf{v}}}(s), \tag{3.19}$$

where $\mathcal{K}_{\mathbf{v}}:=\{\mathbf{v}+\lambda(\mathbf{y}-\mathbf{v}) : \mathbf{y}\in\mathcal{P},\lambda\in\mathbb{R}_{\geq 0}\}$ is the vertex cone of $\mathcal{P}$ at the vertex $\mathbf{v}$.

Proof. We begin with the Brianchon-Gram identity [7]:

$$1_{\mathcal{P}}(\mathbf{x})=\sum_{\mathcal{F}\subseteq\mathcal{P}}(-1)^{\dim\mathcal{F}}1_{\mathcal{K}_{\mathcal{F}}}(\mathbf{x}), \tag{3.20}$$

where the sum is taken over all nonempty faces $\mathcal{F}$ of $\mathcal{P}$ and $\mathcal{K}_{\mathcal{F}}$ is the tangent cone attached to $\mathcal{F}$ defined by $\mathcal{K}_{\mathcal{F}}:=\{\mathbf{x}+\lambda(\mathbf{y}-\mathbf{x}) : \mathbf{x}\in\mathcal{F},\mathbf{y}\in\mathcal{P},\lambda\in\mathbb{R}_{\geq 0}\}$.

Then we take the convolution of both sides with ϕ_ϵ, multiply by z^m, and finally sum over all $m \in \mathbb{Z}^d$ to obtain

$$\sum_{m\in\mathbb{Z}^d} (1_{\mathcal{P}} * \phi_\epsilon)(m) z^m = \sum_{m\in\mathbb{Z}^d} \sum_{\mathcal{F}\subseteq\mathcal{P}} (-1)^{\dim\mathcal{F}} (1_{\mathcal{K}_\mathcal{F}} * \phi_\epsilon)(m) z^m. \tag{3.21}$$

We wish to take the limit as $\epsilon \to 0$ of both sides of equation (3.21), but we first note that the infinite lattice sums are absolutely convergent due to the presence of the damping function ϕ_ϵ and hence we can take the limit inside the sum. Thus, we obtain

$$\begin{aligned} \sum_{m\in\mathbb{Z}^d} \omega_{\mathcal{P}}(m) z^m &= \sum_{m\in\mathbb{Z}^d} \sum_{\mathcal{F}\subseteq\mathcal{P}} (-1)^{\dim\mathcal{F}} \omega_{\mathcal{K}_\mathcal{F}}(m) z^m \\ &= \sum_{\substack{\text{v a vertex} \\ \text{of } \mathcal{P}}} \sum_{m\in\mathbb{Z}^d} \omega_{\mathcal{K}_\text{v}}(m) z^m + \sum_{\substack{\mathcal{F}\subseteq\mathcal{P} \\ \dim\mathcal{F}>0}} (-1)^{\dim\mathcal{F}} \sum_{m\in\mathbb{Z}^d} \omega_{\mathcal{K}_\mathcal{F}}(m) z^m. \end{aligned}$$

With the substitution $z^m = e^{2\pi i \langle s, m\rangle}$, we have shown that

$$\alpha_{\mathcal{P}}(s) = \sum_{\substack{\text{v a vertex} \\ \text{of } \mathcal{P}}} \alpha_{\mathcal{K}_\text{v}}(s) + \sum_{\substack{\mathcal{F}\subseteq\mathcal{P} \\ \dim\mathcal{F}>0}} (-1)^{\dim\mathcal{F}} \alpha_{\mathcal{K}_\mathcal{F}}(s). \tag{3.22}$$

Therefore, it remains to show that $\alpha_{\mathcal{K}_\mathcal{F}}(s) = 0$ for every face $\mathcal{F}$ of $\mathcal{P}$ with $\dim\mathcal{F} > 0$. To this end, consider such a $\alpha_{\mathcal{K}_\mathcal{F}}(s)$. Since $\mathcal{K}_\mathcal{F}$ is also a cone, we can write $\mathcal{K}_\mathcal{F}$ as the disjoint union of its relative open faces $\mathcal{G}^\circ$ and obtain

$$\alpha_{\mathcal{K}_\mathcal{F}}(s) = \sum_{m\in\mathbb{Z}^d} \omega_{\mathcal{K}_\mathcal{F}}(m) z^m = \sum_{\mathcal{G}\subseteq\mathcal{F}} \sum_{m\in\mathbb{Z}^d\cap\mathcal{G}^\circ} \omega_{\mathcal{K}_\mathcal{F}}(m) z^m. \tag{3.23}$$

Since $\omega_{\mathcal{K}_\mathcal{F}}(m)$ is constant on the relative interior of each face $\mathcal{G}$ of $\mathcal{F}$, we denote $\omega_{\mathcal{K}_\mathcal{F}}(m)$ by $\omega_\mathcal{G}$ when $m \in \mathcal{G}^\circ$. Then we have

$$\alpha_{\mathcal{K}_\mathcal{F}}(s) = \sum_{\mathcal{G}\subseteq\mathcal{F}} \omega_\mathcal{G} \sum_{m\in\mathbb{Z}^d\cap\mathcal{G}^\circ} z^m. \tag{3.24}$$

Recall that $\dim\mathcal{F} > 0$, and so $\dim\mathcal{G} > 0$ for every face $\mathcal{G}$ of $\mathcal{F}$. Therefore, $\mathcal{G}^\circ$ contains a line and by theorem 3.1 in [6]

$$\sum_{m\in\mathbb{Z}^d\cap\mathcal{G}^\circ} z^m = 0. \tag{3.25}$$

Thus, by equation (3.24), $\alpha_{\mathcal{K}_\mathcal{F}}(s) = 0$ for every face $\mathcal{F}$ of $\mathcal{P}$ with $\dim\mathcal{F} > 0$. ■

3.3 Solid Angle Reciprocity

We now introduce a measure of discrete volume:

$$A_{\mathcal{P}}(t) := \sum_{m \in \mathbb{Z}^d} \omega_{t\mathcal{P}}(m), \tag{3.26}$$

where $\omega_{t\mathcal{P}}(m)$ is the generalized solid angle measure at $m \in \mathbb{Z}^d \cap t\mathcal{P}$ defined in (3.4). Our next theorem is a generalization of the solid angle analogue of Macdonald's reciprocity, which states that

$$A_{\mathcal{P}}(t) = (-1)^{\dim \mathcal{P}} A_{\mathcal{P}}(-t). \tag{3.27}$$

for rational convex polytopes [7]. First, we define a generalized function for $s \in \mathbb{C}^d$ by

$$A_{\mathcal{P}}(t, s) := \sum_{m \in \mathbb{Z}^d} \omega_{t\mathcal{P}}(m) e^{2\pi i \langle m, s \rangle}. \tag{3.28}$$

We will show that $A_{\mathcal{P}}(t, s)$ is an entire function of t which satisfies the reciprocity relation $A_{\mathcal{P}}(-t, s) = (-1)^{\dim \mathcal{P}} A_{\mathcal{P}}(t, -s)$. Furthermore, the following proof extends Macdonald's reciprocity to *real* convex polytopes via $A_{\mathcal{P}}(t) = \lim_{s \to 0} A_{\mathcal{P}}(t, s)$.

Theorem 3.3.1 (Generalized Macdonald's Reciprocity). *Suppose $\mathcal{P}$ is a real convex d-polytope in $\mathbb{R}^d$. Then*
(1) *For $t \in \mathbb{R}$ and $s \in \mathbb{C}^d$, $A_{\mathcal{P}}(t, s)$ satisfies*

$$A_{\mathcal{P}}(-t, s) = (-1)^d A_{\mathcal{P}}(t, -s). \tag{3.29}$$

(2) *Furthermore, if $\mathcal{P}$ is a simple d-polytope, $t \in \mathbb{R}$ and $s \in \mathbb{C}^d$, then the analytic continuation of $A_{\mathcal{P}}(t, s)$ to an entire function of t is given by*

$$A_{\mathcal{P}}(t, s) = \lim_{\epsilon \to 0} \sum_{\substack{\mathbf{v} \text{ a vertex} \\ \text{of } \mathcal{P}}} \frac{|\det \mathcal{K}(\mathbf{v})|}{(-2\pi i)^d} \sum_{m \in \mathbb{Z}^d} \frac{\exp(2\pi i t \langle \mathbf{v},\ m + s \rangle)\ \hat{\phi}_\epsilon(m + s)}{\prod_{j=1}^d \langle \mathbf{w}_j(\mathbf{v}), m + s \rangle}. \tag{3.30}$$

Proof. Since solid angles are additive and we can assume a triangulation of a polytope, it suffices to prove this theorem for a real simplex $\mathcal{P}$. We will use the fact that

$$\omega_{t\mathcal{P}}(m) = \lim_{\epsilon \to 0} \left(1_{t\mathcal{P}} * \phi_\epsilon\right)(m), \tag{3.31}$$

for an appropriate choice of ϕ_ϵ with $\phi_\epsilon(-x) = \phi_\epsilon(x)$. Then we have

$$\begin{aligned} A_{\mathcal{P}}(t,s) &:= \sum_{m \in \mathbb{Z}^d} \omega_{t\mathcal{P}}(m) e^{2\pi i \langle m,\, s\rangle} && (3.32) \\ &= \lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}^d} \left(1_{t\mathcal{P}} * \phi_\epsilon\right)(m) e^{2\pi i \langle m,\, s\rangle} && (3.33) \\ &= \lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}^d} (\widehat{1_{t\mathcal{P}} * \phi_\epsilon})(m+s) && (3.34) \\ &= \lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}^d} \hat{1}_{t\mathcal{P}}(m+s) \hat{\phi}_\epsilon(m+s). && (3.35) \end{aligned}$$

We used Poisson summation in the (3.34). Next, we use an extension of Brion's theorem for *real* polytopes due to Barvinok [2] to obtain

$$A_{\mathcal{P}}(t,s) = \lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}^d} \left(\sum_{\substack{\mathbf{v} \text{ a vertex} \\ \text{of } \mathcal{P}}} \hat{1}_{t\mathbf{v}+\mathcal{K}(\mathbf{v})}(m+s) \right) \hat{\phi}_\epsilon(m+s). \tag{3.36}$$

Brion's theorem allows us to write $\hat{1}_{t\mathcal{P}}$ as the sum of Fourier-Laplace transforms over the tangent cones at the vertices of $t\mathcal{P}$. Therefore, if $\mathbf{v} + \mathcal{K}(\mathbf{v})$ is the tangent cone at the vertex $\mathbf{v}$ of $\mathcal{P}$, where $\mathcal{K}(\mathbf{v})$ is a simple cone with apex at the origin, then $t(\mathbf{v} + \mathcal{K}(\mathbf{v})) = t\mathbf{v} + \mathcal{K}(\mathbf{v})$ is the tangent cone at the vertex $t\mathbf{v}$ of $t\mathcal{P}$, since a cone whose apex is the origin does not change under dilation. Using the formula for the Fourier-Laplace transform of a simple cone

$$\begin{aligned} A_{\mathcal{P}}(t,s) &= \lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}^d} \left(\sum_{\substack{\mathbf{v} \text{ a} \\ \text{vertex} \\ \text{of } \mathcal{P}}} \frac{|\det \mathcal{K}(\mathbf{v})| \ \exp(2\pi i \langle t\mathbf{v},\ m+s\rangle)}{(-2\pi i)^d \prod_{j=1}^d \langle \mathbf{w}_j(\mathbf{v}), m+s\rangle} \right) \hat{\phi}_\epsilon(m+s) && (3.37) \\ &= \lim_{\epsilon \to 0} \sum_{\substack{\mathbf{v} \text{ a} \\ \text{vertex} \\ \text{of } \mathcal{P}}} \frac{|\det \mathcal{K}(\mathbf{v})|}{(-2\pi i)^d} \sum_{m \in \mathbb{Z}^d} \frac{\exp(2\pi i t \langle \mathbf{v},\ m+s\rangle) \hat{\phi}_\epsilon(m+s)}{\prod_{j=1}^d \langle \mathbf{w}_j(\mathbf{v}), m+s\rangle}. && (3.38) \end{aligned}$$

We note that the only place a t appears in this last equation is in the exponent of an exponential. Hence, $A_{\mathcal{P}}(t,s)$ is an entire function of t, because we can differentiate inside the $\sum$ sign due to the "fast enough" convergence provided by $\hat{\phi}_\epsilon$. This proves part (2).

Now for the proof of part (1), we evaluate the analytic continuation of $A_{\mathcal{P}}(t,s)$ at $-t$ to obtain

$$A_{\mathcal{P}}(-t,s) = \lim_{\epsilon\to 0} \sum_{\substack{\mathbf{v} \text{ a vertex} \\ \text{of } \mathcal{P}}} \frac{|\det \mathcal{K}(\mathbf{v})|}{(-2\pi i)^d} \sum_{m\in\mathbb{Z}^d} \frac{e^{2\pi i(-t)\langle \mathbf{v},\, m+s\rangle} \hat{\phi}_\epsilon(m+s)}{\prod_{j=1}^d \langle \mathbf{w}_j(\mathbf{v}), m+s\rangle} \tag{3.39}$$

$$= \lim_{\epsilon\to 0} \sum_{\substack{\mathbf{v} \text{ a vertex} \\ \text{of } \mathcal{P}}} \frac{|\det \mathcal{K}(\mathbf{v})|}{(-2\pi i)^d} \sum_{n\in\mathbb{Z}^d} \frac{e^{2\pi i t\langle \mathbf{v},\, n-s\rangle} \hat{\phi}_\epsilon(-n+s)}{\prod_{j=1}^d \langle \mathbf{w}_j(\mathbf{v}), -n+s\rangle} \tag{3.40}$$

$$= (-1)^d \lim_{\epsilon\to 0} \sum_{\substack{\mathbf{v} \text{ a vertex} \\ \text{of } \mathcal{P}}} \frac{|\det \mathcal{K}(\mathbf{v})|}{(-2\pi i)^d} \sum_{n\in\mathbb{Z}^d} \frac{e^{2\pi i t\langle \mathbf{v},\, n-s\rangle} \hat{\phi}_\epsilon(n-s)}{\prod_{j=1}^d \langle \mathbf{w}_j(\mathbf{v}), n-s\rangle} \tag{3.41}$$

$$= (-1)^d A_{\mathcal{P}}(t,-s). \tag{3.42}$$

We again used the fact that the lattice sum is invariant under the substitution $m=-n$ and that $\hat{\phi}_\epsilon(-z) = \hat{\phi}_\epsilon(z)$, for all $z \in \mathbb{C}^d$. ■

Corollary 3.3.2. *Suppose $\mathcal{P}$ is a real convex d-polytope in $\mathbb{R}^d$ with d odd. Then*

$$A_{\mathcal{P}}(0,0) = 0.$$

Proof. By Theorem 3.3.1, we have

$$A_{\mathcal{P}}(0,0) = (-1)^d A_{\mathcal{P}}(0,0) = -A_{\mathcal{P}}(0,0).$$

■

We pause for a moment to discuss the subtlety involved in computing $A_{\mathcal{P}}(t)$ using the previous theorem. We know that $A_{\mathcal{P}}(t)$ is an entire function of t and in fact is a quasi-polynomial in t when $\mathcal{P}$ is a rational polytope

[34]. The introduction of the complex parameter s in $A_{\mathcal{P}}(t,s)$ prevents the denominators of $\hat{1}_{\mathbf{v}+\mathcal{K}(\mathbf{v})}$ from being zero. So one might wonder if $A_{\mathcal{P}}(t) = \lim_{s\to 0} A_{\mathcal{P}}(t,s)$ even exists. It is Brion's theorem that tells us that when we add up $\hat{1}_{\mathbf{v}+\mathcal{K}(v)}(m+s)$ at every vertex $\mathbf{v}$, magically all of the singularities in $s \in \mathbb{C}^d$ cancel.

To compute $A_{\mathcal{P}}(t)$ from (3.30), we write all of the rational functions on the right-hand side over one denominator and use L'Hôpital's rule to compute the limit as $s \to 0$. The following example will illustrate this procedure.

Example 3.3.1. Let $\mathcal{P}$ be the triangle in $\mathbb{R}^2$ with vertices $\mathbf{v}_1 = (0,0), \mathbf{v}_2 = (0,1)$ and $\mathbf{v}_3 = (\sqrt{3},0)$.

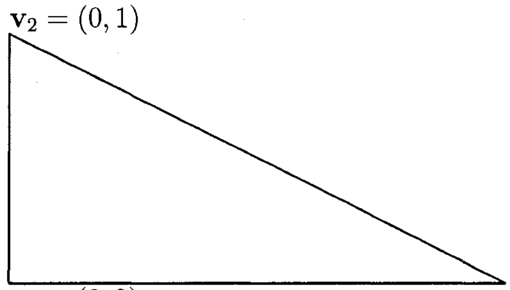

Figure 3.1: The triangle $\mathcal{P}$.

To calculate $A_{\mathcal{P}}(t)$, we use equation (3.30) in Theorem 3.3.1 and we begin by evaluating the determinant of the tangent cone at each vertex. We have

$$|\det \mathcal{K}(\mathbf{v}_1)| = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1,$$

$$|\det \mathcal{K}(\mathbf{v}_2)| = \det\begin{pmatrix} 0 & \sqrt{3} \\ -1 & -1 \end{pmatrix} = \sqrt{3},$$

$$\text{and } |\det \mathcal{K}(\mathbf{v}_3)| = \det\begin{pmatrix} -\sqrt{3} & -1 \\ 1 & 0 \end{pmatrix} = 1.$$

We also need to evaluate

$$\frac{e^{2\pi it\langle \mathbf{v},\, m+s\rangle}}{\prod_{j=1}^{2}\langle \mathbf{w}_j(\mathbf{v}), m+s\rangle}, \tag{3.43}$$

for each of the vertex cones $\mathcal{K}_{\mathbf{v}_1}, \mathcal{K}_{\mathbf{v}_2}$, and $\mathcal{K}_{\mathbf{v}_3}$. Then (3.43) equals

$$\frac{1}{(m_1+s_1)(m_2+s_2)}, \quad \frac{-e^{2\pi it(m_2+s_2)}}{(m_2+s_2)(\sqrt{3}(m_1+s_1)-m_2-s_2)}, \quad \text{and}$$

$$\frac{e^{2\pi it\sqrt{3}(m_1+s_1)}}{(m_1+s_1)(\sqrt{3}(m_1+s_1)-m_2-s_2)}, \quad \text{for } \mathbf{v}_1, \mathbf{v}_2, \text{ and } \mathbf{v}_3 \text{ respectively.}$$

Thus

$$\begin{aligned}
A_{\mathcal{P}}(t,s) &= \lim_{\epsilon\to 0} \sum_{\substack{\mathbf{v} \text{ a vertex}\\ \text{of } \mathcal{P}}} \frac{|\det \mathcal{K}(\mathbf{v})|}{(-2\pi i)^2} \sum_{m\in\mathbb{Z}^2} \frac{e^{2\pi it\langle \mathbf{v},\, m+s\rangle}\, \hat{\phi}_\epsilon(m+s)}{\prod_{j=1}^{2}\langle \mathbf{w}_j(\mathbf{v}), m+s\rangle} \\
&= \lim_{\epsilon\to 0} \frac{1}{-4\pi^2} \sum_{(m_1,m_2)\in\mathbb{Z}^2} \hat{\phi}_\epsilon(m+s) \left(\frac{1}{(m_1+s_1)(m_2+s_2)} \right. \\
&\quad \left. - \frac{\sqrt{3}e^{2\pi it(m_2+s_2)}}{(m_2+s_2)(\sqrt{3}(m_1+s_1)-m_2-s_2)} + \frac{e^{2\pi it\sqrt{3}(m_1+s_1)}}{(m_1+s_1)(\sqrt{3}(m_1+s_1)-m_2-s_2)} \right) \\
&= \lim_{\epsilon\to 0} \frac{1}{-4\pi^2} \sum_{(m_1,m_2)\in\mathbb{Z}^2} \hat{\phi}_\epsilon(m+s) \cdot \frac{f(t,s)}{g(t,s)},
\end{aligned}$$

where

$$\frac{f(t,s)}{g(t,s)} = \frac{\sqrt{3}(m_1+s_1)-m_2-s_2-\sqrt{3}(m_1+s_1)e^{2\pi it(m_2+s_2)}+(m_2+s_2)e^{2\pi it\sqrt{3}(m_1+s_1)}}{(m_1+s_1)(m_2+s_2)(\sqrt{3}(m_1+s_1)-m_2-s_2)}.$$

All that remains is to use L'Hôpital's rule to calculate

$$\lim_{s\to 0} \frac{f(t,s)}{g(t,s)}.$$

In order to take the derivative with respect to s, we first let $s = \sigma(x_1, x_2)$ for some fixed $(x_1, x_2) \neq 0$ and then take the derivative with respect to σ. Since t appears in an exponential in the numerator, each iteration of L'Hôpital's rule will produce a factor of t in the numerator. It is known that for a rational d-polytope, $A_{\mathcal{P}}(t)$ is a quasi-polynomial in t of degree d. Therefore, in general, one must apply L'Hôpital's rule d times for a d-polytope. Thus

$$
\begin{aligned}
\lim_{s\to 0}\frac{f(t,s)}{g(t,s)} &= \lim_{\sigma\to 0}\frac{f(t,\sigma)}{g(t,\sigma)}\\
&= \lim_{\sigma\to 0}\frac{f'(t,\sigma)}{g'(t,\sigma)}\\
&= \lim_{\sigma\to 0}\frac{f''(t,\sigma)}{g''(t,\sigma)}\\
&= \frac{f''(t,0)}{g''(t,0)}\\
&= \frac{-6\pi^2 m_2 x_1^2 t^2 e^{2\pi i t\sqrt{3}m_1} + 2\pi i\sqrt{3}x_1x_2t(e^{2\pi i t\sqrt{3}m_1} - e^{2\pi i t m_2}) + 2\pi^2\sqrt{3}m_1x_2^2t^2e^{2\pi i t m_2}}{-x_2(2m_2x_1 + m_1x_2) + \sqrt{3}x_1(m_2x_1 + 2m_1x_2)},
\end{aligned}
$$

where we used Mathematica in these last steps. We can now choose (x_1, x_2) to be any non-zero vector as long as the denominator is never zero. Therefore, we let $(x_1, x_2) = (1, 1)$ and we have

$$
\begin{aligned}
A_{\mathcal{P}}(t) &= \lim_{s\to 0} A_{\mathcal{P}}(t,s)\\
&= \lim_{\epsilon\to 0}\frac{1}{-4\pi^2}\sum_{(m_1,m_2)\in\mathbb{Z}^2}\hat{\phi}_\epsilon(m)\cdot\frac{f''(t,0)}{g''(t,0)}\\
&= \lim_{\epsilon\to 0}\frac{1}{-4\pi^2}\sum_{(m_1,m_2)\in\mathbb{Z}^2}\hat{\phi}_\epsilon(m)\cdot\\
&\quad \frac{-6\pi^2 m_2 t^2 e^{2\pi i t\sqrt{3}m_1} + 2\pi i\sqrt{3}t(e^{2\pi i t\sqrt{3}m_1} - e^{2\pi i t m_2}) + 2\pi^2\sqrt{3}m_1t^2e^{2\pi i t m_2}}{-2m_2 - m_1 + \sqrt{3}(m_2 + 2m_1)}.
\end{aligned}
$$

When

$$
\phi_\epsilon(s) = \epsilon^{-\frac{d}{2}}\exp\left(\frac{-\pi}{\epsilon}\langle s,\, s\rangle\right) = \epsilon^{-1}\exp\left(\frac{-\pi}{\epsilon}\left(s_1^2 + s_2^2\right)\right), \tag{3.44}
$$

it follows that

$$
\hat{\phi}_\epsilon(m_1, m_2) = \epsilon^{-\frac{1}{2}}\exp\left(-\pi\epsilon\left(m_1^2 + m_2^2\right)\right). \tag{3.45}
$$

Since $\hat{\phi}_\epsilon(m_1, m_2)$ provides absolute convergence, we can break up the series for $A_{\mathcal{P}}(t)$ and use equation (3.45) to obtain the following:

$$A_{\mathcal{P}}(t) = t^2\left(\lim_{\epsilon\to 0}\frac{\epsilon^{-\frac{1}{2}}}{-4\pi^2}\sum_{(m_1,m_2)\in\mathbb{Z}^2}\frac{\left(2\pi^2\sqrt{3}m_1e^{2\pi itm_2}-6\pi^2m_2e^{2\pi it\sqrt{3}m_1}\right)e^{-\pi\epsilon\left(m_1^2+m_2^2\right)}}{-2m_2-m_1+\sqrt{3}(m_2+2m_1)}\right)$$
$$+\ t\left(\lim_{\epsilon\to 0}\frac{\epsilon^{-\frac{1}{2}}}{-4\pi^2}\sum_{(m_1,m_2)\in\mathbb{Z}^2}\frac{2\pi i\sqrt{3}(e^{2\pi it\sqrt{3}m_1}-e^{2\pi itm_2})e^{-\pi\epsilon\left(m_1^2+m_2^2\right)}}{-2m_2-m_1+\sqrt{3}(m_2+2m_1)}\right).$$

■

In the previous example, we note that $A_{\mathcal{P}}(0)=0$ and the dimension, $d=2$, is even. This leads to the following conjecture:

Conjecture 1. *Suppose $\mathcal{P}$ is a real convex d-polytope for any dimension d. Then*

$$A_{\mathcal{P}}(0,0)=0.$$

3.4 The Generating Function of $A_{\mathcal{P}}(t)$

We conclude this final chapter with an extension of an identity for the solid-angle series of a d-polytope $\mathcal{P}$, defined by

$$\mathrm{solid}_{\mathcal{P}}(z) := \sum_{t\geq 0} A_{\mathcal{P}}(t)z^t.$$

This series encodes the solid-angle sum over all dilates of $\mathcal{P}$ simultaneously and the identity we wish to extend is the following [7]:

Theorem 3.4.1. *Suppose $\mathcal{P}$ is an integral d-polytope. Then*

$$\mathrm{solid}_{\mathcal{P}}\left(\frac{1}{z}\right) = (-1)^d\mathrm{solid}_{\mathcal{P}}(z).$$

To extend this theorem to real simple polytopes, we first generalize our definition of $\mathrm{solid}_{\mathcal{P}}$ with the parameter $s=(s_1,\ldots,s_d,s_{d+1})\in\mathbb{C}^{d+1}$:

$$\mathrm{Solid}_{\mathcal{P}}(s) := \sum_{t\geq 0} A_{\mathcal{P}}(t,s_1,\ldots,s_d)e^{2\pi its_{d+1}}.$$

By theorem 3.3.1, if $\mathcal{P}$ is a simple polytope, then

$$\begin{aligned}
\mathrm{Solid}_{\mathcal{P}}(s) &= \sum_{t\geq 0} A_{\mathcal{P}}(t, s_1, \dots, s_d) e^{2\pi i t s_{d+1}} \\
&= (-1)^d \sum_{t\geq 0} A_{\mathcal{P}}(-t, -s_1, \dots, -s_d) e^{2\pi i t s_{d+1}} \\
&= (-1)^d \sum_{t\geq 0} \lim_{\epsilon\to 0} \sum_{\substack{\mathbf{v} \text{ a vertex} \\ \text{of } \mathcal{P}}} \frac{|\det \mathcal{K}(\mathbf{v})|}{(-2\pi i)^d} \sum_{m\in\mathbb{Z}^d} \frac{e^{2\pi i(-t)\langle \mathbf{v},\, m-\tilde{s}\rangle}\, \hat{\phi}_\epsilon(m-\tilde{s})}{\prod_{j=1}^d \langle \mathbf{w}_j(\mathbf{v}), m-\tilde{s}\rangle} e^{2\pi i t s_{d+1}},
\end{aligned}$$

where $\tilde{s} = (s_1, \dots, s_d) \in \mathbb{C}^d$. The presence of $\hat{\phi}_\epsilon$ in the inner-most sum ensures uniform convergence and allows us to bring the sum over t inside. The resulting equation is

$$\mathrm{Solid}_{\mathcal{P}}(s) = (-1)^d \lim_{\epsilon\to 0} \sum_{\substack{\mathbf{v} \text{ a vertex} \\ \text{of } \mathcal{P}}} \frac{|\det \mathcal{K}(\mathbf{v})|}{(-2\pi i)^d} \sum_{m\in\mathbb{Z}^d} \frac{\sum_{t\leq 0} e^{2\pi i t\langle \mathbf{v},\, m-\tilde{s}\rangle + 2\pi i(-t)s_{d+1}}\, \hat{\phi}_\epsilon(m-\tilde{s})}{\prod_{j=1}^d \langle \mathbf{w}_j(\mathbf{v}), m-\tilde{s}\rangle}.$$

Theorem 9.2 in [7] gives us the identity "$\sum_{t\in\mathbb{Z}} z^t = 0$," at the rational function level. We use this identity to rewrite the inner sum $\sum_{t\leq 0} e^{2\pi i t\langle \mathbf{v},\, m-\tilde{s}\rangle + 2\pi i(-t)s_{d+1}}$ as -1 times the same sum over $t \geq 1$ to obtain

$$\begin{aligned}
\mathrm{Solid}_{\mathcal{P}}(s) &= (-1)^{d+1} \lim_{\epsilon\to 0} \sum_{\substack{\mathbf{v} \text{ a vertex} \\ \text{of } \mathcal{P}}} \frac{|\det \mathcal{K}(\mathbf{v})|}{(-2\pi i)^d} \sum_{m\in\mathbb{Z}^d} \frac{\sum_{t\geq 1} e^{2\pi i t\langle \mathbf{v},\, m-\tilde{s}\rangle + 2\pi i(-t)s_{d+1}}\, \hat{\phi}_\epsilon(m-\tilde{s})}{\prod_{j=1}^d \langle \mathbf{w}_j(\mathbf{v}), m-\tilde{s}\rangle} \\
&= (-1)^{d+1} \sum_{t\geq 1} \lim_{\epsilon\to 0} \sum_{\substack{\mathbf{v} \text{ a vertex} \\ \text{of } \mathcal{P}}} \frac{|\det \mathcal{K}(\mathbf{v})|}{(-2\pi i)^d} \sum_{m\in\mathbb{Z}^d} \frac{e^{2\pi i t\langle \mathbf{v},\, m-\tilde{s}\rangle + 2\pi i t(-s_{d+1})}\, \hat{\phi}_\epsilon(m-\tilde{s})}{\prod_{j=1}^d \langle \mathbf{w}_j(\mathbf{v}), m-\tilde{s}\rangle} \\
&= (-1)^{d+1} \mathrm{Solid}_{\mathcal{P}}(-s) + (-1)^d A_{\mathcal{P}}(0, -\tilde{s}).
\end{aligned}$$

We have just shown the following:

Theorem 3.4.2. *Suppose $\mathcal{P}$ is a simple d-polytope. Then* $\mathrm{Solid}_{\mathcal{P}}$ *satisfies the identity*

$$\mathrm{Solid}_{\mathcal{P}}(s_1, \dots, s_{d+1}) = (-1)^{d+1} \mathrm{Solid}_{\mathcal{P}}(-s_1, \dots, -s_{d+1}) + (-1)^d A_{\mathcal{P}}(0, -s_1, \dots, -s_d).$$

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APPENDIX A

We recall here some well known, and some not-so-well known facts about Harmonic Analysis.

Fact 1.

$$\omega_{p,\mathcal{K}}(\mathbf{x}) = \lim_{\epsilon \to 0} \left(1_{\mathcal{K}} * \phi_\epsilon\right)(\mathbf{x}), \quad \textit{for all } \mathbf{x} \in \mathbb{R}^d.$$

Fact 2. $\widehat{(f * g)}(x) = \hat{f}(x)\hat{g}(x)$.

Fact 3. *If* $f(l) = 1_{r\mathcal{P}}(l + x)e^{-2\pi\langle l,\ y\rangle}$*, then* $\hat{f}(m) = \hat{1}_{r\mathcal{P}}(m + iy)e^{-2\pi i\langle x,\ m+iy\rangle}$.

Proof.

$$\begin{aligned}
\hat{f}(m) &= \int_{\mathbb{R}^d} 1_{r\mathcal{P}}(u + x)e^{-2\pi\langle u,\ y\rangle}e^{2\pi i\langle u,\ m\rangle}du \\
&= \int_{\mathbb{R}^d} 1_{r\mathcal{P}}(u + x)e^{2\pi i\langle u,\ m+iy\rangle}du \\
&= \int_{\mathbb{R}^d} 1_{r\mathcal{P}}(w)e^{2\pi i\langle w-x,\ m+iy\rangle}dw \\
&= e^{-2\pi i\langle x,\ m+iy\rangle}\int_{\mathbb{R}^d} 1_{r\mathcal{P}}(w)e^{2\pi i\langle w,\ m+iy\rangle}dw \\
&= e^{-2\pi i\langle x,\ m+iy\rangle}\hat{1}_{r\mathcal{P}}(m + iy).
\end{aligned}$$

■

Fact 4. *(Continuous Brion Theorem)*

Suppose $\mathcal{P}$ *is a simple rational convex d-polytope. For a vertex cone* $K_{\mathbf{v}}$ *of* $\mathcal{P}$*, fix a set of generators* $\mathbf{w}_1(\mathbf{v}), \ldots, \mathbf{w}_d(\mathbf{v}) \in \mathbb{Z}^d$*. Then*

$$\int_{\mathcal{P}} \exp(\mathbf{x} \cdot \mathbf{z})d\mathbf{x} = (-1)^d \sum_{\mathbf{v} \textit{ a vertex of } \mathcal{P}} \frac{\exp(\mathbf{v} \cdot \mathbf{z})|\det(\mathbf{w}_1(\mathbf{v}), \ldots, \mathbf{w}_d(\mathbf{v}))|}{\prod_{k=1}^d (\mathbf{w}_k(\mathbf{v}) \cdot \mathbf{z})}.$$

Fact 5. *Let* $g(x) = f(\frac{x}{\epsilon})$*. Then* $\hat{g}(x) = |\epsilon|^d \hat{f}(\epsilon x)$*.*

Fact 6. *Given a sublattice* $\mathbb{L}$ *of* $\mathbb{Z}^d$*, say* $\mathbb{L} = M^T\mathbb{Z}^d$*,*

$$\sum_{n\in\mathbb{L}} F(n) = \frac{1}{|\det M|} \sum_{g\in\mathbb{Z}^d/M\mathbb{Z}^d} \sum_{l\in\mathbb{Z}^d} F(l)e^{2\pi i\langle M^{-1}g,\, l\rangle}.$$

Proof. The lemma is equivalent to the statement that the "delta function" for the sublattice $M^{\mathrm{T}}\mathbb{Z}^d$ of $\mathbb{Z}^d$ is:

$$1_{M^{\mathrm{T}}\mathbb{Z}^d}(l) = \frac{1}{|\det M|} \sum_{g\in\mathbb{Z}^d/M\mathbb{Z}^d} e^{2\pi i\langle l, M^{-1}g\rangle},$$

since $\sum_{n\in\mathbb{Z}^d} F(n) = \sum_{l\in\mathbb{Z}^d} F(l)1_{\mathbb{L}}(l)$. This statement follows from the orthogonality relations for the character $g \mapsto e^{2\pi i\langle\, l, M^{-1}g\rangle}$ on the finite abelian group $\mathbb{Z}^d/M\mathbb{Z}^d$. That is, in one direction, if $l \in \mathbb{L} = M^{\mathrm{T}}\mathbb{Z}^d$, then $M^{-\mathrm{T}}l \in \mathbb{Z}^d$ and so $\langle\, l, M^{-1}g\rangle = \langle M^{-\mathrm{T}}l, g\rangle \in \mathbb{Z}$, so that $e^{2\pi i\langle l,\, M^{-1}g\rangle} = 1$ and we have

$$1_{M^{\mathrm{T}}\mathbb{Z}^d}(l) = \frac{1}{|\det M|} \sum_{g\in\mathbb{Z}^d/M\mathbb{Z}^d} 1 = \frac{1}{|\det M|} \cdot |\det M| = 1.$$

In the other direction, if $M^{\mathrm{T}}l \notin \mathbb{Z}^d$, then $g \mapsto e^{2\pi i\langle\, l, M^{-1}g\rangle}$ gives a non-trivial character, and hence $\sum_{g\in\mathbb{Z}^d/M\mathbb{Z}^d} e^{2\pi i\langle\, l, M^{-1}g\rangle} = 0$, by the orthogonality of characters on the finite abelian group $\mathbb{Z}^d/M\mathbb{Z}^d$. ■

Fact 7. *For* $Re(s) > 0$*,* $\alpha \in \mathbb{R}$*, and* $\tau \in H$*, the complex upper half plane:*

$$\sum_{m\in\mathbb{Z}} \frac{e^{2\pi i m\alpha}}{(\tau+m)^s} = \frac{(-2\pi i)^s}{\Gamma(s)} \sum_{n=1}^{\infty} (n - \{\alpha\})^{s-1} e^{2\pi i\tau(n-\{\alpha\})}.$$

Fact 8.

Suppose that $F(x) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n x}$ *and* $G(x) = \sum_{n\in\mathbb{Z}} b_n e^{2\pi i n x}$ *converge absolutely for all* $x \in \mathbb{R}$*. Then* $\sum_{n\in\mathbb{Z}} a_n b_n e^{2\pi i n x} = \int_0^1 F(x-t)G(t)dt$*.*

Proof.

$$\begin{aligned}
\int_0^1 F(x-t)G(t)dt &= \int_0^1 \left(\sum_{n\in\mathbb{Z}} a_n e^{2\pi i n(x-t)}\right)\left(\sum_{m\in\mathbb{Z}} b_m e^{2\pi i m t}\right) dt \\
&= \int_0^1 \sum_{\substack{n\in\mathbb{Z}\\ m\in\mathbb{Z}}} a_n b_m e^{2\pi i[n(x-t)+mt]} dt \\
&= \sum_{\substack{n\in\mathbb{Z}\\ m\in\mathbb{Z}}} a_n b_m e^{2\pi i n x} \int_0^1 e^{2\pi i t(m-n)} dt \\
&= \sum_{\substack{n\in\mathbb{Z}\\ m\in\mathbb{Z}}} a_n b_m e^{2\pi i n x} \delta_{nm} \\
&= \sum_{n\in\mathbb{Z}} a_n b_n e^{2\pi i n x}.
\end{aligned}$$

■

Fact 9.

For $x \in \mathbb{R}$ *and* $0 < t < 1$, $\{x-t\} = \begin{cases} \{x\} - t & \text{if } t \le \{x\} \\ \{x\} - t + 1 & \text{if } t > \{x\} \end{cases}$.

Proof. We note that $0 \le \{x\} < 1$ for all $x \in \mathbb{R}$. The key to this proof is the following identity:

$$\lfloor x \rfloor = x - \{x\}, \text{ for all } x \in \mathbb{R}, \tag{A.1}$$

where $\lfloor x \rfloor$ is the greatest integer $\le x$. If $t \le \{x\}$, then $\lfloor x \rfloor = \lfloor x - t \rfloor$. Thus, by (A.1), we have $x - \{x\} = x - t - \{x - t\}$, which implies that $\{x - t\} = \{x\} - t$. Now if $t > \{x\}$, it follows that $\lfloor x \rfloor = \lfloor x - t \rfloor + 1$. Then (A.1) gives us $x - \{x\} = x - t - \{x - t\} + 1$, which implies that $\{x - t\} = \{x\} - t + 1$. ■