Ph.D. Comprehensive Examination
Complex Analysis

January 2023

Part I. Do three of these problems.

I.1. The function $f(z) = z^2$ maps the complex upper half-plane $\mathcal{H}_+ = \{z \in \mathbb{C} : \mathrm{Im}\,(z) > 0\}$ bijectively into $G = \mathbb{C} \setminus \mathbb{R}_+$. Write the inverse map $f^{-1} : G \to \mathcal{H}_+$ in terms of the *principal branch* of the square root, denoted by $\sqrt{\cdot}$.

I.2. Compute using complex integration

$$\int_0^\infty \frac{x dx}{x^3 + 1}$$

I.3. Let

$$K = \{z \in \mathbb{C} : 1 \le |z| \le 2\}.$$

Show that every function analytic on some open set G containing K, can always be approximated uniformly on K by functions analytic in $A = \mathbb{C} \setminus \{0\}$.

I.4. Suppose that a sequence of analytic functions $f_n : \mathcal{H}_+ = \{z \in \mathbb{C} : \mathrm{Im}\,(z) > 0\} \to \mathbb{C}$ satisfies $\mathrm{Im}\,(f_n(z)) > 0$ for all $z \in \mathcal{H}_+$ and all $n \ge 1$. Suppose that for each $z \in \mathcal{H}_+$ the limit

$$f(z) = \lim_{n\to\infty} f_n(z)$$

exists. Prove that $f(z)$ is analytic in $\mathcal{H}_+$ and that convergence is uniform on compact subsets of $\mathcal{H}_+$.

Part II. Do two of these problems.

II.1. Let $G = \{z \in \mathbb{C} : |z| > 1\}$. Suppose $f : G \to \mathbb{C}$ is analytic and there exists a sequence $z_n \to \infty$, such that $z_n^2 f'(z_n) \to 0$. Prove that f cannot be injective on G. **You may not use the Great Picard's theorem!**

II.2. Let f be analytic in $B^-(0, R) = \{z \in \mathbb{C} : |z| \le R\}$ with $f(0) = 0$, $f'(0) \ne 0$, and $f(z) \ne 0$ for $0 < |z| \le R$. Put

$$\rho = \min\{|f(z)| : |z| = R\} > 0.$$

Define $g : B(0, \rho) \to \mathbb{C}$ by

$$g(\omega) = \frac{1}{2\pi i} \int_{|z|=R} \frac{z f'(z)}{f(z) - \omega} dz.$$

Show that $z = g(\omega)$ is the unique solution of $f(z) = \omega$, $|z| < R$, provided $\omega \in B(0, \rho)$.
Hint: Use argument principle to show uniqueness of solution and the residue theorem to show that the solution is $g(\omega)$.

II.3. Let $G = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \pi\}$ and suppose $f : G \to \mathbb{C}$ is analytic. Suppose that there exist constants $M > 0$, $A > 0$, and $a < 1/2$, such that

$$\limsup_{z\to w} |f(z)| \le M, \quad \forall w \in \partial G,$$

and

$$|f(z)| \le e^{Ae^{a|\mathrm{Re}\, z|}}, \quad \forall z \in G.$$

Use Phragmén-Lindelöf theorem to prove that $|f(z)| \le M$ for all $z \in G$.