Part I. Do three of these problems.

I.1. Let ω be a real number $\neq 0$. Compute

$$\int_{-\infty}^{\infty} \frac{e^{ix\omega}}{1+x^2}\, dx$$

using residues.

I.2. Let $G \subset \mathbb{C}$ be open, let $a \in G$. Suppose $r > 0$ is such that the Taylor series at a of every analytic function on G converges in the disc

$$B(a;r) = \{z \in \mathbb{C} : |z-a| < r\}.$$

Show that $B(a;r) \subset G$.

I.3. Let $G \subset \mathbb{C}$ be a connected open set and $f : G \to \mathbb{C}$ analytic. Suppose that there is $a \in G$ such that $f(a) = 0$, and that $f(z) \notin (0,\infty)$ for all $z \in G$. Show that $f \equiv 0$.

I.4. Find the image of the disk $\{z \in \mathbb{C} : |z-i| < 2\}$ under the Möbius transformation

$$w = \frac{z+i}{z-2i}.$$

Part II. Do two of these problems.

II.1. Let $D = B(0;1)$ be the unit disc. Suppose $f : D \to D$ is analytic and invertible. Show that f is a Möbius transformation.

II.2. Let u be a solution of $\Delta\Delta u = 0$ on the unit disc D. Show that there are holomorphic functions $f, g : D \to \mathbb{C}$ such that

$$u(z) = \operatorname{Re}\big(f(z) + \overline{z}g(z)\big).$$

II.3. Suppose $f(z)$ is analytic inside a bounded domain G and continuous on the closure of G. Suppose the number $M > 0$ is such that $|f(z)| \geq M$ for all $z \in \partial G$. Suppose there exists a point $z_0 \in G$ such that $|f(z_0)| < M$. Prove that $f(z)$ must have a zero in G.