Ph.D. Comprehensive Examination
Complex Analysis
August 2023

Part I. Do three of these problems.

I.1. Suppose $h(x, y)$ is a harmonic function on an open set $G \subset \mathbf{R}^2$. Prove that $f(z) = h_x - ih_y$ is analytic in G, regarded as a subset of $\mathbb{C}$.

I.2. Let $L(z)$ denote the principal branch of the logarithm. What is the radius of convergence of the power series of $L(z)$ centered at $z_0 = -1 + i$? Justify your answer.

I.3. Let D denote the open unit disk in $\mathbb{C}$. Suppose f is an entire function, such that $f(\overline{D}) \subset \overline{D}$. Suppose, in addition, $f(0) = 1$. What can you say about $f(z)$?

I.4. What is the image of the unit disk under the map $f(z) = (3 + 4i)z^2 + 6i$.

Part II. Do two of these problems.

II.1. Prove that there exists no continuous function $\phi \in C([0, 1])$, such that

$$\int_0^1 \phi(x)e^{-xt}dx = e^{-2t}, \quad \forall t \in (a, b),$$

no matter what real numbers $a < b$ are.

II.2. Suppose $G \subset \mathbb{C}$ and $\Omega \subset \mathbb{C}$ are open and connected. Suppose f is analytic in Ω and g is analytic in G. Suppose also that $\phi : G \to \Omega$ is continuous and satisfies $f(\phi(z)) = g(z)$. Prove that $\phi(z)$ must be analytic in G, provided $f(w)$ is not a constant function.

(a) First prove the statement under the additional assumption that $f'(w) \neq 0$ for all $w \in \Omega$.
(b) Use part (a) to prove the statement without the additional assumption.

II.3. How many roots does $f(z) = z^4 - z^3 + z + 3$ have in the exterior of the unit disk?