Ph.D. Comprehensive Examination
Complex Analysis
Fall 2000

Part I. Do three of these problems.

I.1. Show that if $u(x, y)$ is a harmonic function in a simply connected region, D then, then u is the real part of a function that is analytic in D.

I.2. Let $f(z)$ be analytic on $\{z : 0 < |z| < 2\}$ and suppose that for $n = 0, 1, 2, \ldots$

$$\int_{|z|=1} z^n f(z)\, dz = 0.$$

Show that f has a removable singularity at $z = 0$.

I.3. Suppose that $f(z)$ is an entire function satisfying $|f(z)| > 1$ when $|z| > 1$. Prove that $f(z)$ is a polynomial.

I.4. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire and that f has exactly k zeros in the open disc $\{z : |z| < 1\}$ but none on the circle $\{z : |z| = 1\}$. Show that there exists $\varepsilon > 0$ such that any entire functon g that satisfies $|f(z) - g(z)| < \varepsilon$ on the circle $|z| = 1$ must also have exactly k zeros in the open disc $\{z : |z| < 1\}$.

Part II on next page

Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part II. Do two of these problems.

II.1. Let $\alpha > 0$. Prove that

(1) if $f \in L^1(0,1)$ then

$$f_\alpha(x) = \int_0^x (x-t)^{\alpha-1} f(t)\, dt$$

exists a.e. and is integrable on (0,1);

(2) if $f \in L^p(0,1)$ then f_α is continuous in (0,1) for $\alpha > 1/p$.

II.2. Let $1 \le p,\ q \le \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Prove tha $fg \in L^r(\mathbb{R}^n)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

II.3. Prove that

(1)

$$\log \frac{1}{1-x} = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

(2)

$$\int_0^1 \log \frac{1}{1-x}\, dx = 1.$$