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PROBABILITY AND MATHEMATICAL FINANCE
PART I: LONG RANGE SELF-AVOIDING WALKS ABOVE CRITICAL DIMENSION
PART II: FINITE HORIZON OPTIMAL INVESTMENT AND CONSUMPTION WITH TRANSACTION COST

Author: (Please type)
Yun Cheng

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Dissertation Examining Committee: (please type)	Read and Approved By: (Signatures)
Wei Shih Yang Dissertation Advisory Committee Chairperson	Wei Shih Yang
Mete Soner	Mete Soner
Jian Guo Liu	Jian Liu
Tzong-Yow Lee	T. Y. Lee
Ranganathan Srinivasan	R. Srinivasan
John T. Schiller Examining Committee Chairperson	J. Schiller If Member of the Dissertation Examining Committee

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(Dean of the Graduate School)

PROBABILITY AND MATHEMATICAL FINANCE

PART I: LONG RANGE SELF-AVOIDING WALKS ABOVE CRITICAL DIMENSION

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A Dissertation
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the Temple University Graduate Board

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in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

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by
Yun Cheng
August, 2000

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ABSTRACT

PROBABILITY AND MATHEMATICAL FINANCE

PART I: LONG RANGE SELF-AVOIDING WALKS ABOVE CRITICAL DIMENSION

PART II: FINITE HORIZON OPTIMAL INVESTMENT AND CONSUMPTION WITH TRANSACTION COST

Yun Cheng
DOCTOR OF PHILOSOPHY

Temple University, August, 2000

Professor Wei-Shih Yang, Chair

The subject of this thesis work consists of two parts. Part I is in the area of probability, where I obtained the limiting distribution of long range self-avoiding random walks above critical dimension. Part II is in the area of mathematical finance, where regularity and free boundary results were obtained on the finite horizon optimal investment and consumption with transaction cost.

ACKNOWLEDGEMENTS

I wish to thank my advisor Professor Wei-Shih Yang, who has been enlightening me with wonderful ideas for all these years. It is certain that this work could not be finished had it not been for the infinite patience from Professor Yang. Thank you so much, Professor Yang.

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Special thanks go to Professor Mete Soner, who put extra effort teaching me the powerful viscosity solution method, spending time with me to discuss the difficult points and sharing with me the critical ideas that eventually lead to the solution of the problem.

Finally, I wish to thank Professor Lee Tzong-Yow and Professor Ranganathan Srinivasan, who patiently read through my thesis and provided me with strong encouragements.

To my parents, and my wife.

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Part I

LONG RANGE SELF-AVOIDING RANDOM WALKS ABOVE CRITICAL DIMENSION

CHAPTER 1

INTRODUCTION

A self-avoiding walk is a path on Z^d lattice that does not visit the same site more than once. The model originated in chemistry several decades ago as a model for long-chain polymer molecules. Since then it has become an important model in statistical physics, as it exhibits critical behavior analogous to that occurring in the Ising model and related systems as percolation.

In spite of its simple definition, many of the questions about the self-avoiding walks are difficult to resolve as it does not respond well to standard probabilistic methods. Computer simulations have played an important role in the development of the theory by providing computational conjectures. The lace expansion is by far the only theoretical method that has led to rigorous results such as existence of critical exponents and mean field behavior.

The lace expansion was first introduced by Brydges and Spencer [1], who used it to study weakly self-avoiding walk when the dimension is above the critical dimension. It was then developed (among others) by Slade [29], Hara and Slade [15], [16]to study the strictly self-avoiding walks, also under the assumption that the dimension is above the critical dimension.

At the heart of the lace expansion method are two components. One is a convolution equation in which the two point function involves with the perturbation function. The other is the Feynman diagram scheme which provides estimates on the perturbation function. To apply the lace expansion method, one needs the help of some assumptions on the model so that the two point functions could be shown to converge to their simple random walk counterparts.

My thesis concerns the limiting distributions of long range self-avoiding random walks. We call a random walk long range if each step of the walk has infinite range. In this situation, Yang and Klein [30] has shown that if the one-step walk follows the discrete Cauchy distribution, the weakly self-avoiding walk will follow the standard Cauchy distribution, when the lattice dimension is above the critical dimension 2. The difference between my work and [30] is that I consider the strictly self-avoiding walks, while [30] considers weakly self-avoiding walks.

In chapter 2, I will give a self-contained formulation of the lace expansion theory, with enough detail for our long range random walk study.

In chapter 3, I consider the self-avoiding long range random walk in high dimension. We show that if the one-step distribution follows the discrete Cauchy distribution, the limiting distribution will follow the classical Cauchy distribution.

In chapter 4, I consider the case in which the dimension $d = 3$, and assume the random walk follows the spread-out discrete Cauchy distribution. We show that in this case the limiting distribution is still the Cauchy distribution.

CHAPTER 2

THE LACE EXPANSION

2.1 The Brydges-Spencer Theorem

Let $X_1, X_2, ...$ be independent identically distributed random variables on lattice Z^d, and $W(T) = \sum_{i=0}^{T} X_i$ be its finite sum. We define

$$\mathcal{B}_\tau[a,b] = \{st : s < t, |s-t| \leq \tau, s, t \in [a,b]\}, \tag{2.1}$$

$$K_\tau[a,b](\omega) = \prod_{st \in \mathcal{B}_\tau[a,b]} (1 + U_{st}(\omega)), \qquad K_\tau[a,a] = 1, \tag{2.2}$$

where $U_{st}(\omega) = -1$ if $\omega(s) = \omega(t)$ and $U_{st}(\omega) = 0$ otherwise.

Clearly $K_\tau[a,b](\omega) = 1$ if the random walk $(\omega(s) : a \leq s \leq b)$ is self-avoiding within memory τ; $K_\tau[a,b](\omega) = 0$ if the random walk is not self-avoiding. Also, $\tau = 0$ corresponds to simple random walk, as in this situation the set $\mathcal{B}[a,b] = \emptyset$.

We call the elements of $\mathcal{B}[a,b]$ *edges*, and subsets of $\mathcal{B}[a,b]$ *graphs*.

A graph $G \subset \mathcal{B}_\tau[a,b]$ is called a *connected graph* if it satisfies the following:

1. For each integer $m \in (a,b)$, there exists $st \in G$ such that $m \in (s,t)$.

2. There exist bonds in G that connect to a and b respectively.

By convention, a single point that does not connect to any other points is also a connected graph.

A *lace* graph is by definition a minimally connected graph. We will use $\mathcal{L}_\tau^N[a,b]$ to denote those laces on $[a,b]$ that have exactly N edges and that each of the edges has its length no larger than τ.

Given a connected graph G on $[a,b]$, the following is a standard procedure to find the representative lace $\mathcal{L}(G)$ corresponding to G:

1. Find the first edge s_1t_1: set $s_1 = a$ and let $t_1 = \max(t : s_1t \in G)$.

2. Assume we have already found edges $s_1t_1, s_2t_2, ...s_nt_n$. We choose edge $s_{n+1}t_{n+1}$ by setting $t_{n+1} = \max(t : st \in G, s < t_n)$, and $s_{n+1} = \min(s : st_n \in G)$.

3. Continue the process in 2 until $t_{n+1} = b$.

Figure 2.1 is the illustration of the lace $N = 3$. We note that $s_1 < s_2 < t_1 \leq s_3 < t_2 < t_3$ and it is possible that $t_1 = s_3$.

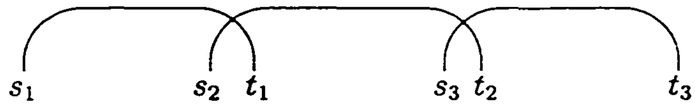

Figure 2.1: Lace Structure for $N = 3$

Two connected graphs $G_1, G_2 \subset \mathcal{B}_\tau[a,b]$ are said to be *compatible* with each other iff $\mathcal{L}(G_1) = \mathcal{L}(G_2)$. Clearly, the *compatible* relation among connected graphs in $\mathcal{B}_\tau[a,b]$ is an equivalent relation.

The lace graphs acts as the 'skeleton' in their equivalent classes. This fact will help us categorize the terms in $K_\tau[a,b]$, as is seen in the following.

For a fixed lace graph $L \subset \mathcal{B}_\tau[a,b]$, let

$$G(L) = \cup\{G : G \subset \mathcal{B}_\tau[a,b], \mathcal{L}(G) = L\}.$$

It is easy to verify that $G(L)$ is still a connected graph on $\mathcal{B}_\tau[a,b]$ and is compatible with L. Let us denote $\mathcal{C}(L) = G(L) \setminus L$, then

$$\sum_{\{G:\mathcal{L}(G)=L\}} \prod_{st\in G} U_{st} = \prod_{st\in L} U_{st} \prod_{st\in\mathcal{C}(L)} (1 + U_{st}). \tag{2.3}$$

We now introduce the following "perturbation" term J_τ:

$$J_\tau^N[a,b] = \sum_{L\in\mathcal{L}_\tau^N[a,b]} \prod_{st\in L} U_{st} \prod_{st\in\mathcal{C}(L)} (1+U_{st}), \tag{2.4}$$

$$J_\tau[a,b] = \sum_{N=1}^{\infty} J_\tau^N[a,b], \tag{2.5}$$

then clearly we have

$$J_\tau[a,b] = \sum_{\text{G connected on [a,b]}} \prod_{st\in G} U_{st}. \tag{2.6}$$

Proposition 2.1 *For $T \geq 1$, we have*

$$K_\tau[0,T] = K_\tau[1,T] + \sum_{j=1}^{T} J_\tau[0,j]K_\tau[j,T]. \tag{2.7}$$

Proof:

From the definition of $K_\tau[0,T]$, we know

$$K_\tau[0,T] = \sum_G \prod_{st\in G} U_{st}, \tag{2.8}$$

where the summation index G runs through all the graphs of $\mathcal{B}_\tau[a,b]$.

Given a graph G (which is not necessarily connected), we denote its first connected component by G_1. Then

$$K_\tau[0,T] = \sum_{G_1} \prod_{st\in G_1} U_{st} \sum_G \prod_{st\in G\setminus G_1} U_{st},$$

where G_1 runs through all connected graphs starting at 0, and G runs through all graphs that has G_1 as its first component.

Now we can categorize G_1 according to its lace representative and use (2.3)-(2.6) to get

$$\begin{aligned} K_\tau[0,T] &= \sum_{j=0}^{T}\{\sum_{N=1}^{\infty} \sum_{L\in\mathcal{L}_\tau^N[0,j]} \prod_{st\in L} U_{st} \prod_{st\in\in\mathcal{C}(L)} (1+U_{st})\}K_\tau[j,T] \\ &= \sum_{j=0}^{T} J_\tau[0,j]K_\tau[j,T] \\ &= K_\tau[1,T] + \sum_{j=1}^{T} J_\tau[0,j]K_\tau[j,T]. \end{aligned}$$

Q.E.D.

The following definitions will be of fundamental importance.

For $x \in Z^d$, we denote

$$N_\tau(x,T) = E(K_\tau[0,T]I_{(W_T=x)}) \qquad N_\tau(x,0) = \delta_{0,x} \tag{2.9}$$

$$\hat{N}_\tau(k,T) = \sum_{x\in Z^d} \exp\{ik\cdot x\}N_\tau(x,T), \qquad k\in[-\pi,\pi]^d. \tag{2.10}$$

For $z \in C$, the so called two point function is defined by

$$N_\tau(x,z) = \sum_{T=0}^{\infty} E(K_\tau[0,T]I_{(W_T=x)})z^T. \tag{2.11}$$

The Fourier-Laplace transformation of $N_\tau(x,T)$ is given by

$$\hat{N}_\tau(k,z) = \sum_{x\in Z^d} \exp\{ik\cdot x\}N_\tau(x,z)z^T, \qquad k\in[-\pi,\pi]^d. \tag{2.12}$$

For $\tau = 0$, $\hat{N}_0(k,z)$ can be computed explicitly.

$$\begin{aligned}
\hat{N}_0(k,z) &= \sum_{T=0}^{\infty}\sum_{x\in Z^d} \exp\{ik\cdot x\}E\{I_{(W(T)=x)}\}z^T \\
&= \sum_{T=0}^{\infty}\{E\{e^{ik\cdot W(1)}\}z\}^T \\
&= \frac{1}{1-z\hat{D}(k)}.
\end{aligned} \tag{2.13}$$

where $\hat{D}(k) = \sum_{y\in Z^d} \exp\{ik\cdot y\}E(I_{(W_1=y)})$ for $k\in[-\pi,\pi]^d$.

We denote the convergence radius of $\hat{N}_\tau(k,z)$ by $r_\tau(k)$. It is easy to see $r_\tau(k) \ge r_\tau(0) \ge r_0(0) = 1$.

Note that from (2.9) to (2.12), we use the same N to denote different functions. They are distinguished from each other by their arguments. We will also denote the Fourier transform of a function N by $\hat{N}$.

Similar to the above definitions, let us define the Laplace and Fourier-Laplace transformation on J_τ:

$$\Pi_\tau^N(x,z) = (-1)^N \sum_{T=1}^{\infty} E(J_\tau^N[0,T]I_{(W(T)=x)})z^T, \tag{2.14}$$

$$\Pi_\tau(x,z) = \sum_{N=1}^{\infty}(-1)^N \Pi_\tau^N(x,z), \tag{2.15}$$

$$\hat{\Pi}_\tau(k,z) = \sum_{x\in Z^d} \exp\{ik\cdot x\}\Pi_\tau(x,z). \tag{2.16}$$

From (2.7), we have $J_\tau[0,T] \leq K_\tau[0,T]$. Thus the convergence radius of $\hat{\Pi}_\tau(k=0,z)$ is no less than $r_\tau(0)$. We will see in our application problems that the former is actually strictly bigger than the latter.

Theorem 2.1 *(Brydges and Spencer)*

For any z such that $\hat{\Pi}_\tau(k=0,z)$ and $\hat{N}_\tau(x,z)$ converges absolutely, we have

$$\begin{aligned} N_\tau(x,z) &= \delta_{0,x} + z\sum_{y\in Z^d} N_\tau(x-y,z)E(I_{(W_1=y)}) \\ &\quad + \sum_{v\in Z^d}\Pi_\tau(v,z)N_\tau(x-v,z), \end{aligned} \tag{2.17}$$

$$\hat{N}_\tau(k,z) = \frac{1}{1 - z\hat{D}(k) - \hat{\Pi}_\tau(k,z)}. \tag{2.18}$$

Proof: For $T \geq 1$, by (2.7) we have

$$
\begin{aligned}
N_\tau(x,z) &= \delta_{0,x} + \sum_{T=1}^{\infty} E(K_\tau[0,T] I_{(W_T=x)}) z^T \\
&= \delta_{0,x} + \sum_{T=1}^{\infty} E\{\{K_\tau[1,T] + \sum_{j=1}^{T} J_\tau[0,j] K_\tau[j,T]\} I_{(W_T=x)}\} z^T \\
&= \delta_{0,x} + z \sum_{y \in Z^d} \sum_{T=1}^{\infty} E(K[1,T] I_{(W_1=y, W_T=x)}) z^{T-1} \\
&\quad + \sum_{v \in Z^d} \sum_{T=1}^{\infty} E\{\sum_{j=1}^{T} J[0,j] I_{(W_j=v)} K[j,T] I_{(W_T=x)}\} z^T \\
&= \delta_{0,x} + z \sum_{y \in Z^d} N_\tau(x-y,z) E(I_{(W_1=y)}) \\
&\quad + \sum_{v \in Z^d} \sum_{T=1}^{\infty} \sum_{j=1}^{T} E\{J[0,j] I_{(W_j=v)}\} z^j E^v\{K[j,T] I_{(W_T=x)}\} z^{T-j} \\
&= \delta_{0,x} + z \sum_{y \in Z^d} N_\tau(x-y,z) E(I_{(W_1=y)}) \\
&\quad + \sum_{v \in Z^d} \Pi_\tau(v,z) N_\tau(x-v,z). \qquad (2.19)
\end{aligned}
$$

In the above reasoning, the penultimate equality came from the independent increment property of the random walk.

Taking Fourier transform on both sides of (2.18), we have

$$\hat{N}_\tau(k,z) = 1 + \hat{N}_\tau(k,z)\hat{D}(k)z + \hat{\Pi}_\tau(k,z)\hat{N}_\tau(k,z). \qquad (2.20)$$

This leads to (2.18).

Q.E.D.

By comparing (2.18) and (2.13) we observe that $\hat{\Pi}_\tau(k,z)$ characterizes how self-avoiding walk deviates from the simple random walk. If the self-avoiding walk is very close to the simple random walk, then we expect $\hat{\Pi}_\tau(k,z)$ to be very small.

2.2 The Feynman Diagram

In order to estimate $\hat{\Pi}_\tau(k,z)$, we need an efficient way of computing the $J_\tau^N[a,b]$ as defined in (2.4). Feynman diagram provides us the right tool in this aspect.

We observe that for a fixed $L \in \mathcal{L}_\tau^N[a,b]$, in order that a random walk ω contribute to the summation in the definition of $J_\tau^N[a,b]$, it must satisfy the following:

$$\omega(s) = \omega(t), \text{ for all } st \in L; \quad \omega(s) \neq \omega(t), \text{ for all } st \in \mathcal{C}(L). \tag{2.21}$$

A careful examination of the lace structure as in Figure 2.1 will convince us that ω must follow a path as shown in Figure 2.2.

Figure 2.2 illustrates the Feynman diagrams up to $N = 6$. The edges in the graph represents sub-walks, which are numbered according to their occurring precedence. The solid edges represent those sub-walks that have distinct terminal points; the dashed edges represent sub-walks that might have same starting and terminating points. For a diagram of order N, the points are numbered by $x_1, \ldots, x_{2N-1}$. For a given sub-walk i, its terminal points are denoted by $n(i-1)$ and $n(i)$ which are chosen from $x_1, \ldots, x_{2N-1}$.

Proposition 2.2 *Let $N \geq 1$ be fixed, $f_1, f_2, \ldots, f_{2N-1}$ be nonnegative even functions defined on Z^d. Define*

$$F_N(x) = f_1 * f_2 f_3 * f_4, \ldots, * f_{2N-2} f_{2N-1}(x), \quad x \in Z^d,$$

where $*$ represents discrete convolution operator on Z^d. Then we have:

$$\sum_{x_1, x_2 \ldots x_{N-1}} \prod_{i=1}^{2N-1} f_i(n(i) - n(i-1)) = F_N(0). \tag{2.22}$$

Proof: For the case $N = 2$, we have

$$\sum_{x_1} f_1(x_1) f_2(x_1) f_3(x_1) = f_1 * f_2 f_3(0).$$

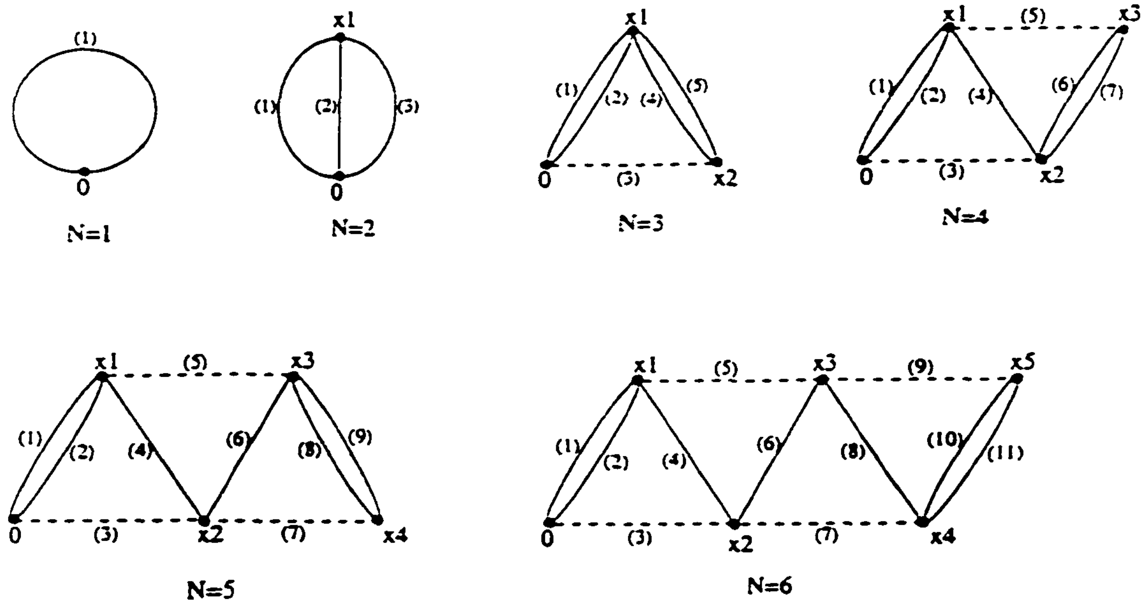

Figure 2.2: Feynman Diagrams

Proposition 2.2 *Let $N \geq 1$ be fixed, $f_1, f_2, \ldots, f_{2N-1}$ be nonnegative even functions defined on Z^d. Define*

$$F_N(x) = f_1 * f_2 f_3 * f_4, \ldots, * f_{2N-2} f_{2N-1}(x), \quad x \in Z^d,$$

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$$\sum_{x_1, x_2 \ldots x_{N-1}} \prod_{i=1}^{2N-1} f_i(n(i) - n(i-1)) = F_N(0). \tag{2.22}$$

Proof: For the case $N = 2$, we have

$$\sum_{x_1} f_1(x_1) f_2(x_1) f_3(x_1) = f_1 * f_2 f_3(0).$$

Suppose we know (2.22) holds true for $N-1$, we consider the case N:

$$\sum_{x_1 \ldots x_{N-1}} f_1(x_1) \ldots f_{2N-4}(x_{2N-2} - x_{2N-3})$$
$$f_{2N-3}(x_{N-1} - x_{N-3}) f_{2N-2}(x_{N-1} - x_{N-2}) f_{2N-1}(x_{N-1} - x_{N-2})$$

Clearly, for any fixed f on Z^d,

$$\|f\|_\infty \le \|f\|_2 \le \|f\|_1. \tag{2.23}$$

Proposition 2.3 *Let f, g be functions defined on Z^d, we have the following inequalities hold true:*

$$\|f * g\|_\infty \le \|f\|_1\|g\|_\infty, \tag{2.24}$$

$$\|f * g\|_\infty \le \|f\|_2\|g\|_2, \tag{2.25}$$

$$\|f * g\|_2 \le \|f\|_1\|g\|_2. \tag{2.26}$$

Proof: We only prove (2.26).

$$\begin{aligned}
\|f * g\|_2 &\le \sum_{x\in Z^d}\sum_{y\in Z^d}(|f(y)||g(x-y)|)^2 \\
&\le \sum_{x\in Z^d}\sum_{y\in Z^d}(|f(y)|^{\frac{1}{2}})^2\sum_{y\in Z^d}(|f(y)|^{\frac{1}{2}}g(x-y))^2 \\
&= \{\sum_{y\in Z^d}|f(y)|\}^2\sum_{x\in Z^d}g(x)^2 \\
&= \|f\|_1^2\|g\|_2^2.
\end{aligned}$$

Q.E.D.

Proposition 2.4 *For fixed $N \ge 1$, let $f_1, f_2, \ldots, f_{2N-1}$ and F_N be defined as in Proposition 2.2, we have*

$$F_N(0) \le \|f_\alpha\|_\infty \prod_{i=1,i\neq\alpha}^{2N-1} \|f_i\|_2, \tag{2.27}$$

where α could be any integer between 1 and $2N-1$.

Proof: For the case α being odd, we have

$$
\begin{aligned}
F_N(0) &\leq \|f_1\|_2\|f_2 f_3 * f_4 \cdots f_{2N-1}\|_2 \qquad \text{(by (2.25))} \\
&\leq \|f_1\|_2\|f_2\|_2\|f_3 * f_4 \cdots f_{2N-1}\|_\infty \\
&\leq \|f_1\|_2\|f_2\|_2\|f_3\|_2\|f_4 f_5 \cdots f_{2N-1}\|_2 \\
&\leq \cdots\cdots \\
&\leq \prod_{i=1}^{\alpha-1}\|f_i\|_2\|f_\alpha * f_{\alpha+1} \cdots f_{2N-1}\|_\infty \\
&\leq \prod_{i=1}^{\alpha-1}\|f_i\|_2\|f_\alpha\|_\infty\|f_{\alpha+1} f_{\alpha+2} \cdots f_{2N-1}\|_1 \qquad \text{(by (2.24))} \\
&\leq \prod_{i=1}^{\alpha-1}\|f_i\|_2\|f_\alpha\|_\infty\|f_{\alpha+1}\|_2\|f_{\alpha+2} * f_{\alpha+3} \cdots f_{2N-1}\|_2 \\
&\leq \prod_{i=1, i\neq\alpha}^{\alpha+1}\|f_i\|_2\|f_\alpha\|_\infty\|f_{\alpha+2}\|_2\|f_{\alpha+3} \cdots f_{2N-1}\|_1 \qquad \text{(by (2.26))} \\
&\leq \prod_{i=1, i\neq\alpha}^{\alpha+2}\|f_i\|_2\|f_\alpha\|_\infty\|f_{\alpha+3}\|_2\|f_{\alpha+4} * \cdots f_{2N-1}\|_2 \\
&\leq \cdots\cdots \\
&\leq \prod_{i=1, i\neq\alpha}^{2N-1}\|f_i\|_2\|f_\alpha\|_\infty.
\end{aligned}
$$

If α is even, we have

$$
\begin{aligned}
F_N(0) &= \prod_1^{\alpha-1}\|f_i\|_2\|f_\alpha f_{\alpha+1} \cdots f_{2N-1}\|_2 \qquad \text{(same as } \alpha \text{ odd case)} \\
&\leq \prod_1^{\alpha-1}\|f_i\|_2\|f_\alpha\|_\infty\|f_{\alpha+1} * f_{\alpha+2} \cdots f_{2N-1}\|_2 \\
&\leq \cdots\cdots \\
&\leq \prod_{i=1, i\neq\alpha}^{2N-1}\|f_i\|_2\|f_\alpha\|_\infty. \qquad \text{(same as } \alpha \text{ odd case)}
\end{aligned}
$$

Q.E.D.

Let us denote

$$
G_\tau^\alpha(x,z) = \sum_{T=\alpha}^{\tau-1} N_\tau(x,T)z^T, \qquad \alpha = 0, 1. \tag{2.28}
$$

The following proposition provides estimates on $\hat{\Pi}_\tau(k,z)$ using the norms of $G_\tau^\alpha(x,z)$.

Proposition 2.5

$$|\hat{\Pi}_\tau(k,z)| \leq \|G_\tau^1(x,z)\|_\infty |z| + \|G_\tau^1(x,z)\|_\infty \cdot \sum_{N=2}^{\infty} \|G_\tau^1(x,z)\|_2^N \|G_\tau^0(x,z)\|_2^{N-2} \quad (2.29)$$

$$|\partial_z \hat{\Pi}_\tau(k,z)| \leq \|\frac{\partial}{\partial z}(zG_\tau^1(x,z))\|_\infty + \sum_{N=2}^{\infty} CN \|\partial_z G_\tau^1(x,z)\|_\infty \|G_\tau^1(x,z)\|_2^N \cdot \|G_\tau^0(x,z)\|_2^{N-2} \quad (2.30)$$

$$|\partial_k \hat{\Pi}_\tau(k,z)| \leq \sum_{N=2}^{\infty} N \|xG_\tau^1(x,z)\|_\infty \|G_\tau^1(x,z)\|_2^N \cdot \|G_\tau^0(x,z)\|_2^{N-2} \quad (2.31)$$

$$|\partial_z \partial_k \hat{\Pi}_\tau(k,z)| \leq \sum_{N=2}^{\infty} CN^2 \|xG_\tau^1(x,z)\|_\infty \|G_\tau^1(x,z)\|_2^{N-1} \cdot \|G_\tau^0(x,z)\|_2^{N-2} \|\partial_z G_\tau^1(x,z)\|_2 \quad (2.32)$$

Proof: We first prove (2.29).

By definition,

$$|\hat{\Pi}_\tau(k,z)| \leq \sum_{N=1}^{\infty} \sum_{T=1}^{\infty} \sum_{x} E(|J_\tau^N[0,T]| I_{(W_T=x)}) z^T.$$

For the case $N=1$, we observe that there is only one lace graph in the set

$\mathcal{L}^1_\tau[0,T]$, and moreover, to make $U_{0T} = -1$, we must have $T \geq 2$. Thus

$$\begin{aligned}
\hat{\Pi}^1_\tau(k,z) &\leq \sum_{T=2}^{\infty} |E\{ \sum_{L\in\mathcal{L}^1_\tau[0,T]} \prod_{st\in L} U_{st} \prod_{s't'\in\mathcal{C}(L)} (1+U_{s't'}) I_{(W_T=0)}\} z^T| \\
&\leq \sum_{T=2}^{\tau} E\{ \prod_{s't'\in\mathcal{C}(L)} (1+U_{s't'}) I_{(W_T=0)}\} |z|^T \\
&= \sum_{x\neq 0} \sum_{T=2}^{\tau} E\{ I_{(W_1=x)} E[\prod_{s't'\in\mathcal{C}(L)} (1+U_{s't'}) I_{(W_T=0)} || X_1]\} |z|^T \\
&\leq |z| \sum_{x\neq 0} E\{ I_{(W_1=x)} \sum_{T=1}^{\tau-1} E(K[0,T] I_{(W_T=x)} |z|^T\} \\
&\leq |z| \| G^1_\tau(x,z) \|_\infty .
\end{aligned}$$

For $N \geq 2$, we have:

$$|\hat{\Pi}^N_\tau(k,z)| \leq \sum_{x_1,\ldots,x_{N-1}\in Z^d} \sum_{T=3}^{\infty} \sum_{T_1,\ldots,T_{2N-1}} E\{ \prod_{st\in\mathcal{C}(L)} (1+U_{st}) I_{A(x,N,T)}\} z^T, \quad (2.33)$$

where $0 < T_1 \leq T_2 \leq \cdots \leq T_{2N-1} = T$ are possible terminal times for lace graphs having $x_1, \ldots, x_{N-1}$ as its terminal locations; L is the particular lace that has $T_1 \leq T_2 \leq \cdots \leq T_{2N-1} = T$ as its time terminals; and $A(x,N,T) = \{\omega : \omega \text{ passes } n(1), \ldots, n(N-1) \text{ at time } T_1 \leq T_2 \leq \cdots \leq T_{2N-1} = T\}$.

The expectation in (2.33) can be estimated by:

$$\begin{aligned}
E\{ \prod_{st\in\mathcal{C}(L)} (1+U_{st}) I_{A(x,N,T)}\} &\leq E\{ \prod_{i=1}^{2N-1} K[T_{i-1}, T_i] I_{A_i}\} \\
&= \prod_{i=1}^{2N-1} E\{K[T_{i-1}, T_i] I_{A_i}\} \\
&= \prod_{i=1}^{2N-1} N_\tau(n(i)-n(i-1), T_i - T_{i-1}),
\end{aligned}$$

where $A_i = \{\omega : \omega(T_i - T_{i-1}) = n(i) - n(i-1)\}$.

For fixed $x_1, \ldots, x_{N-1} \in Z^d$, let us denote

$$H_N(x_1,\ldots,x_{N-1}) = \sum_{T=3}^{\infty} \sum_{T_1,\ldots,T_{2N-1}} \prod_{i=1}^{2N-1} N_\tau(n(i)-n(i-1), T_i - T_{i-1}) z^{T_i - T_{i-1}}.$$

Taking into consideration the Feynman diagram, we observe that each of the terms in the summation of $H_N(x_1, \ldots, x_{N-1})$ belongs to the expansion of the product

$$\prod_{i=1}^{N+1} G_\tau^1(n(i) - n(i-1), z) \prod_{j}^{N-1} G_\tau^0(n(j) - n(j-1), z),$$

where the $N+1$ G_τ^1's correspond to the $N+1$ solid edges in the Feynman diagram and the $N-2$ G_τ^0's correspond to the $N-2$ dashed edges in the Feynman diagram. Thus by Proposition 2.4 we have

$$\begin{aligned} |\hat{\Pi}_\tau^N(k, z)| &\le \sum_{x_1, \ldots, x_{N-1} \in Z^d} H_N(x_1, \ldots, x_{N-1}) \\ &\le \sum_{x_1, \ldots, x_{N-1}} \prod_{i=1}^{N+1} G_\tau^1(n(i) - n(i-1), z) \prod_{j=1}^{N-2} G_\tau^0(n(j) - n(j-1), z) \\ &\le \|G_\tau^1(x, z)\|_\infty \|G_\tau^1(x, z)\|_2^N \|G_\tau^0(x, z)\|_2^{N-2}. \end{aligned}$$

Taking summation over N on $|\hat{\Pi}_\tau^N(k, z)|$ we obtain (2.29).

We can modify the above procedure to prove (2.30). The modification that has to be made is to multiply T in the z^{T-1} terms. We can estimate this T by taking $T = \sum_{i=1}^{2N-1}(T_i - T_{i-1})$ and assign each $T_i - Ti - 1$ to the corresponding sub-walks. Notice that $G_\tau^0(x, z) = \delta_{0x} + G_\tau^1(x, z)$, we can conclude that the sub-walk appended by $T_i - T_{i-1}$ is bounded by the term $\|\partial_z G_\tau^1(x, z)\|_\infty$.

To prove (2.31), we have to multiply the kth coordinate of x_{2N-1} in each of the terms. We can estimate this by taking $|x_{2N-1}| \le \sum_{i=1}^{N-1} |x_i - x_{i-1}|$ and assign each of the $|x_i - x_{i-1}|$'s into the corresponding solid edge related sub-walks.

Finally, we combine the previous methods to prove (2.32). Notice that $x_i G_\tau^0(x, z) = x_i G_\tau^0(x, z)$, we can always choose a sub-walk different from the one that is appended with $T_i - T_{i-1}$ and append it with $x_i - x_{i-1}$.

Q.E.D.

CHAPTER 3

HIGH DIMENSIONAL LIMITING DISTRIBUTION

In this chapter, we assume the i.i.d. random sequence $X_1, X_2, \ldots, X_n, \ldots$ satisfy the simple discrete Cauchy distribution:

$$P\{X_1 = \pm n e_j\} = \frac{3}{d\pi^2}\frac{1}{n^2}, \quad \text{for } n \in N \setminus \{0\}, \tag{3.1}$$

where $\{e_j : 1 \le j \le d\}$ are unit vectors in Z^d.

(The appendix contains detailed computation results on discrete Cauchy distribution.)

The probability that concentrates on those T-step self-avoiding walks with memory τ is given by

$$< \cdot >_{T,\tau} = \sum_x \cdot N_\tau(x,T) / \sum_x N_\tau(x,T). \tag{3.2}$$

Our goal is to show that there exists a sufficiently large dimension d_0, such that for $d \ge d_0$,

$$\lim_{T\to\infty} < \exp\{ik \cdot \sum_{j=1}^{T} X_j/T\} >_{T,T} = \exp\{-\frac{3}{d\pi}\sum_{j=1}^{d}|k_j|\}, \quad k \in [\pi,\pi]^d. \tag{3.3}$$

That is, the scaled random sum $\sum_{j=1}^{T} X_j/T$ converges weakly to the classical Cauchy distribution.

3.1 Simple Long Range Random Walks

For $d > T$, a $2T$-step walk that returns to the origin must stay in a T dimensional subspace in Z^d. Thus the following is true:

$$N_0(0, 2T) \le \binom{d}{T}\left(\frac{T}{d}\right)^{2T} \le \frac{T^{2T}}{T!d^T}. \tag{3.4}$$

For arbitrary $x \in Z^d$ and $n \in N$, we have

$$N_0(x, 2T+n) = \frac{1}{(2\pi)^d}\int \hat{D}(k)^{2T+n} e^{ik\cdot x} dk \le \int \hat{D}(k)^{2T} dk = N_0(0, 2T). \tag{3.5}$$

(Note: Unless otherwise stated, the integration domain is $[-\pi, \pi]^d$.)

We will also use the following lower bound of $1 - \hat{D}(k)$:

$$1 - \hat{D}(k) \ge \frac{3|k|^2}{2d\pi^2}, \quad k \in [-\pi, \pi]^d. \tag{3.6}$$

Proposition 3.1 *For any integer $m \ge 0$, we have*

$$\sum_{T=1}^{\infty} T^3 N_0(0, 2T) \le O(d^{-1}), \quad d \to \infty.$$

Proof: From (3.4) (3.5), we have

$$\begin{aligned} \sum_{T=1}^{3} T^3 N_0(0, 2T) &\le O(d^{-1}), \\ \sum_{T=4}^{d-1} T^3 N_0(0, 2T) &\le (d-5)(d-1)^3 N_0(0, 8) \le O(d^{-1}). \end{aligned}$$

For $|\rho| < 1$, we will need the following estimation:

$$\begin{aligned} \sum_{T=d}^{\infty} 2T(2T-1)(2T-2)\rho^{2T-3} &\le \sum_{i=2d}^{\infty} i(i-1)(i-2)|\rho|^{i-3} \\ &= \left(\sum_{i=2d}^{\infty} x^i\right)^{(3)}\Big|_{x=|\rho|} \\ &\le Cd^3 \frac{|\rho|^{2d-3}}{(1-|\rho|)^4}. \end{aligned}$$

Now we compute the case $T \geq d$,

$$\begin{aligned}
\sum_{T=d}^{\infty} T^3 N_0(0,2T) &= \frac{1}{(2\pi)^d}\int \sum_{T=d}^{\infty} T^3 \hat{D}(k)^{2T} dk \\
&\leq \frac{C}{(2\pi)^d}\int \sum_{T=d}^{\infty}(2T)(2T-1)(2T-2)\hat{D}(k)^{2T-3}\hat{D}(k)^3 dk \\
&\leq \frac{Cd^3}{(2\pi)^d}\int |\hat{D}(k)|^3 \frac{|\hat{D}(k)|^{2d-3}}{(1-|\hat{D}(k)|)^4} dk \\
&\leq \frac{Cd^3}{(2\pi)^d}\int \exp\{-\frac{1}{d}\sum_{i=1}^{d}(\frac{3|k_i|}{\pi}-\frac{3k_i^2}{2\pi^2})(2d)\} \\
&\qquad \cdot\{\frac{1}{d}\sum_{i=1}^{d}(\frac{3|k_i|}{\pi}-\frac{3k_i^2}{2\pi^2})\}^{-4} dk \\
&\leq \frac{Cd^7}{(2\pi)^d}\int \exp\{-\frac{3|k|^2}{\pi^2}\}(\frac{3|k|^2}{2\pi^2})^{-4} dk \\
&\leq \frac{Cd^7}{(2\pi)^d}\int_0^{\infty} \exp\{\frac{-3\rho^2}{\pi^2}\}\rho^{-8}\rho^{d-1} d\rho\omega_{d-1} \\
&\leq \frac{Cd^7}{(2\pi)^d}\frac{d\pi^{d/2}}{\Gamma(\frac{d+2}{2})}\Gamma(\frac{d}{2}-5) \\
&\leq O(d^{-1}).
\end{aligned}$$

Q.E.D.

Proposition 3.2 *For* $|z| \leq 1$, $\nu = 0, 1$

$$\|\partial_z^{\nu}\sum_{T=1}^{\infty} N_0(x,T)z^T\|_{\infty} \leq O(d^{-1}), \tag{3.7}$$

$$\|\partial_z^{\nu}\sum_{T=1}^{\infty} N_0(x,T)z^T\|_2^2 \leq O(d^{-1}). \tag{3.8}$$

Proof:

$$\begin{aligned}
|\partial_z \sum_{T=1}^{\infty} N_0(x,T)z^T| &\leq \sum_{T=1}^{\infty} T N_0(x,T) \\
&\leq N_0(x,1) + \sum_{n=0}^{\infty}(2n+1)N_0(x,2n+1) \\
&\qquad + \sum_{n=1}^{\infty}(2n)N_0(x,2n).
\end{aligned}$$

Clearly $N_0(x,1) \leq O(d^{-1})$, $N_0(x,2n+1) \leq N_0(0,2n)$ and $N_0(x,2n) \leq N_0(0,2n)$. The previous proposition leads us to (3.8).

$$\begin{aligned}
\|\partial_z \sum_{T=1}^{\infty} N_0(x,T)z^T\|_2^2 &\leq \|\sum_{T=1}^{\infty} TN_0(x,T)\|_2^2 = \sum_{S,T=1}^{\infty}\sum_x STN_0(x,S)N_0(x,T) \\
&= \sum_{n=2}^{\infty}\sum_{S+T=n}(ST)N_0(0,n) \leq \sum_{n=2}^{\infty} n^3 N_0(0,n) \\
&\leq O(d^{-1}).
\end{aligned}$$

Q.E.D.

Proposition 3.3 *For* $|z| \leq 1$

$$1 \leq \|N_\tau(x,z)\|_2^2 \leq 1 + O(d^{-1}) \tag{3.9}$$

Proof:

$$\begin{aligned}
\|N_\tau(x,z)\|_2^2 &\leq \|\delta_{0,x} + \sum_{T=1}^{\infty} N_0(x,T)\|_2^2 \\
&= \sum_{x\in Z^d}\{\delta_{0,x}^2 + 2\delta_{0,x}\sum_{T=1}^{\infty} N_0(x,T) + [\sum_{T=1}^{\infty} N_0(x,T)]^2\} \\
&= 1 + 2\sum_{T=1}^{\infty} N_0(0,T) + \|\sum_{T=1}^{\infty} N_0(x,T)\|_2^2
\end{aligned}$$

An application of Proposition (3.1), (3.2) completes the proof.

Q.E.D.

Proposition 3.4 *For fixed integer* $m \geq 1$

$$\sup_{d\geq 1}\{\frac{1}{(2\pi)^d}\int (1-\hat{D}(k))^{-m} dk\} < \infty. \tag{3.10}$$

Proof:

$$\begin{aligned}
\frac{1}{(2\pi)^d}\int (1-\hat{D}(k))^{-m} dk &\leq C\cdot\frac{1}{(2\pi)^d}\int_0^{\sqrt{d}} (\frac{d}{\rho^2})^m \rho^{d-1} d\rho\, \omega_{d-1} \\
&\quad + \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d\setminus\{k:|k|\leq\sqrt{d}\}} (1-\hat{D}(k))^{-m} dk \\
&= I_1 + I_2.
\end{aligned}$$

$$
\begin{aligned}
I_1 &= C\frac{1}{(2\pi)^d}\frac{d\pi^{d/2}}{\Gamma(\frac{d+2}{2})}d^m\int_0^{\sqrt{d}}\rho^{d-1-2m}d\rho \leq O(1), \\
I_2 &\leq C\frac{1}{(2\pi)^d}\cdot\int_{[-\pi,\pi]^d-\{k:|k|\leq\sqrt{d}\}}(\frac{d}{|k|^2})^m dk \leq O(1),
\end{aligned}
$$

Q.E.D.

Proposition 3.5 *For* $|z| \leq 1$, $\nu = 0, 1$

$$
\||x|\partial_z^\nu N_0(x,z)\|_\infty \leq O(d^{-1}) \tag{3.11}
$$

Proof: It suffices to prove the case $\nu = 1$. Let $0 < \rho < 1$, $\partial_z N_0(x,\rho)$ is summable w.r.t. x and thus has its Fourier transform:

$$
\partial_z \hat{N}_0(k,\rho) = \frac{\hat{D}(k)}{(1-\rho\hat{D}(k))^2}.
$$

Although $\partial_z \hat{N}_0(k,\rho)$ is not differentiable at the origin, it is Lipschitz continuous w.r.t. each k. It has a.s. derivative w.r.t. k_1

$$
\partial_{k_1}\partial_z\hat{N}(k,\rho) = \frac{\partial_{k_1}\hat{D}(k)}{(1-\rho\hat{D}(k)^2} + \frac{2\rho\hat{D}(k)\,\partial_{k_1}\hat{D}(k)}{(1-\rho\hat{D}(k))^3}.
$$

We can apply the integration by part formulae in the following calculation.

$$
\begin{aligned}
ix_1\partial_z N_0(x,\rho) &= \frac{1}{(2\pi)^d}\int ix_1\exp(-ik\cdot x)\partial_z\hat{N}_0(k,\rho)dk \\
&= \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^{d-1}}\exp(-i\bar{k}\cdot\bar{x})d\bar{k}\int_{-\pi}^{\pi}ix_1e^{-ik_1x_1}\partial_z\hat{N}_0(k,\rho)dk_1 \\
&= \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^{d-1}}\exp(-i\bar{k}\cdot\bar{x})\{-ie^{-ik_1x_1}\partial_z\hat{N}_0(k,\rho)|_{-\pi}^{\pi} \\
&\quad+\int_{-\pi}^{\pi}e^{-ik_1x_1}\partial_{k_1}\partial_z\hat{N}_0(k,\rho)dk_1\}d\bar{k} \\
&= \frac{1}{(2\pi)^d}\int\exp(-ik\cdot x)\partial_{k_1}\partial_z\hat{N}_0(k,\rho)dk
\end{aligned}
$$

Applying Proposition (3.4) and using the fact $\partial_{k_1}\hat{D}(k) \leq O(d^{-1})$ we arrive at the result.

Q.E.D.

3.2 The Convergence of Lace Expansion

Proposition 3.6 *For fixed $\tau < \infty$, the norms $||G^1_\tau(x,\rho)||_2$, $||\partial_z G^1_\tau(x,\rho)||_2$, $|||x|^u \partial^\nu_z G^1_\tau(x,\rho)||_\infty$ ($\nu = 0, 1, |u| \leq 1$) are continuous w.r.t ρ if $\rho \leq r_\tau(0)$.*

Proof: For the two L_2 norms, we only prove the second one. By definition

$$\begin{aligned}
\sum_{x\in Z^d}\{\partial_z G^1_\tau(x,\rho)\}^2 &= \sum_{x\in Z^d}\sum_{T_1,T_2=1}^{\tau-1} T_1T_2N_\tau(x,T_1)N_\tau(x,T_2)\rho^{T_1+T_2-2} \\
&\leq \sum_{n=0}^{\infty}\sum_{j=0}^{n}(\sum_x (j+1)N_\tau(x,j)\rho^j) \\
&\qquad (\sum_x (n-j+1)N_\tau(x,n-j)\rho^{n-j}) \\
&\leq \{\sum_{n=0}^{\infty}(n+1)N_\tau(k=0,n)\rho^n\}^2.
\end{aligned}$$

Thus for $\rho < r_\tau(0)$, $||\partial_z G^1_\tau(x,\rho)||_2$ is finite and continuous w.r.t. ρ. Moreover, notice that the coefficients of the above series are nonnegative, the norm $||\partial_z G^1_\tau(x,\rho)||_2$ is actually left continuous at $r_\tau(0)$.

For the infinite norm $|||x|^u \partial^\nu_z G^1_\tau(x,\rho)||_\infty$, we only prove the case $\nu = 1$.

$$\begin{aligned}
||x_1\partial_z G^1_\tau(x,\rho)||_\infty &= \sup_{x\in Z^d}|x_1|\partial_z G^1_\tau(x,\rho) \\
&= \sup_{x\in Z^d}|x_1|\sum_{T=1}^{\tau} TN_\tau(x,T)\rho^{T-1} \qquad (3.12)
\end{aligned}$$

Since $TN_\tau(x,T) \geq 0$ and ρ^{T-1} are convex functions w.r.t. ρ for $0 \leq \rho \leq r_\tau(0)$, $\sum_{T=1}^{\tau} TN_\tau(x,T)\rho^{T-1}$ are also convex functions(for each fixed x). Thus (3.12) is also a convex function w.r.t. ρ. We suffice to show (3.12) is finite for $\rho \leq r_\tau(0)$.

$$\begin{aligned}
\sup_{x\in Z^d}|x_1|\sum_{T=1}^{\tau} TN_\tau(x,T)\rho^{T-1} &\leq \sum_{T=1}^{\tau}\sup_{x\in Z^d}|x_1|TN_\tau(x,T)\rho^{T-1} \\
&\leq \sum_{T=1}^{\tau}\sup_{x\in Z^d}|x_1|TN_0(x,T)\rho^{T-1} \\
&\leq \sum_{T=1}^{\tau}\frac{Tr_\tau(0)^{T-1}}{(2\pi)^d}\int T|\hat{D}(k)|^{T-1}|\partial_{k_1}\hat{D}(k)|dk \\
&< \infty
\end{aligned}$$

Notice that in the above derivation, the assumption $\tau < \infty$ is necessary.

Q.E.D.

Lemma 3.1 *For a sequence of nonnegative numbers $(a_n)_{n=1}^{\infty}$ satisfying $a_{n+m} \le a_n a_m$ for all $n, m \ge 0$, we have*

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = \inf_{n\ge 1} a_n^{\frac{1}{n}} \tag{3.13}$$

Proof: Without lose of generality, we assume $a_n > 0$ for all $n \ge 1$. So we can denote $b_n = \log a_n$, then $b_{n+m} \le b_n + b_m$ for all $n, m \ge 1$.

For any $\epsilon > 0$, choose N such that $b_N \le \inf_{n\ge 1} b_n/n + \epsilon$. Let $n = kN + r$ with $0 \le r \le N - 1$. Then

$$\frac{b_n}{n} \le \frac{kb_N + b_r}{n} = \frac{kN}{n}\frac{b_N}{N} + \frac{b_r}{n} \to \frac{b_N}{N}, \quad \text{as } n \to \infty.$$

Q.E.D.

Proposition 3.7 *Let $\chi_\tau(z) = \sum_{T=0}^{\infty} E\{K_\tau[0,T]\}z^T$. Then for $0 \le \rho < r_\tau(0)$, $\chi(\rho) < \infty$, and $\chi(r_\tau(0)) = \infty$.*

Proof: We verify that the sequence $\{E\{K_\tau[0,T]\}\}_{T=1}^{\infty}$ possess property (3.13):

$$\begin{aligned} E\{K_\tau[0,T_1+T_2]\} &\le E\{K_\tau[0,T_1]K_\tau[T_1,T_1+T_2]\} \\ &= \sum_{x\in Z^d} E\{E^{W(x)}(K_\tau[0,T_2])\, K_\tau[0,T_1]I_{(W(T_1)=x)}\} \\ &= EK_\tau[0,T_1]E\{K_\tau[0,T_2]\}. \end{aligned}$$

It follows $\lim_{n\to\infty} EK_\tau[0,T]^{\frac{1}{n}} = \mu$ exists and $0 < \mu < \infty$, $\chi(\rho) < \infty$ for $0 \le \rho < r_\tau(0)$. Moreover from Lemma 3.1 we know $E\{K_\tau[0,T]\} \ge \mu^T$ for all T, which implies $\chi(r_\tau(0)) = \infty$.

Q.E.D.

Theorem 3.1 *There exists a universal dimension d_0 such that for all $d \ge d_0$, $\rho \in [0, r_\tau(0)]$ and $\tau > 0$,*

statement $P_4 \Longrightarrow$ statement P_2.

Here $P_a (a = 2, 4)$ is the following assertion:

$$P_a : \begin{cases} \|\partial_z^\nu G_\tau^1(x,\rho)\|_2^2 \le ak_0 d^{-1}, & \nu = 0,1, \\ \||x^u|\partial_z^\nu G_\tau^1(x,\rho)\|_\infty \le ak_0 d^{-1}, & |u| \le 1, \nu = 0,1, \end{cases} \tag{3.14}$$

where k_0 is a universal constant that does not depend on τ or d.

Proof: From Proposition 3.2, 3.5 and the fact $G_\tau^1(x,\rho) \le G_0^1(x,\rho)$, we know it suffices to consider the case $\rho \in [1, r_\tau(0)]$. Let us define

$$F_\tau(k,z) = 1 - z\hat{D}(k) - \hat{\Pi}_\tau(k,z), \tag{3.15}$$

then for $1 \le \rho \le r_\tau(0)$ we have

$$F_\tau(k,\rho) = 1 - \rho\hat{D}(0) - \hat{\Pi}_\tau(0,\rho) + \rho(1 - \hat{D}(k)) + \hat{\Pi}_\tau(0,\rho) - \hat{\Pi}_\tau(k,\rho) \tag{3.16}$$

By (2.18) and Lemma (4.2), we know $F_\tau(0,\rho) = 1 - \rho\hat{D}(0) - \hat{\Pi}_\tau(0,\rho) \ge 0$ for $\rho \in [1, r_\tau(0)]$ and the equality holds iff $\rho = r_\tau(0)$.

$$|\hat{\Pi}_\tau(0,\rho) - \hat{\Pi}_\tau(k,\rho)| \le |\nabla_k \hat{\Pi}_\tau(\theta k,\rho)|\,|k| \qquad 0 < \theta < 1 \tag{3.17}$$

Our assumption P_4 and (2.31) imply that $|\nabla_k \hat{\Pi}_\tau(\theta k,\rho)| \le O(d^{-2})$. (In this proof, the constant C will refer to a quantity that does not depend on the k_0, τ or d, while the quantity $O(\cdot)$ my depend on k_0.)

From the fact that

$$\frac{3|k|}{2\pi d} \le 1 - \hat{D}(k) \le \frac{3|k|}{2\pi\sqrt{d}} \tag{3.18}$$

we conclude

$$F_\tau(k,\rho) \ge \rho(1 - \hat{D}(k)) - O(d^{-2}) \cdot |k| \ge C(1 - \hat{D}(k)) \tag{3.19}$$

Notice that we may require a bigger d_0 to guarantee the inequality (3.19) holds true, with the constant C irrelevant to k_0, τ and d. However, this increasing of d_0 is done in a deterministic manner, and it depends only on k_0. Our future deductions will be correct as long as they do not depend on the choosing of k_0.

We now show that $||\partial_z^\nu G_\tau^1(x,\rho)||_2^2 \le 2k_0 d^{-1}$, with $\nu = 0, 1$.

For $\nu = 1$ we have

$$\begin{aligned} ||\partial_z G_\tau^1(x,\rho)||_2^2 &\le ||\partial_z N_\tau(x,\rho)||_2^2 \\ &= \frac{1}{(2\pi)^d}||\partial_z \hat{N}_\tau(k,\rho)||_2^2 \\ &= \frac{1}{(2\pi)^d}\int\{\frac{\hat{D}(k)+\partial_z\Pi_\tau(k,\rho)}{F_\tau(k,\rho)^2}\}^2 dk \\ &= \frac{1}{(2\pi)^d}\int\frac{\hat{D}(k)^2+2\hat{D}(k)\partial_z\Pi_\tau(k,\rho)+\partial_z\Pi_\tau(k,\rho)^2}{F_\tau(k,\rho)^4}dk \end{aligned}$$

We will estimate the terms in the right side integral separately. The first term is estimated by:

$$\begin{aligned} \frac{1}{(2\pi)^d}\int\frac{\hat{D}(k)^2}{F_\tau(k,\rho)^4}dk &\le C\frac{1}{(2\pi)^d}\int\frac{\hat{D}(k)^2}{(1-\hat{D}(k))^4}dk \\ &= ||\partial_z N_0(x,\rho=1)||_2^2 \\ &\le Cd^{-1}. \end{aligned}$$

To estimate the second term, we notice that by (2.31) and our assumption P_4,

$$|\partial_z\Pi_\tau(k,\rho)| \le O(d^{-1}). \quad \rho \le r_\tau(0), k \in [-\pi,\pi]^d.$$

Thus we have

$$\begin{aligned} \frac{1}{(2\pi)^d}\int\frac{2\hat{D}(k)\partial_z\Pi_\tau(k,\rho)}{F_\tau(k,\rho)^4}dk &\le O(d^{-1})\frac{1}{(2\pi)^d}\int\frac{\hat{D}(k)}{F_\tau(k,\rho)^4}dk \\ &\le O(d^{-1})\{\int\frac{\hat{D}(k)^2}{(1-\hat{D}(k))^4}dk\}^{\frac{1}{2}} \\ &\quad \cdot\{\int\frac{1}{(1-\hat{D}(k))^4}dk\}^{\frac{1}{2}} \\ &\le O(d^{-\frac{3}{2}}). \end{aligned}$$

The third term is estimated similarly:

$$\frac{1}{(2\pi)^d}\int\frac{\partial_z\Pi_\tau(k,\rho)^2}{F_\tau(k,\rho)^4}dk \le O(d^{-2}).$$

Now we show $\||x^u|\partial_z^\nu G_\tau^1(x,\rho)\|_\infty \le 2k_0 d^{-1}$. For simplicity, we only consider the case $\nu = 1$ and $u = (1, 0, \dots, 0)$.

$$\begin{aligned}
|x_1 \partial_z G_\tau^1(x,\rho)| &\le |x_1 \partial_z N_\tau(x,\rho)| \\
&= \left|\frac{1}{(2\pi)^d}\int x_1 \exp(-ik\cdot x)\partial_z N_\tau(k,\rho)dk\right| \\
&\le \frac{1}{(2\pi)^d}\int |\partial_{k_1}\partial_z N_\tau(k,\rho)|dk
\end{aligned}$$

(The integration by parts calculation is legal since $\partial_z N_\tau(k,\rho)$ is absolutely continuous w.r.t. k_1.)

$$\begin{aligned}
\partial_{k_1}\partial_z N_\tau(k,\rho) &= \frac{\partial_{k_1}\hat{D}(k) + \partial_{k_1}\partial_z\hat{\Pi}_\tau(k,\rho)}{(1-\rho\hat{D}(k) - \hat{\Pi}_\tau(k,\rho))^2} \\
&+ \frac{2\rho\hat{D}(k)\,\partial_{k_1}\hat{D}(k) + \hat{D}(k)\partial_{k_1}\hat{\Pi}_\tau(k,\rho)}{(1-\rho\hat{D}(k) - \hat{\Pi}_\tau(k,\rho))^3} \\
&+ \frac{\rho\partial_z\hat{\Pi}_\tau(k,\rho)\,\partial_{k_1}\hat{D}(k) + \partial_z\hat{\Pi}_\tau(k,\rho)\,\partial_{k_1}\hat{\Pi}_\tau(k,\rho)}{(1-\rho\hat{D}(k) - \hat{\Pi}_\tau(k,\rho))^3} \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \qquad (3.20)
\end{aligned}$$

Since $\partial_{k_1}\hat{D}(k) \le C(d^{-1})$, we easily have $I_1, I_3 \le Cd^{-1}$ and $I_5 \le O(d^{-2})$. By (2.31) (2.32) and assumption P_4, we know

$$\begin{aligned}
\partial_{k_1}\hat{\Pi}_\tau(k,\rho) &\le O(d^{-2}), \\
\partial_{k_1}\partial_z\hat{\Pi}_\tau(k,\rho) &\le O(d^{-2}).
\end{aligned}$$

Thus $I_2, I_4, I_6 \le O(d^{-2})$.

Combining the above estimates and put them in (3.20) we finished the proof.

Q.E.D.

Corollary 3.1 *There exists a universal constant k_0 (which does not depend on τ) such that for $\rho \in [0, r_\tau(0)])$, the following inequalities hold:*

$$\|\partial_z^\nu G_\tau^1(x,\rho)\|_2^2 \le 2k_0 d^{-1} \qquad \nu = 0, 1 \qquad (3.21)$$

$$\||x|^u \partial_z^\nu G_\tau^1(x,\rho)\|_\infty \le 2k_0 d^{-1} \qquad |u| \le 1, \nu = 0, 1 \qquad (3.22)$$

Proof: Proposition 3.2, 3.5 mandates that there exists a k_0, such that inequalities (3.21) and (3.22) hold for $\rho \leq 1$. By Proposition 3.6, $\|\partial_z^\nu G_\tau^1(x,\rho)\|_2^2$ and $\||x|^u \partial_z^\nu G_\tau^1(x,\rho)\|_\infty$ are continuous function of ρ for $\rho \leq r_\tau(0)$. Theorem 3.1 further states that that if we choose k_0 sufficiently large, then (3.21) and (3.22) hold for $\rho \leq r_\tau(0)$.

Q.E.D.

Corollary 3.2 $\hat{\Pi}(k,z)$ *is continuously differentiable w.r.t.* k *and analytic w.r.t.* z *for* $z < r_\tau(0)$. *Moreover, the following inequalities hold for* $|z| \leq r_\tau(0)$

$$|\partial_z^\nu \Pi_\tau(k,z)| \leq O(d^{-1}), \qquad \nu = 0,1, \tag{3.23}$$

$$|\partial_k^u \Pi_\tau(k,z)| \leq O(d^{-2}), \qquad |u| = 1, \tag{3.24}$$

$$|\partial_k^u \partial_z \Pi_\tau(k,z)| \leq O(d^{-2}), \qquad |u| = 1. \tag{3.25}$$

This corollary is a direct consequence of Proposition 2.5 and Corollary 3.1.

Lemma 3.2 *For* $\epsilon \in (0,1]$, *we have*

$$\sup_{1 \leq T \leq \tau} \frac{1}{T^\epsilon}(1 + \frac{\epsilon \log \tau}{2\tau})^T < C. \tag{3.26}$$

Proof: Let us consider for fixed ϵ and $T \in [1,\tau]$, the function

$$f_\tau(T) = \frac{1}{T^\epsilon}(1 + \frac{\epsilon \log \tau}{2\tau})^T.$$

$$g_\tau(T) = \log f(T) = -\epsilon \log T + T \log(1 + \frac{\epsilon \log \tau}{2\tau})$$

$g_\tau(T)$ attains its minimum at

$$T_{\min} = \frac{\epsilon}{\log(1 + \epsilon \log \tau / 2\tau)}.$$

Thus,

$$\begin{aligned} \sup_{1 \leq T \leq \tau} g_\tau(T) &= \max\{g_\tau(\tau), g_\tau(1)\} \\ &= g_\tau(1) \leq 2. \end{aligned}$$

Q.E.D.

Lemma 3.3

$$\sup_{\tau\geq 1} r_\tau(0) < \infty. \tag{3.27}$$

Proof: Let us denote

$$A = \{\omega : X_1(\omega) = X_2(\omega) = \ldots = (1, 0, \ldots, 0)\}.$$

Then

$$E\{K_\tau[0,T]\} \geq E\{K_\tau[0,T]I_A(\omega)\} \geq (\frac{3}{(d\pi)^2})^T.$$

Thus $\limsup_{T\to\infty}\{E(K_\tau[0,T])\}^{\frac{1}{T}} \geq 3/(d\pi)^2$.

Q.E.D.

Theorem 3.2 *For sufficiently large d_0, $\Pi_\tau(k,z)$ is analytic in*

$$B(D_\tau(\frac{1}{2})) = \{z : |z| < D_\tau(\frac{1}{2}) = r_\tau(0)(1 + \log\tau/2\tau)\}.$$

Moreover, in $B(D_\tau(\frac{1}{2}))$, the inequalities (3.23)-(3.25) still hold.

Proof: We have for $|z| \leq D_\tau(\frac{1}{2})$,

$$\begin{aligned} |G^1_\tau(x,z)| &\leq |G^1_\tau(x, D_\tau(\frac{1}{2}))| \leq \sum_{T=1}^{\tau-1} N_\tau(x,T)\{D_\tau(\frac{1}{2})\}^T \\ &\leq r_\tau \sum_{T=1}^{\tau-1} T N_\tau(x,T) r_\tau^{T-1}\{\frac{1}{T}(1+\frac{\log\tau}{2\tau})^T\} \\ &\leq r_\tau \sup_{1\leq T\leq\tau}\{\frac{1}{T}(1+\frac{\log\tau}{2\tau})^T\}|\partial_z G^1_\tau(x,r_\tau)| \\ &\leq O(d^{-1}). \end{aligned}$$

Similarly, we can prove for $z \in B(D_\tau(\frac{1}{2}))$,

$$|\partial_z G^1_\tau(x,z)| \leq C\partial_z G^1_\tau(x,r_\tau)|,$$

$$|xG^1_\tau(x,z)| \leq |xG^1_\tau(x,r_\tau)|.$$

Inequalities (3.23)-(3.25) follows from Proposition 2.5.

Q.E.D.

3.3 The Limiting Distribution

Proposition 3.8 *Suppose the dimension d and memory τ are sufficiently large, and $|k| \leq d\pi \log \tau / 12\tau$, then $N_\tau(k, z)$ has a simple pole at $r_\tau(k) \in (0, D_\tau(\frac{1}{4}))$, and is otherwise analytic in $|z| \leq D_\tau(\frac{1}{2})$. The pole $r_\tau(k)$ is continuously differentiable w.r.t. k.*

Proof: Since $F_\tau(k, z)$ has a Taylor series whose coefficients are all real numbers, the zeroes of $F_\tau(k, z)$ must occur in conjugate pairs.

Suppose z_1, z_2 be zero points of $F_\tau(k, z)$, we have

$$(z_2 - z_1)\hat{D}(k) = \Pi_\tau(k, z_1) - \Pi_\tau(k, z_2) = (z_1 - z_2)\int_0^1 \partial_z \hat{\Pi}_\tau(k, z_2 + t(z_1 - z_2))dt$$

If $|k|$ is small, $\hat{D}(k) \sim 1$, then $(z_2 - z_1)\hat{D}(k) \sim z_2 - z_1$. On the other hand, by theorem 3.2, $\partial_z \hat{\Pi}_\tau(k, z_1 + t(z_2 - z_1)) \sim O(d^{-1})$. We conclude that for small $|k|$ and large d, $F_\tau(k, z)$ has at most one zero, and the only zero must lie on the real axis.

Next we show $F_\tau(k, z)$ does have a zero. We notice that for $(k, z) \in \{(k, z) : |k| < \epsilon, |z| < D_\tau(\frac{1}{2})\}$, $F_\tau(k, z)$ is a continuous function w.r.t. k, z, and $\partial_z F_\tau(k, z)$ is also continuous w.r.t. k, z. Moreover for large d,

$$\partial_z F_\tau(k = 0, z) = -1 - \partial_z \hat{\Pi}_\tau(k = 0, z) \neq 0,$$

$$F_\tau(k = 0, z = r_\tau(0)) = 0.$$

Apply the implicit function theorem we can find a unique continuous $r_\tau(k)$ defined on a sufficiently small ball $k \in B_\epsilon(0)$, such that $F_\tau(k, r_\tau(k)) = 0$.

Since $\hat{D}(k)$ contains the form $|k|$, $r_\tau(k)$ is probably not differentiable at $k = 0$. However, $r_\tau(k)$ is continuously differentiable for $k \in B_\epsilon(0) - \{0\}$. Thus for $k \neq 0$ and $|k|$ small

$$\begin{aligned} \partial_{k_i} r_\tau(k) &= -\frac{\partial_{k_i} F_\tau(k, r_\tau(k))}{\partial_z F_\tau(k, r_\tau(k))} \\ &= \frac{r_\tau(k)\{\frac{3}{d\pi} sgn(k_i) - \frac{3k_i}{\pi^2}\} - \partial_{k_i} \hat{\Pi}_\tau(k, r_\tau(k))}{\hat{D}(k) + \partial_z \hat{\Pi}_\tau(k, r_\tau(k))} \end{aligned} \tag{3.28}$$

(3.28) is valid as long as $|r_\tau(k)| < D_\tau(\frac{1}{2})$.

Next we show if $|k| \le d\pi \log \tau / 12\tau$, then

$$r_\tau(k) \in (0, D_\tau(\frac{1}{4})). \tag{3.29}$$

Under our assumption on $|k|$, we observe from (3.28) that $\partial_{k_i} r_\tau(k) \sim \frac{3}{d\pi} r_\tau(k) sgn(k_i)$. Thus

$$\frac{d}{dt} r_\tau(tk) = \sum_{i=1}^{d} \partial_{k_i} r_\tau(tk) \sim \frac{3}{d\pi} r_\tau(tk) \sum_{i=1}^{d} sgn(k_i) k_i (> 0). \tag{3.30}$$

$$r_\tau(k) = r_\tau(0) + \int_0^1 \frac{d}{dt} r_\tau(tk) dt \approx r_\tau(0) + \frac{3}{d\pi} r_\tau(k)|k| \tag{3.31}$$

Thus for $k \ne 0$, $r_\tau(k) > r_\tau(0)$, and

$$r_\tau(k) \le r_\tau(0)(1 - \frac{3}{d\pi}|k|)^{-1} \le r_\tau(0)(1 + \frac{3}{d\pi}|k|) \le r_\tau(0)(1 + \log\tau/4\tau). \tag{3.32}$$

Finally, we show $F_\tau(k, z)$ has a simple zero at $r_\tau(k)$.

This could be seen from the following:

$$\begin{aligned} F_\tau(k,z) &= F_\tau(k,z) - F_\tau(k, r_\tau(k)) \\ &= -(z - r_\tau(k))\hat{D}(k) - (\hat{\Pi}_\tau(k,z) - \hat{\Pi}_\tau(k, r_\tau(k))) \\ &\approx -(z - r_\tau(k))\{1 + \int_0^1 \partial_z \hat{\Pi}_\tau(k, r_\tau(k) + t(z - r_\tau(k))dt\} \\ &= -(z - r_\tau(0))(1 + O(d^{-1})). \end{aligned}$$

Q.E.D.

Proposition 3.9 *For sufficiently large d*

$$\lim_{n\to\infty} \{\frac{r_n(k/n)}{r_n(0)}\}^n = \exp\{\frac{3}{d\pi} \sum_{i=1}^{d} |k_i|\}. \tag{3.33}$$

Proof: for n sufficiently large, $|k/n| \leq d\pi logn/12n$, so the results in the previous proposition hold.

$$\begin{aligned} r_n(\frac{k}{n}) &= r_n(0) + \int_0^1 \frac{d}{dt}\{r_n(t\frac{k}{n})\}dt \\ &= r_n(0) + \int_0^1 \sum_{i=1}^{d} \frac{3}{d\pi} r_n(t\frac{k_i}{n}) sgn(k_i)\frac{k_i}{n}dt \\ &= r_n(0) + \sum_{i=1}^{d}[\frac{3}{d\pi} r_n(0)\frac{|k_i|}{n} + o(\frac{k_i}{n})]. \end{aligned}$$

The above estimation leads to the desired result in this proposition.

Q.E.D.

Theorem 3.3 *There exists a dimension $d_0 > 0$, such that for $d > d_0$, we have*

$$\lim_{T\to\infty} \frac{\hat{N}_T(k/T, T)}{\hat{N}_T(k=0, T)} = \exp\{-\frac{3}{d\pi}\sum_{i=1}^{d}|k_i|\}. \tag{3.34}$$

Proof: We assume T be sufficiently large so that $|k/T| \leq d\pi logT/12T$.

$$\begin{aligned} \hat{N}_T(\frac{k}{T}, T) &= \frac{1}{2\pi i}\int_{|z|=\frac{1}{2}} \frac{\hat{N}_T(k/T, z)}{z^{T+1}}dz \\ &= \frac{1}{2\pi i}\int_{|z|=D_\tau(\frac{1}{2})} - \int_{|z-r_T(k/T)|=\epsilon(T)} \frac{\hat{N}_T(k/T, z)}{z^{T+1}}dz \\ &= I_1 + I_2. \end{aligned}$$

Where $\epsilon(T) \leq \log T/4T$, thus $\{z : |z - r_\tau(k)| = \epsilon(T)\} \subset \{z : |z| \leq D_T(\frac{1}{2})\}$.

For $|z| \leq D_\tau(\frac{1}{2})$, We have $F_\tau(k, z) = (z - r_\tau(k))H_\tau(k, z)$ where $H_\tau(k, z)$ is an analytic function in this domain with

$$H_\tau(k, r_\tau(k)) = \partial_z F_\tau(k, r_\tau(k)) = -\hat{D}_\tau(k) - \partial_z\hat{\Pi}_\tau(k, z).$$

Notice that $\partial_z\hat{\Pi}_\tau(k=0, z) = 0$, we have for sufficiently large T

$$H_T(k/T, z) \sim -1.$$

Thus

$$\int_{|z|=D_\tau(\frac{1}{2})} |\hat{N}_T(k/T, z)|dz \leq C\int_{|z|=D_\tau(\frac{1}{2})} |\frac{1}{z - r_\tau(k)}|dz \leq |\log\tau|.$$

We can estimate I_1 by

$$\begin{aligned} I_1 &\le C\{\frac{r_T(k/T)}{D_T(1/2)}\}^{T+1} r_T(k/T)^{-T-1} \int_{|z|=D_T(\frac{1}{2})} |\hat{N}_T(k/T,z)|dz \\ &\le O(T^{-\frac{1}{4}}) r_T(k/T)^{-T-1} \end{aligned}$$

I_2 could be estimated using the residue theorem

$$\begin{aligned} I_2 &= -\frac{1}{2\pi i} \int_{|z-r_T(k/T)|\epsilon(T)} \frac{1}{(z-r_T(k/T))H_T(k/T,z)z^{T+1}} dz \\ &= r_T(k/T)^{-T-1}. \end{aligned}$$

Thus

$$\begin{aligned} \lim_{T\to\infty} \langle \exp{(ik\cdot x)} \rangle_T &= \lim_{T\to\infty} \frac{\hat{N}_T(k/T,T)}{\hat{N}_T(0,T)} \\ &= \lim_{T\to\infty} \{\frac{r_T(0)}{r_T(k/T)}\}^{T+1} \\ &= \exp\{-\frac{3}{d\pi}\sum_{i=1}^{d} |k_i|\}. \end{aligned}$$

Q.E.D.

CHAPTER 4

THE 3-DIMENSIONAL SPREAD-OUT MODEL

4.1 The Spread-Out Simple Random Walk

In this chapter, we will assume the i.i.d. random variables $X_1, \ldots, X_i, \ldots$ satisfy the following spread-out discrete Cauchy distribution:

$$P\{X_1 = me_j\} = \frac{1}{L}\frac{1}{\pi^2 n^2}, \tag{4.1}$$

where $\{e_j, j = 1, 2, 3d\}$ are the unit vectors on Z^3; L is a sufficiently large positive integer; and m is an integer satisfying $(n-1)L < |m| \leq nL$ for some integer $n > 0$.

We call L the diffusion parameter. It will act as the driving force for our lace expansion.

We now compute the characteristic function of X_1 with diffusion parameter L:

$$\begin{aligned}\hat{D}_L(k) &= E\exp\{ik \cdot X\} \\ &= \sum_{j=1}^{3}\sum_{n=1}^{\infty}\sum_{r=0}^{L-1}\{e^{ik_j(nL-r)} + e^{ik_j(-nL+r)}\}\frac{1}{L\pi^2 n^2} \\ &= \frac{1}{3}\sum_{j=1}^{3}\{f_1(k_j)f_2(k_j) + f_3(k_j)f_4(k_j)\},\end{aligned} \tag{4.2}$$

where for $j = 1, 2, 3$,

$$f_1(k_j) = \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(nLk_j)}{n^2}, \tag{4.3}$$

$$f_2(k_j) = \frac{1}{L}\sum_{r=0}^{L-1}\cos(k_j r), \tag{4.4}$$

$$f_3(k_j) = \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(nLk_j)}{n^2}, \tag{4.5}$$

$$f_4(k_j) = \frac{1}{L}\sum_{r=0}^{L-1}\sin(k_j r). \tag{4.6}$$

From Appendix .2 we know

$$f_1(k_j) = 1 - (\frac{3}{\pi}L|k_j| - \frac{3}{2\pi^2}L^2{k_j}^2), \quad \text{if } Lk_j \in [-\pi, \pi]. \tag{4.7}$$

By Fourier analysis, it can be shown that

$$f_3(k_j) = \frac{6}{\pi^2}\{-(\log 2)Lk_j - \int_0^{Lk_j}\log(\sin\frac{t}{2})dt\}, \quad \text{if } Lk_j \in [-\pi, \pi]. \tag{4.8}$$

We also have the following fundamental identities:

$$f_2(k_j) = \frac{1}{L}\frac{\sin(\frac{2L-1}{2}k_j) + \sin(\frac{k_j}{2})}{2\sin(\frac{k_j}{2})}, \tag{4.9}$$

$$f_4(k_j) = \frac{1}{L}\frac{\cos(\frac{k_j}{2}) - \cos(\frac{2L-1}{2}k_j)}{2\sin(\frac{k_j}{2})}. \tag{4.10}$$

Lemma 4.1 *$f_1(k_j)$ has the following fundamental properties:*

1. *For $k_j \in [-\pi, \pi]$, $f_1(k_j) = f_1(-k_j)$; $f_1(k_j)$ has period $2\pi/L$.*

2. *For $k_j \in [0, \pi/L]$, $f_1(k_j)$ is convex, strictly decreasing, with $f_1(0) = 1$, $f_1((1 - \frac{\sqrt{3}}{3})\frac{\pi}{L}) = 0$, $f_1(\pi/2L) = -1/8$, $f_1(\pi/L) = -\frac{1}{2}$.*

3. *For $k_j \in [\pi/L, (1 + \frac{\sqrt{3}}{3})\pi/L]$, we have $f_1(k_j)$ be strictly increasing, with $f_1((1 + \frac{\sqrt{3}}{3})\pi/L) = 0$.*

4. *For* $|k_j| \leq (1 - \frac{\sqrt{3}}{3})\pi/L$,

$$\frac{3+\sqrt{3}}{2\pi} L|k_j| \leq 1 - f_1(k_j) \leq \frac{3L|k_j|}{\pi}.$$

Lemma 4.2 *$f_2(k_j)$ has the following fundamental properties:*

1. *For* $k_j \in [-\pi, \pi]$, $f_2(k_j) = f_2(-k_j)$, $|f_2(k_j)| \leq 1$.

2. *If* $|k_j| \leq (1 - \frac{\sqrt{3}}{3})\pi/L$, *then*

$$0 \leq f_2(k_j) \leq 1 - \frac{1}{6\pi^2} L^2 {k_j}^2. \tag{4.11}$$

3. *For sufficiently large* L, *if* $(1-\frac{\sqrt{3}}{3})\frac{\pi}{L} \leq |k_j| \leq \pi/2L$, *then* $|f_2(k_j)| \leq 0.76$; *if* $|k_j| \geq \pi/2L$, *then* $|f_2(k_j)| \leq 0.64$; *if* $|k_j| \geq 3\pi/2L$, *then* $|f_2(k_j)| \leq 0.22$.

Proof: To prove (4.11), we have for $|k_j| \leq \pi/L$,

$$\begin{aligned} 1 - \frac{1}{L}\sum_{r=0}^{L-1}\cos(k_j r) &= \frac{1}{L}\sum_{r=0}^{L-1}\{1-\cos(k_j r)\} \\ &= \frac{1}{L}\sum_{r=0}^{L-1} 2(\sin\frac{k_j r}{2})^2 \\ &\geq \frac{1}{6\pi^2}L^2{k_j}^2. \end{aligned}$$

To prove item 3, we start with (4.9).

We have for any $\epsilon > 0$, there exists L_ϵ, such that for $L \geq L_\epsilon$, if $|k_j| \geq (1 - \frac{\sqrt{3}}{3})\frac{\pi}{L}$, then

$$|f_2(k_j)| \leq \frac{1}{L}\{\frac{1+\epsilon}{(1-\frac{\sqrt{3}}{3})\frac{\pi}{L}} + \frac{1}{2}\} \leq 0.76.$$

The last step is by letting $\epsilon \to 0$, $L \to \infty$.

Similarly, we have the results for $|k_j| \geq \pi/2L$ and $|k_j| \geq 3\pi/2L$.

Q.E.D.

Lemma 4.3 *$f_3(k_j)$ has the following fundamental properties:*

1. *For* $k_j \in [-\pi, \pi]$, $f_3(k_j) = -f_3(-k_j)$; $f_3(k_j)$ *has period* $2\pi/L$.

2. *For $k_j \in [0, \pi/3L]$, $f_3(k_j)$ is increasing, with*

$$f_3(\frac{\pi}{3L}) \leq 0.67. \quad (4.12)$$

For $k_j \in [\pi/3L, \pi/L]$, $f_3(k_j)$ is decreasing, with

$$f_3(\frac{\pi}{L}) \geq -0.27. \quad (4.13)$$

3. *For $0 \leq k_j \leq \pi/L$, we have*

$$\frac{6}{\pi^2} Lk_j\{-\log(Lk_j) + 1\} \leq f_3(k_j) \leq \frac{6}{\pi^2} Lk_j\{-\log(Lk_j) + 2\}. \quad (4.14)$$

Proof: We first prove item 2. It is easy to see for $k_j \in [0, \pi/L]$, $f_3(k_j)$ is increasing on $[0, \pi/3L]$ and decreasing on $[\pi/3L, \pi/L]$. Using the fact that $\sin(t/2) \geq 3t/2\pi$ whenever $t \in [0, \pi/3]$, we get (4.12). Similarly, using the fact that $\sin(t/2) \leq t/2$ we get (4.13).

(4.14) is achieved by estimating $f_3'(k_j)$ on $k_j \in [0, \pi/L]$.

Q.E.D.

Lemma 4.4 *$f_4(k_j)$ has the following fundamental properties:*

1. *For $k_j \in [-\pi, \pi]$, $f_4(k_j) = -f_4(-k_j)$; $|f_4(k_j)| \leq 1$.*

2. *For $0 \leq k_j \leq \pi/2L$, we have*

$$0 \leq \frac{Lk_j}{2\pi} \leq f_4(k_j) \leq \frac{Lk_j}{2}. \quad (4.15)$$

Let us denote for $k \in [-\pi, \pi]^3$, $\|k\|_\infty = \max\{|k_j| : j = 1, 2, 3\}$.

Proposition 4.1 *For $\|k\|_\infty \leq (1 - \frac{\sqrt{3}}{3})\pi/L$, we have*

$$0 \leq \hat{D}_L(k) \leq 1 - \frac{1}{3}\sum_{j=1}^{3} 0.05L|k_j| \quad (4.16)$$

For $\|k\|_\infty \geq (1 - \frac{\sqrt{3}}{3})\pi/L$, we have $\hat{D}_L(k) \leq a = 0.99$.

In particular, there exists $C > 0$, such that for $k \in [-\pi, \pi]^d$,

$$1 - \hat{D}_L(k) \geq C|k|. \quad (4.17)$$

Proof: By Lemma 4.1 - Lemma 4.4, we have for $||k||_\infty \le (1-\frac{\sqrt{3}}{3})\pi/L$,

$$\begin{aligned}\hat{D}_L(k) &\le \frac{1}{3}\sum_{j=1}^{3}\{(1-\frac{3+\sqrt{3}}{2\pi}L|k_j|)(1-\frac{L^2{k_j}^2}{6\pi^2}) \\ &\quad +(\frac{6}{\pi^2}L|k_j|(-\log(L|k_j|)+2))(\frac{L|k_j|}{2})\} \\ &\le 1-\frac{1}{3}\sum_{j=1}^{3}0.05L|k_j|.\end{aligned}$$

The estimates in Lemma 4.1 - Lemma 4.4 also render us the upper bound

$$\hat{D}_L(k) \le 0.99, \quad \text{if } ||k||_\infty \ge (1-\frac{\sqrt{3}}{3})\pi/L. \tag{4.18}$$

Q.E.D.

For a function f defined on $[-\pi,\pi]^d$, we denote $||f||_1 = \frac{1}{(2\pi)^d}\int |f(k)|dk$.

Proposition 4.2 *There exists a $C>0$, $L_0>0$, such that for any $n\ge 1$, $L\ge L_0$, the following inequality holds:*

$$||\hat{D}_L(k)^n||_1 \le a^n + Cn^{-3}L^{-3}. \tag{4.19}$$

Proof: For $||k||_\infty \le (1-\frac{\sqrt{3}}{3})\pi/L$, with L sufficiently large, we have

$$\frac{1}{3}\sum_{j=1}^{3}0.05L|k_j| \le 0.1 < 1.$$

Thus

$$\begin{aligned}\frac{1}{(2\pi)^3}\int_{[-\pi,\pi]^d}|\hat{D}_L(k)|^n dk &\le \frac{1}{(2\pi)^3}\{\int_{||k||_\infty\le(1-\frac{\sqrt{3}}{3})\pi/L}|\hat{D}_L(k)|^n dk \\ &\quad +\int_{(1-\frac{\sqrt{3}}{3})\pi/L\le||k||_\infty\le\pi}|\hat{D}_L(k)|^n dk\} \\ &= I_1+I_2.\end{aligned}$$

From the previous proposition, we know $I_2 \le a^n$.

$$\begin{aligned} I_1 &\le \frac{1}{(2\pi)^3}\int_{\|k\|_\infty \le (1-\frac{\sqrt{3}}{3})\pi/L} (1-\frac{1}{3}\sum_{j=1}^{3} 0.05L|k_j|)^n dk \\ &\le \frac{1}{(2\pi)^3}\int_{R^3} \exp\{-n\frac{1}{3}\sum_{j=1}^{3} 0.05L|k_j|\}dk \\ &\le \frac{1}{(2\pi)^3}\frac{C}{L^3}\int_0^\infty \exp\{-Cn\rho\}\rho^2 d\rho \\ &\le CL^{-3}n^{-3}. \end{aligned}$$

Q.E.D.

Proposition 4.3 *For $i = 1, 2, 3$,*

$$\|\partial_{k_i}\hat{D}_L(k)\|_1 \le C(\log L)^2. \tag{4.20}$$

Proof: From (4.2), we have

$$\begin{aligned} \|\partial_{k_i}\hat{D}_L(k)\|_1 \le \frac{1}{3}\sum_{j=1}^{3}\sum_{n=0}^{L-1}\frac{2}{(2\pi)^3}\int_{n\pi/L}^{(n+1)\pi/L} &\{|f_1'(k_i)||f_2(k_i)| + |f_1(k_i)||f_2'(k_i)| \\ &+ |f_3'(k_i)||f_4(k_i)| + |f_3(k_i)||f_4'(k_i)|\} \end{aligned} \tag{4.21}$$

For $k_j \in [n\pi/L, (n+1)\pi/L]$, $n \ge 1$, (4.7), (4.9) and (4.10) imply that

$$|f_1(k_j)| \le 1, \quad |f_1'(k_j)| \le CL, \tag{4.22}$$

$$|f_2(k_j)| \le \frac{C}{n}, \quad |f_2'(k_j)| \le C(\frac{L}{n} + \frac{L}{n^2}), \tag{4.23}$$

$$|f_4(k_j)| \le \frac{C}{n}, \quad |f_4'(k_j)| \le C(\frac{L}{n} + \frac{L}{n^2}). \tag{4.24}$$

Using (4.8), we can also compute:

$$\begin{aligned} \int_{n\pi/L}^{(n+1)\pi/L} |f_3'(k_j)|dk_j &= \int_0^{\pi/L} \frac{6}{\pi^2}| - (\log 2)L - L\log(\sin(\frac{Lk_j}{2}))|dk_j \\ &\le C\log L. \end{aligned} \tag{4.25}$$

$$\int_{n\pi/L}^{(n+1)\pi/L} |f_3(k_j)|dk_j \le \frac{C}{L}. \tag{4.26}$$

Combing (4.23) - (4.26) and using the fact

$$\sum_{n=1}^{L} \frac{1}{n} \leq \log L,$$

we arrive at the result.

Q.E.D.

Proposition 4.4

$$\|\hat{D}_L(k)\|_1 \leq CL^{-1}(\log L)^2. \tag{4.27}$$

Proof: We know $\hat{D}_L(k) = \frac{1}{3}\sum_{j=1}^{3}\{f_1(k_j)f_2(k_j) + f_3(k_j)f_4(k_j)\}$, thus it suffices to show

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f_1(k_j)f_2(k_j) + f_3(k_j)f_4(k_j)|dk_j \leq CL^{-1}\log L.$$

This result could be obtained easily using (4.23) - (4.26).

Q.E.D.

Lemma 4.5 *Let $0 < a < 1$ and $C > 0$ be fixed. Then for each $\lambda > -3/\log a$, there exists $L_0 > 0$, such that for $L > L_0$,*

$$a^{\lambda \log L} \leq C(\log L)^{-3}L^{-3}.$$

Proposition 4.5

$$\|N_0^1(x, z = 1)\|_2^2 \;\; \leq \;\; CL^{-1}(\log L)^2, \tag{4.28}$$

Proof:

$$\|N_0^1(x, z = 1)\|_2^2 = \|\sum_{T=1}^{\infty} \hat{D}_L(k)^T\|_2^2 \tag{4.29}$$

$$\leq \{\sum_{T=1}^{\lambda \log L} \|\hat{D}_L(k)^T\|_2 + \sum_{T=\lambda \log L+1}^{\infty} \|\hat{D}_L(k)^T\|_2\}^2$$

$$= (I_1 + I_2)^2. \tag{4.30}$$

For $1 \leq T \leq \log L$, we have

$$\|\hat{D}_L(k)^T\|_2 \leq \|\hat{D}_L(k)\|_2 = \{N_0(0, T = 2)\}^{\frac{1}{2}} \leq CL^{-\frac{1}{2}}. \tag{4.31}$$

Thus $I_1 \le CL^{-1/2}(\log L)^1$.

To estimate I_2, we apply (4.19) and the previous lemma to get

$$I_2 \le \sum_{T=\lambda \log L+1}^{\infty} CT^{-\frac{3}{2}}L^{-\frac{3}{2}} \le CL^{-\frac{3}{2}}. \tag{4.32}$$

Thus $(I_1 + I_2)^2 \le CL^{-1}(\log L)^2$.

Q.E.D.

Proposition 4.6 *For $\nu = 0, 1$, we have*

$$\|\partial_z^\nu N_0^1(x, z = 1)\|_\infty \le CL^{-1}(\log L)^{\nu+1}. \tag{4.33}$$

For $i = 1, 2, 3$,

$$\|x_i N_0^1(x, z = 1)\|_\infty \le C(\log L)^{\nu+4}. \tag{4.34}$$

Proof:

$$\begin{aligned}
\|\partial_z^\nu N_0^1(x, z = 1)\|_\infty &\le \sum_{T=1}^{\lambda \log L} T^\nu \|N_0^1(x, T)\|_\infty + \|\sum_{\lambda \log L+1}^{\infty} T^\nu N_0^1(x, T)\|_\infty \\
&\le \sum_{T=1}^{\lambda \log L} T^\nu \|\hat{D}_L(k)\|_1 + \sum_{\lambda \log L+1}^{\infty} T^\nu \|\hat{D}_L(k)^T\|_1 \\
&\le C\{(\log L)^{\nu+1}\frac{1}{L} + \frac{1}{L^2}\} \\
&\le CL^{-1}(\log L)^{\nu+1}.
\end{aligned}$$

$$\begin{aligned}
\|x_i N_0^1(x, z = 1)\|_\infty &\le \|x_i N_0(x, T = 1)\|_\infty + \sum_{T=2}^{\lambda \log L} T\|\partial_{k_i}\hat{D}_L(k)\|_1 \\
&\quad + \sum_{T=\lambda \log L}^{\infty} T\|\partial_{k_i}\hat{D}_L(k)(\hat{D}_L(k))^{T-1}\|_1 \\
&\le C(1 + (\log L)^{\nu+4} + L^{-2}) \\
&\le C(\log L)^{\nu+4}
\end{aligned}$$

Q.E.D.

4.2 The Convergence of Lace Expansion

Similar to Proposition 3.6, we know for $\tau < \infty$, $\nu = 0, 1$ the norms $||\partial_z^\nu G_\tau^1(x,\rho)||_2$, $||x_i G_\tau^1(x,\rho)||_\infty$ are continuous w.r.t.ρ for $\rho \le r_\tau(0)$. Also, corresponding result to that of Proposition 3.7 also holds.

Let us denote $N_\tau^1(x,z) = N_\tau(x,z) - \delta_{0x}$.

Proposition 4.7 *For any $\rho \in (0, r_\tau(0)]$ and integer $m > 0$,*

$$\partial_z^m N_\tau(x,\rho) \le m!\rho^{-m} \underbrace{N_\tau^1 * \ldots * N_\tau^1}_{m} * N_\tau(x,\rho). \tag{4.35}$$

Proof:

$$\begin{aligned}
\partial_z^m N_\tau(x,\rho) &= \sum_{T=m}^{\infty} T\ldots(T-m+1)N_\tau(x,T)\rho^{T-m} \\
&= m!\rho^{-m}\sum_{T=m}^{\infty}\begin{pmatrix} T \\ m \end{pmatrix} N_\tau(x,T)\rho^T \\
&= m!\rho^{-m}\sum_{T=m}^{\infty}\sum_{0<T_1<\ldots<T_m\le T} N_\tau(x,T)\rho^T \\
&\le m!\rho^{-m}\sum_{T=m}^{\infty}\sum_{0<T_1<\ldots<T_m\le T}\sum_{y_1,\ldots,y_m\in Z^d} \\
&\quad \{\prod_{i=1}^{m} N_\tau(y_i - y_{i-1}, T_i - T_{i-1})\rho^{T_i - T_{i-1}}\} N_\tau(x - y_m, T - T_m) \\
&= m!\rho^{-m}\underbrace{N_\tau^1 * \ldots * N_\tau^1}_{m} * N_\tau(x,\rho).
\end{aligned}$$

Q.E.D.

Theorem 4.1 *There exist an universal integer $L_0 > 0$, such that for all $L \ge L_0$, $\tau > 0$ and $\rho \in [0, r_\tau(0)]$, statement P_4 implies statement P_2.*

Here statement $P_a (a = 2, 4)$ denotes the following assertion:

$$P_a : \begin{cases} ||G_\tau^1(x,\rho)||_2^2 \le ak_0 L^{-1}(\log L)^4, \\ ||\partial_z^\nu G_\tau^1(x,\rho)||_\infty \le ak_0 L^{-1}(\log L)^{\nu+2}, \quad \nu = 0, 1, \\ ||x_i G_\tau^1(x,\rho)||_\infty \le ak_0(\log L)^2, \end{cases} \tag{4.36}$$

where k_0 is a universal constant that does not depend on τ or L.

Proof: Let us assume that statement P_4 holds. We first consider the L^2 norm.

$$\|G^1_\tau(x,\rho)\|_2^2 \le \|N^1_\tau(x,\rho)\|_2^2 = \|N_\tau(x,\rho)\|_2^2 - 1 = \|\frac{1}{\hat{F}_\tau(k,\rho)}\|_2^2 - 1. \quad (4.37)$$

Since $\hat{F}_\tau(0, r_\tau(0)) = 0$, and $r_\tau(0) \ge 1$, we have

$$\begin{aligned} \hat{F}_\tau(k, r_\tau(0)) &= \hat{F}_\tau(k, r_\tau(0)) - \hat{F}_\tau(0, r_\tau(0)) \\ &\ge (1 - \hat{D}_L(k)) + \hat{\Pi}_\tau(0, r_\tau(0)) - \hat{\Pi}_\tau(k, r_\tau(0)). \end{aligned} \quad (4.38)$$

We try to estimate $\hat{\Pi}_\tau(0, r_\tau(0)) - \hat{\Pi}_\tau(k, r_\tau(0))$.

Observe the definition of $\hat{\Pi}^N_\tau(k,z)$, we see

$$\begin{aligned} \hat{\Pi}^1_\tau(0, r_\tau(0)) &= \hat{\Pi}^1_\tau(k, r_\tau(0)), \\ \hat{\Pi}^2_\tau(0, r_\tau(0)) &\ge \hat{\Pi}^2_\tau(k, r_\tau(0)). \end{aligned}$$

Thus

$$\hat{\Pi}_\tau(0, r_\tau(0)) - \hat{\Pi}_\tau(k, r_\tau(0)) \ge \sum_{N=3}^{\infty} \{\hat{\Pi}^N_\tau(0, r_\tau(0)) - \hat{\Pi}^N_\tau(k, r_\tau(0))\}.$$

Using Proposition 4.5, 4.6 and assumption P_4, we have for $N \ge 3$,

$$\begin{aligned} |\hat{\Pi}^N_\tau(0, r_\tau(0)) - \hat{\Pi}^N_\tau(k, r_\tau(0))| &\le |\nabla_k \hat{\Pi}^N_\tau(\theta k, r_\tau(0))||k| \\ &\le CN\|xG^1_\tau(x,z)\|_\infty \|G^1_\tau(x,z)\|_2^N \\ &\quad \|G^0_\tau(x,z)\|_2^{N-2}|k| \\ &\le CN(\log L)^4(L^{-1}(\log L)^4)|^{\frac{N}{2}}|k|, \end{aligned}$$

which implies

$$\begin{aligned} \hat{\Pi}_\tau(0, r_\tau(0)) - \hat{\Pi}_\tau(k, r_\tau(0)) &\ge -CL^{-\frac{3}{2}}(\log L)^{10}|k| \\ &\ge -CL^{-\frac{3}{2}}(\log L)^{10}(1 - \hat{D}_L(k)). \end{aligned} \quad (4.39)$$

The second inequality is by (4.17).

Substitute this estimate into (4.38) and then (4.37) we get

$$\hat{F}_\tau(k, r_\tau(0)) \ge (1 - CL^{-\frac{3}{2}}(\log L)^{10})(1 - \hat{D}_L(k)). \quad (4.40)$$

$$||G_\tau^1(x,\rho)||_2^2 \le (1 - CL^{-\frac{3}{2}}(\log L)^{10})^2 || \frac{1}{(1-\hat{D}_L(k))^2} ||_1 - 1. \tag{4.41}$$

By Proposition 4.2, 4.4,

$$\begin{aligned} || \frac{1}{(1-\hat{D}_L(k))^2} ||_1 &\le 1 + \sum_{T=2}^{\lambda \log L} \int T(\hat{D}_L(k)) dk + \sum_{\lambda \log L}^{\infty} \int T(\hat{D}_L(k))^{T-1} dk \\ &\le 1 + CL^{-1}(\log L)^4 + CL^{-2} \end{aligned}$$

Thus for sufficiently large L, $||G_\tau^1(x,\rho)||_2^2 \le CL^{-1}(\log L)^4$.

Now we consider the norm $||x_i G_\tau^1(x,\rho)||$.

From the representation

$$\hat{N}_\tau^1(k,\rho) = \frac{\rho \hat{D}_L(k) + \hat{\Pi}_\tau(k,\rho)}{1 - \rho \hat{D}_L(k) - \hat{\Pi}_\tau(k,\rho)},$$

we have,

$$\begin{aligned} ||x_i G_\tau^1(x,\rho)||_\infty &\le ||x_i N_\tau^1(x,\rho)||_\infty \\ &\le C \int |\partial_{k_i} \hat{N}_\tau^1(k,\rho)| dk \\ &\le C \int \{ | \frac{\rho \partial_{k_i} \hat{D}_L(k) + \partial_{k_i} \hat{\Pi}_\tau(k,\rho)}{1 - \rho \hat{D}_L(k) - \hat{\Pi}_\tau(k,\rho)} | \\ &\quad + | \frac{\rho^2 \hat{D}_L(k) \partial_{k_i} \hat{D}_L(k) + \rho \hat{D}_L(k) \partial_{k_i} \hat{\Pi}_\tau(k,\rho)}{(1 - \rho \hat{D}_L(k) - \hat{\Pi}_\tau(k,\rho))^2} | \\ &\quad + | \frac{\rho \hat{\Pi}_\tau(k,\rho) \partial_{k_i} \hat{D}_L(k) + \hat{\Pi}_\tau(k,\rho) \partial_{k_i} \hat{\Pi}_\tau(k,\rho)}{(1 - \rho \hat{D}_L(k) - \hat{\Pi}_\tau(k,\rho))^2} | dk \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned}$$

To estimate the above terms, we first derive some lower bounds for $\hat{F}_\tau(k,\rho)$.

Similar to (4.40), we have for $\rho \in [1, r_\tau(0)]$,

$$\hat{F}_\tau(k,\rho) \ge C(1 - \hat{D}_L(k)). \tag{4.42}$$

Thus for $\nu = 1, 2$

$$\begin{aligned} \frac{1}{\{1 - \rho \hat{D}_L(k) - \hat{\Pi}_\tau(k,\rho)\}^\nu} &\le C\{1 + \sum_{T=1}^{\lambda \log L} T^{\nu-1} |\hat{D}_L(k)|^T \\ &\quad + \sum_{T=\lambda \log L + 1}^{\infty} T^{\nu-1} |\hat{D}_L(k)|^T \}. \end{aligned}$$

Now by Proposition 4.3, 4.4, we have

$$I_1 \leq C(\log L)^2 + C(\log L)^3 L^{-1} + CL^{-2}.$$

By assumption P_4 and (2.31), we know $\partial_{k_i}\hat{\Pi}_\tau(k,\rho) \leq O(L^{-1}(\log L)^{10})$, so

$$I_2 \leq O(L^{-1}(\log L)^{10}).$$

Similar to the above, we can obtain:

$$\begin{aligned} I_3 &\leq CL^{-1}(\log L)^3 + CL^{-2}, \\ I_4 &\leq O(L^{-1}(\log L)^n) + CL^{-2}, \\ I_5 &\leq O(L^{-1}(\log L)^n). \\ I_6 &\leq O(L^{-2}(\log L)^n) + C, \end{aligned}$$

where n is some positive integer.

We thus finished the case of $||x_i G_\tau^1(x,\rho)||$.

Next, we consider $||G_\tau^1(x,\rho)||_\infty$ and $||\partial z G_\tau^1(x, r_\tau(0))||_\infty$. We have

$$||G_\tau^1(x, r_\tau(0))||_\infty \leq ||G_\tau^1(x,\rho=1)||_\infty + (r_\tau(0)-1))||\partial_z G_\tau^1(x, r_\tau(0))||_\infty.$$

We know from Proposition 4.6 that $||G_\tau^1(x,\rho=1)||_\infty \leq CL^{-1}\log L$.

From the identity $1 - r_\tau(0) - \hat{\Pi}_\tau(0, r_\tau(0)) = 0$ and assumption P_4 we have that

$$r_\tau(0) - 1 \leq |\hat{\Pi}_\tau(0, r_\tau(0))| \leq O(L^{-1}\log L).$$

On the other hand, by Proposition 5.1 and assumption P_4, we have

$$\begin{aligned} \partial_z G_\tau^1(x, r_\tau(0)) &\leq \partial_z G_\tau^0(x, r_\tau(0)) \\ &\leq \frac{1}{r_\tau(0)} G_\tau^1 * G_\tau^0(x, r_\tau(0)) \\ &= \frac{1}{r_\tau(0)}\{G_\tau^1 * G_\tau^1(x, r_\tau(0)) + G_\tau^1(0, r_\tau(0))\} \\ &\leq C||\hat{N}_\tau^1(k, r_\tau(0))||_2^2 + ||G_\tau^1(x, r_\tau(0))||_\infty \\ &\leq C(L^{-1}(\log L)^2) + O(L^{-1}(\log L)^2). \end{aligned} \tag{4.43}$$

Thus under our assumption P_4, we must have

$$||G^1_\tau(x, r_\tau(0))||_\infty \leq C(L^{-1}(\log L)^2) + O(L^{-2}(\log L)^3) \leq C(L^{-1}(\log L)^2). \tag{4.44}$$

Once we have proved the above result on $||G^1_\tau(x, r_\tau(0))||_\infty$, we can repeat the computation on (4.43) and in turn obtain the result for $||\partial z G^1_\tau(x, r_\tau(0))||_\infty$.

Q.E.D.

Corollary 4.1 *For* $|z| \leq r_\tau(0)$,

$$|\hat{\Pi}_\tau(k, z)| \leq CL^{-1}(\log L)^2, \tag{4.45}$$

$$|\partial_z \hat{\Pi}_\tau(k, z)| \leq CL^{-1}(\log L)^3, \tag{4.46}$$

$$|\partial k_i \hat{\Pi}_\tau(k, z)| \leq CL^{-1}(\log L)^6. \tag{4.47}$$

4.3 Fractional Derivatives, Limiting Distribution

We will need the following lemma on fractional derivatives, its proof is contained in the Appendix.

Lemma 4.6 *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *have radius of convergence* R. *We denote for any* $|z| < R$, $\epsilon > 0$,

$$\delta_z^\epsilon f(z) = \sum_{n=0}^{\infty} n^\epsilon a_n z^n. \tag{4.48}$$

Then

$$\delta_z^\epsilon f(z) = C_{1-\epsilon} \int_0^\infty f'(ze^{-\lambda^{1/1-\epsilon}}) e^{-\lambda^{1/1-\epsilon}} d\lambda, \tag{4.49}$$

where $C_{1-\epsilon} = [\epsilon\Gamma(\epsilon)]^{-1}$.

Moreover, if $a_n \geq 0$ *for all* n, *then the above equality holds for* $z = R$.

Let us denote for $\lambda > 0$,

$$p_\lambda = r_\tau(0) e^{-\lambda^{1/1-\lambda}}.$$

Proposition 4.8 *For sufficiently large L, there exists $c > 0$ such that for any $k \in [-\pi,\pi]^3$ and $\lambda > 0$,*

$$\hat{F}_\tau(k,p_\lambda) \geq c[1 - e^{-\lambda^{1/1-\lambda}}\hat{D}_L(k)]. \tag{4.50}$$

Proof:

$$\begin{aligned}
\hat{F}_\tau(k,p_\lambda) &= [\hat{F}_\tau(k,p_\lambda) - \hat{F}_\tau(0,p_\lambda)] + [\hat{F}_\tau(0,p_\lambda) - \hat{F}_\tau(0,r_\tau(0))] \\
&= p_\lambda[1 - \hat{D}_L(k)] + [\hat{\Pi}_\tau(0,p_\lambda) - \hat{\Pi}_\tau(k,p_\lambda)] \\
&\quad + \int_{p_\lambda}^{r_\tau(0)} [-\partial_z \hat{F}_\tau(0,\rho)]d\rho.
\end{aligned}$$

By (4.39), we have $\hat{\Pi}_\tau(0,p_\lambda) - \hat{\Pi}_\tau(k,p_\lambda) \geq -O(L^{-1})(1 - \hat{D}_L(k))$.

By Corollary 4.1, we have $-\partial_z \hat{F}_\tau(0,\rho) \geq C > 0$ if L is large. Thus

$$\int_{p_\lambda}^{r_\tau(0)} [-\partial - z\hat{F}_\tau(0,\rho)]d\rho \geq C(r_\tau(0) - p_\lambda).$$

We can conclude that for $\lambda < \lambda_0 < \infty$, (4.50) holds for some $c > 0$.

For $\lambda > \lambda_0$, by $0 < \hat{N}_\tau(k,p_\lambda) \leq \hat{N}_\tau(0,p_{\lambda_0})$, we conclude $\hat{F}_\tau(k,p_\lambda)$ is bounded from below, and thus (4.50) still holds for some $c > 0$.

Q.E.D.

Theorem 4.2 *For sufficiently large L, $|z| \leq r_\tau(0)$ and $\epsilon \in (0,1)$, there exists $C > 0$, satisfying*

$$\begin{aligned}
\|\delta_z^\epsilon \partial_z G_\tau^1(x,z)\|_\infty &\leq CL^{-1}(\log L)^3, \qquad (4.51)\\
\|\delta_z^\epsilon G_\tau^1(x,z)\|_2 &\leq CL^{-\frac{1}{2}}(\log L)^3, \qquad (4.52)\\
\|x_i \delta_z^\epsilon G_\tau^1(x,z)\|_\infty &\leq C(\log L)^4. \qquad (4.53)
\end{aligned}$$

Proof: We first prove (4.51).

$$\begin{aligned}
\delta_z^\epsilon G_\tau^1(x,r_\tau(0)) &= C_{1-\epsilon} r_\tau(0) \int_0^\infty \partial_z^2 G_\tau^1(x,p_\lambda) e^{-\lambda^{1/1-\epsilon}} d\lambda \\
&\leq C_{1-\epsilon} r_\tau(0) \int_0^\infty 2p_\lambda{}^{-2} G_\tau^1 * G_\tau^1 * G_\tau^0(x,p_\lambda) e^{-\lambda^{1/1-\epsilon}} d\lambda \\
&\leq C \int_0^\infty p_\lambda{}^{-2} e^{-\lambda^{1/1-\epsilon}} \int |\hat{G}_\tau^1(k,p_\lambda)|^2 |\hat{G}_\tau^0(k,p_\lambda)| dk d\lambda.
\end{aligned} \tag{4.54}$$

From the fact that $|\hat{\Pi}_\tau(k,p_\lambda)| \leq |\partial_z\hat{\Pi}_\tau(k,\theta p_\lambda)|p_\lambda \leq CL^{-1}(\log L)^3 p_\lambda$, we have

$$p_\lambda{}^{-1}\hat{G}^1_\tau(k,p_\lambda) = p_\lambda{}^{-1}\frac{p_\lambda\hat{D}_L(k) + \hat{\Pi}_\tau(k,p_\lambda)}{\hat{F}_\tau(k,p_\lambda)} \leq C\hat{F}_\tau(k,p_\lambda)^{-1}.$$

Thus the right side of (4.54) is bounded by

$$\begin{aligned}
& C\int_0^\infty e^{-\lambda^{1/1-\epsilon}}\int(|\hat{D}_L(k)|^2 + |\partial_z\Pi_\tau(k,\theta p_\lambda)|^2)[\hat{F}_\tau(k,p_\lambda)]^{-3}dkd\lambda \\
\leq\ & C\int(|\hat{D}_L(k)|^2 + |\partial_z\Pi_\tau(k,\theta p_\lambda)|^2) \\
& \quad\cdot\int_0^\infty e^{-\lambda^{1/1-\epsilon}}(1 - e^{-\lambda^{1/1-\epsilon}}\hat{D}_L(k))^{-3}d\lambda dk \\
=\ & C\int(|\hat{D}_L(k)|^2 + |\partial_z\Pi_\tau(k,\theta p_\lambda)|^2) \\
& \quad\cdot\frac{1}{\hat{D}_L(k)}\delta^\epsilon_z[(1 - z\hat{D}_L(k))^{-2}]\big|_{z=1}dk \\
\leq\ & C\int\sum_{T=2}^\infty T(T-1)^\epsilon|\hat{D}_L(k)|^T dk \\
& +C\|\hat{\Pi}_\tau(k,r_\tau(0))\|_\infty\int\sum_{T=2}^\infty T(T-1)^\epsilon|\hat{D}_L(k)|T - 2dk \\
\leq\ & CL^{-1}(\log L)^3.
\end{aligned}$$

To prove (4.52), we have

$$
\begin{aligned}
\|\delta_z^\epsilon G_\tau^1(x, r_\tau(0))\|_2 &= \|C_{1-\epsilon} r_\tau(0) \int_0^\infty \partial_z G_\tau^1(x, p_\lambda) e^{-\lambda^{1/1-\epsilon}} d\lambda\|_2 \\
&\le \|C_{1-\epsilon} r_\tau(0) \int_0^\infty p_\lambda{}^{-1} G_\tau^1 * G_\tau^0(x, p_\lambda) e^{-\lambda^{1/1-\epsilon}} d\lambda\|_2 \\
&= \|C_{1-\epsilon} r_\tau(0) \int_0^\infty p_\lambda{}^{-1} \hat{G}_\tau^1(k, p_\lambda) \hat{G}_\tau^0(k, p_\lambda) e^{-\lambda^{1/1-\epsilon}} d\lambda\|_2 \\
&\le \|C(|\hat{D}_L(k)| + |\partial_z \Pi_\tau(k, \theta p_\lambda)|) \\
&\qquad \int_0^\infty e^{-\lambda^{1/1-\epsilon}} \hat{F}_\tau(k, p_\lambda)^{-2} d\lambda\|_2 \\
&\le \|C(|\hat{D}_L(k)| + |\partial_z \Pi_\tau(k, \theta p_\lambda)|) \\
&\qquad \cdot \int_0^\infty e^{-\lambda^{1/1-\epsilon}} (1 - e^{-\lambda^{1/1-\epsilon}} \hat{D}_L(k))^{-2} d\lambda\|_2 \\
&= \|C(|\hat{D}_L(k)| + |\partial_z \Pi_\tau(k, \theta p_\lambda)|) \\
&\qquad \cdot \frac{1}{\hat{D}_L(k)} \delta_z^\epsilon [(1 - z\hat{D}_L(k))^{-1}]\big|_{z=1}\|_2 \\
&\le C \sum_{T=1}^\infty T^\epsilon \|\hat{D}_L(k)^T\|_2 \\
&\qquad + C\|\Pi_\tau(k, \theta p_\lambda)\|_\infty \sum_{T=1}^\infty T^\epsilon \|\hat{D}_L(k)^{T-1}\|_2 \\
&\le C L^{-\frac{1}{2}} (\log L)^3
\end{aligned}
$$

Finally, we prove (4.53). We have

$$
x_i \delta_z^\epsilon G_\tau^1(x, r_\tau(0)) = C_{1-\epsilon} r_\tau(0) \int_0^\infty x_i \partial_z G_\tau^1(x, p_\lambda) e^{-\lambda^{1/1-\epsilon}} d\lambda \tag{4.55}
$$

Repeat the computations as in (3.20) we know for $\lambda > \lambda_0 > 0$,

$$
|x_i G_\tau^1(x, p_\lambda)| \le C.
$$

For $\lambda \le \lambda_0$, the right side of (4.55) is bounded by

$$
\begin{aligned}
& C\int_0^{\lambda_0}\int \partial_{k_i}\{\hat{G}^1_\tau(k,p_\lambda)\hat{G}^0_\tau(k,p_\lambda)\}dke^{-\lambda^{1/1-\epsilon}}d\lambda \\
\le\ & C\int\{|p_\lambda\partial_{k_i}\hat{D}_L(k)|+|\partial_{k_i}\hat{\Pi}_\tau(k,p_\lambda)|\}\int_0^{\lambda_0}\{\frac{1}{[\hat{F}_\tau(k,p_\lambda)]^2} \\
& +2\frac{|p_\lambda\hat{D}_L(k)|+|\hat{\Pi}_\tau(k,p_\lambda)|}{[\hat{F}_\tau(k,p_\lambda)]^3}\}e^{-\lambda^{1/1-\epsilon}}d\lambda dk \\
\le\ & C\int|\partial_{k_i}\hat{D}_L(k)|\int_0^{\lambda_0}e^{-\lambda^{1/1-\epsilon}}[\hat{F}_\tau(k,p_\lambda)]^{-3}d\lambda dk \\
\le\ & C\int|\partial_{k_i}\hat{D}_L(k)|\int_0^{\lambda_0}e^{-\lambda^{1/1-\epsilon}}[1-p_{\lambda\hat{D}_L(k)}]^{-3}d\lambda dk \\
\le\ & C\int|\partial_{k_i}\hat{D}_L(k)|\frac{1}{\hat{D}_L(k)}\delta_z^\epsilon[(1-z\hat{D}_L(k))^{-2}]\big|_{z=1}dk \\
\le\ & C\int|\partial_{k_i}\hat{D}_L(k)|dk+\int\sum_{T=3}^{\lambda\log L}T(T-1)^\epsilon|\partial_{k_i}\hat{D}_L(k)|\hat{D}_L(k)^{T-2}dk \\
& +\int\sum_{T=\lambda\log L+1}T(T-1)^\epsilon|\partial_{k_i}\hat{D}_L(k)|\hat{D}_L(k)^{T-2}dk \\
\le\ & C(\log L)^4.
\end{aligned}
$$

Q.E.D.

Corollary 4.2 *For L sufficiently large, $\epsilon \in (0,1)$ and $|z| \le r_\tau(0)$, there exists $C > 0$ such that*

$$|\delta_z^\epsilon\partial_z\hat{\Pi}_\tau(k,z)| \le CL^{-1}(\log L)^3, \tag{4.56}$$

$$|\delta_z^\epsilon\partial_{k_i}\hat{\Pi}_\tau(k,z)| \le CL^{-1}(\log L)^7. \tag{4.57}$$

Proof: The fractional derivatives induce an extra T^ϵ term. We can modify the proof of Proposition 2.5 by using the following inequalities:

$$(a_1+a_1+\ldots+a_n)^{1+\epsilon} \le n^\epsilon(a_1^\epsilon+a_2^\epsilon+\ldots+a_n^\epsilon),$$

$$(a_1+a_1+\ldots+a_n)^{\epsilon} \le n^\epsilon(a_1^\epsilon+a_2^\epsilon+\ldots+a_n^\epsilon).$$

Q.E.D.

The proof of the following theorem is similar to Theorem 3.2 and is omitted.

Theorem 4.3 *For sufficiently large L, $\hat{\Pi}_\tau(k,z)$ is analytic in the set*

$$\{z : |z| < D_\tau(\epsilon) = r_\tau(0)(1 + \epsilon \log \tau/2\tau)\}.$$

Moreover, in this set, the inequalities of the Corollary 4.6 still hold with the powers of logL increased by 1.

Theorem 4.4 *There exists a sufficiently large $L_0 > 0$, such that for $L \geq L_0$,*

$$\lim_{T\to\infty} \frac{\hat{N}_T(k/LT,T)}{\hat{N}_T(0,T)} = \exp\{-\frac{1}{\pi}\sum_{i=1}^{3}|k_i|\}. \tag{4.58}$$

Thus $W(T)/LT$ converges weakly to the classical Cauchy distribution.

The proof of this theorem can be obtained by a detailed but straightforward modification of the results in section 3.3.

Part II

FINITE HORIZON OPTIMAL INVESTMENT AND CONSUMPTION WITH TRANSACTION COSTS

CHAPTER 5

INTRODUCTION

In 1969, Merton initiated the study on optimal investment and consumption in [27]. His paper is also widely acknowledged as the landmark work that initiated the study of financial markets via continuous-time, stochastic models. The motivation of his study was to understand the interaction among many agents whose individual investment/consumption actions lead to the market price formulation. Merton chose to study this issue by first understanding the behavior of a single agent acting as a price-taker and seeking to maximize expected utility consumption. In the setting of frictionless market, he was able to obtain a close-form solution to the stochastic control problem.

Merton's model has been generalized in several directions. The utility function has been generalized from power functions to concave increasing functions by [22]. Market coefficients depending in an adapted way on an underlying Brownian motion were treated by [2] and [19]. The existence and uniqueness of equilibrium price was proved in the complete market setting by [7] [9] [10] [18] [20] and [21]. Considerable amount of effort has been exerted on the incomplete market situation, where the investor's portfolio is restricted to a convex subset. Most of the works in this direction use the convex dual martingale method, which was first proposed by [31] and developed among others by [17] [23] and most noticeably [5]. See also [14] [32] for partial differential equation method. The study of equilibrium problem in incomplete markets depends on the ability to characterize individual utility optimization with random endowment streams. Here the convex dual method expires because the value function of the dual problem is no longer convex. Cucuo [4] was able to show the existence of the optimal policy by using the martingale method. (see also [11] for recent advances) Duffie *et al.* [8] used viscosity solution techniques to show the value function of the Hamilton-Jacobi-Bellman (HJB)equation is smooth and provided a feedback form optimal policy.

The introduction of proportional transaction costs to Merton's model was first accomplished by Magill and Constantinides in [26]. Though this paper shows clear insight into the nature of the optimal policy, some of the mathematical tools needed were not available to the authors at the time. It is understood by now that this problem is one of a singular stochastic control, that is, the optimal transaction occur only when the investor's bond/stock ratio is on the boundary of the no-transaction region. In the parlance of stochastic analysis, the singular transaction processes are the local time associated with the boundary of no-transaction region. This boundary of the no-transaction region, which is a free boundary, is not a priori known to the investor and must be solved as part of the problem. Davis and Norman [6] were the first to realize the above problem formulation. Also they were the first to provide a rigorous analysis of the underlying HJB equation. More recently, Shreve and Soner [28] employed the newly developed viscosity solution concept to analyze the HJB equation. Their analysis much clarified the vague points of previous results and provided a framework in which the liquidity premium estimation can be accomplished. Perhaps the most valuable point of their work is that they demonstrated in a clear as a bell fashion, the power of viscosity solution approach in mathematical finance. My thesis work was deeply influenced by their stimulating method. In particular, the viscosity solution method stands as the fundamental base of my analysis.

The fundamental work of viscosity solution theory is due to P.L. Lions, M.G. Crandall, L.C. Evans, R. Jensen and H. Ishii. The survey article by Crandall, Ishii and Lions [3] provides a good account of the viscosity theory.

The application to stochastic control is reported in the book by Fleming and Soner [13]. The classical approach to stochastic control is to construct a classical solution to the HJB equation and use this solution to find the optimal policy. Because of the high nonlinearity of the HJB equations, it is usually very difficult to find such classical solutions directly. By contrast, the viscosity solution approach is to start with a candidate solution - the value function of the control problem, and use the dynamical programming principle to show that it solves the HJB equation in the viscosity sense. Although this viscosity solution is usually too weak to suffice a verification theorem, it still provides us the invaluable connection between stochastic control and partial differential equation theory. The regularity of the value function can often be upgraded when the two theories can fertilize each other fruitfully.

In my knowledge, all of the previous works on inter-temporal utility optimization with transaction costs are restricted to the infinite horizon case. This case has the advantage that the value function can be reduced to one dimension, thanks to the homothetic property. Thus the HJB equation becomes a nonlinear ODE, and the boundaries of the no-transaction region are fixed with respect to time. These facts simplify the analysis significantly. In the finite horizon case being considered here, I have to face the essential difficulties of dealing with nonlinear parabolic PDE and the possibility that the free boundaries are moving with time.

The main results of my work are: the value function is shown to be C^∞ in the non-degenerate no-transaction region, thus solve the optimization problem in that region; also, it is shown that the two transaction free boundaries exist on all time horizon, and moreover we provide one upper bound and two lower bounds for their locations. This is achieved by solving the Skorohod problem locally and using this local solution to provide $\epsilon-$ optimal consumption/transaction strategies.

The main tools I use are viscosity solution method, Campanato space method from partial differential equations and martingale technique. In particular, I develop a "bootstrap" technique to upgrade the regularity of viscosity solution. I hope this technique will also be useful in other problems of similar characteristic.

5.1 The Financial Market

We consider a market in which two securities are traded continuously in a finite time horizon $[0, T]$. One security is the risk free bond whose price $P_0(t)$ evolves according to the equation

$$dP_0(t) = rP_0(t)dt, \quad P_0(0) = p_0; \quad t \in [0, T].$$

Another security is the risky stock whose price $P_1(t)$ follows the equation

$$dP_1(t) = \alpha P_1(t)dt + \sigma P_1(t)dW_t, \quad P_1(0) = p_1; \quad t \in [0, T].$$

Where the process $W = \{W_t, \mathcal{F}_t, 0 \leq t \leq T\}$ is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and the filtration $\{\mathcal{F}_t\}$ satisfies the usual condition. It is assumed the interest rate r, mean rates of return α and the volatility σ are positive constants and $\alpha > r$.

An agent, with an initial position (x, y) in the bond and stock markets at time t, has to make decisions on his control processes $(c(s), L(s), M(s); s \geq t)$ in order to maximize his expected utility function $V(t, x, y)$.

We assume that consumption can only take place at bond market. The consumption rate process will be denoted by $c = \{c(s), \mathcal{F}_s, s \in [t, T]\}$. It is nonnegative and satisfies

$$\int_t^T c(s)ds < \infty, \quad a.s., \quad t \in [0, T].$$

Transactions between bond and stock markets incur proportional transaction cost. The transaction rate from bond (stock) market to stock (bond) market is λ (μ). $0 \leq \lambda, \mu < 1$. In the following we will denote $\{L(s), \mathcal{F}_s, s \in [t, T]\}$ $(\{M(s), \mathcal{F}_s, t \in [t, T]\})$ to be the cumulative transaction value from bond (stock) market to stock (bond) market. They are increasing, a.s. finite RCLL processes.

We denote the agent's position at time s ($t \le s \le T$) in bond and stock market by $(X(s), Y(s))$. Then if the control processes $(c(s), L(s), M(s); s \ge t)$ are given, (X,Y) will evolve according to the following equations:

$$dX(s) = (rX(s) - c(s))ds - dL(s) + (1-\mu)dM(s), \quad (5.1)$$

$$dY(s) = \alpha Y(s)ds + \sigma Y(s)dW(s) + (1-\lambda)dL(s) - dM(s), \quad (5.2)$$

with $X(t-) = x$, $Y(t-) = y$.

To ensure there is no arbitrage opportunity in the market, we require $(X(t), Y(t))$ be in the following solvency region

$$\mathcal{S} = \{(x,y) : (1-\lambda)x + y > 0, x + (1-\mu)y > 0\}.$$

This solvency region $\mathcal{S}$ has two boundaries:

$$\partial_1\mathcal{S} = \{(x,y) : y \le 0, x + \frac{y}{1-\lambda} = 0\},$$

$$\partial_2\mathcal{S} = \{(x,y) : y > 0, x + (1-\mu)y = 0\}.$$

We will denote $Q = [0,T) \times \mathcal{S}$. The parabolic boundary of domain Q will be denoted by $\partial^* Q$.

An admissible control for $(t,x,y) \in \overline{Q}$ is an investment/consumption strategy that keeps the investor's balance in bond and stock markets within the solvency region.

$$\mathcal{A}(t,x,y) = \{(c(s), L(s), M(s)) : (X(s), Y(s)) \in \overline{\mathcal{S}}, \quad \forall s \in [t,T]\}. \quad (5.3)$$

Our utility function has the form $U(c) = c^p/p$ for $c \ge 0$, where $0 < p < 1$. Its convex dual is given by

$$\overline{U}(\overline{c}) = \sup_{c \ge 0}\{U(c) - \overline{c}c\} = \frac{1-p}{p}(\overline{c})^{-\frac{p}{1-p}}.$$

A small investor, with an initial position (x,y) in the bond and stock markets at time t, has to make decisions on his control processes $(c(s), L(s), M(s);$ $s \ge t)$ in order to maximize his expected utility function.

$$V(t,x,y) = \sup_{(c(s),L(s),M(s)) \in \mathcal{A}(t,x,y)} E\int_t^T \frac{c(s)^p}{p} ds, \qquad \forall (t,x,y) \in \overline{Q}, \quad (5.4)$$

under the state processes constraint (5.1) and (5.2).

5.2 The Dynamical Programming Principle

The following proposition states the *dynamical programming principle.* This is the starting point of the dynamical programming method. We refer the reader to [13] for its proof.

Proposition 5.1 *Let* $(t, x, y) \in \overline{Q}$ *and* τ *be any* $(\mathcal{F}_s, s \geq t)$ *- stopping time. Then*

$$V(t,x,y) = \sup_{(c(s),L(s),M(s))\in\mathcal{A}(t,x,y)} E\{\int_t^{T\wedge\tau} U(c(s))ds + I_{(\tau<T)}V(\tau, X(\tau), Y(\tau))\}. \tag{5.5}$$

From the dynamical programming principle, we observe that if position (t, x_2, y_2) could be reached from (t, x_1, y_1) by a direct transaction, then $V(t, x_2, y_2) \leq V(t, x_1, y_1)$. So intuitively, we have

$$\frac{\partial V}{\partial x} - (1-\lambda)\frac{\partial V}{\partial y} \geq 0, \quad \frac{\partial V}{\partial y} - (1-\mu)\frac{\partial V}{\partial x} \geq 0, \quad \forall (t,x,y) \in Q.$$

Let us now formally derive the *Hamilton-Jacobi-Bellman (HJB) equation* for our problem.

For $\varphi \in C^{1,2}(Q)$, we will denote

$$\mathcal{L}\varphi(t,x,y) = -\frac{\partial \varphi}{\partial t} - \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 \varphi}{\partial y^2} - \alpha y \frac{\partial \varphi}{\partial y} - rx\frac{\partial \varphi}{\partial x}, \quad (t,x,y) \in Q.$$

Let us assume tentatively that $V \in C^{1,2}(Q)$, and apply Ito's formulae for RCLL semimartingales to V. We have for any $(\mathcal{F}_s, s \geq t)$ - stopping time $\tau \geq t$,

$$\begin{aligned} V(\tau, X(\tau), Y(\tau)) &= V(t,x,y) + \int_t^\tau \{-\mathcal{L}V - c(s)\frac{\partial V}{\partial x}\}ds \\ &+ \int_t^\tau \{-\frac{\partial V}{\partial x} + (1-\lambda)\frac{\partial V}{\partial y}\}dL^c(s) \\ &+ \int_t^\tau \{-\frac{\partial V}{\partial y} + (1-\mu)\frac{\partial V}{\partial x})\}dM^c(s) \\ &+ \sum_{0\leq s\leq\tau} \{V(s, X(s), Y(s)) - V(s-, X(s-), Y(s-))\} \\ &+ \int_t^\tau \sigma y \frac{\partial V}{\partial y} dW(s), \end{aligned} \tag{5.6}$$

where L^c and M^c are the continuous part of L and M respectively.

Let us further assume that the domain Q could be divided into three regions SS, SMM and NT, such that if the investor is in region SS, it is optimal to make instant transactions from stock market to bond market; if the investor is in SMM, it is optimal to make instant transactions from bond market to stock market; while if the investor is in NT, the optimal strategy is to make no transactions, just make appropriate consumptions.

From Proposition 5.1 and (5.6) we would expect V satisfy the following:

$$\begin{aligned} \mathcal{L}V - \overline{U}(\frac{\partial V}{\partial x}) &= 0, \qquad \forall (t,x,y) \in NT, \\ \frac{\partial V}{\partial x} - (1-\lambda)\frac{\partial V}{\partial y} &= 0, \qquad \forall (t,x,y) \in SMM, \\ \frac{\partial V}{\partial y} - (1-\mu)\frac{\partial V}{\partial x} &= 0, \qquad \forall (t,x,y) \in SS. \end{aligned}$$

We obtain the *Hamilton-Jacobi-Bellman* equation:

$$min\{\mathcal{L}V - \overline{U}(\frac{\partial V}{\partial x}),\ \frac{\partial V}{\partial y} - (1-\mu)\frac{\partial V}{\partial x},\ \frac{\partial V}{\partial x} - (1-\lambda)\frac{\partial V}{\partial y}\} = 0, \quad (t,x,y) \in Q. \tag{5.7}$$

The following boundary condition for V can be derived from the oscillation property of Brownian martingale. (e.g., Remark 2.1 of [28])

Proposition 5.2 *For* $(t,x,y) \in \partial^* Q$, *we have*

$$V(t,x,y) = 0. \tag{5.8}$$

CHAPTER 6

VISCOSITY SOLUTION OF THE HJB EQUATION

6.1 Existence Result

We first derive some fundamental properties of the value function V.

Proposition 6.1 *For $(t,x,y) \in Q$, we have the following lower bounds for V:*

$$V(t,x,y) \geq \begin{cases} \frac{1}{p} f(t)^{1-p}(x+(1-\mu)y)^p, & y \geq 0, \\ \frac{1}{p} f(t)^{1-p}(x+y/(1-\lambda))^p, & y < 0, \end{cases} \tag{6.1}$$

where

$$f(t) = \frac{1-p}{rp}\left(e^{\frac{rp}{1-p}(T-t)} - 1\right). \tag{6.2}$$

Proof: For any $(t,x,y) \in Q$, the investor can always choose to transfer all his money from stock to bond and then try to optimize his expected utility function only within the bond market. This optimization problem could be stated as

$$\sup_{c \in \mathcal{A}_t} E \int_t^T \frac{c(s)^p}{p} ds,$$

$$dX(s) = (rX(s) - c(s))ds, \qquad X(t) = x,$$

where the admissible control set $\mathcal{A}_t = \{c(s) \geq 0 : \text{such that } X(s) \geq 0, \text{ for all } s \geq t\}$.

The dynamical programming equation of this problem is

$$\frac{\partial V}{\partial s} + \sup_{c>0}\{rx\frac{\partial V}{\partial x} - c\frac{\partial V}{\partial x} + \frac{c^p}{p}\} = 0, \qquad (s,x) \in [t,T) \times (0,\infty), \tag{6.3}$$

$$V(s,x) = 0, \qquad \forall (s,x) \in \partial^*[t,T) \times (0,\infty). \tag{6.4}$$

We try the form $V(s,x) = \frac{1}{p}f(s)^{1-p}x^p$ and substitute it in (6.3) (6.4). It is easy to see $f(s)$ has to satisfy the following equation:

$$f'(s) + \frac{rp}{1-p}f(s) + 1 = 0, \qquad f(T) = 0.$$

So

$$f(s) = \frac{1-p}{rp}(e^{\frac{rp}{1-p}(T-s)} - 1).$$

We have obtained a smooth solution of the HJB equation (6.3) (6.4). By verification theorem (e.g., [13] I.5.1), we have

$$V(s,x) = \frac{1}{p}(f(s))^{1-p}x^p.$$

If $y \geq 0$, the initial transaction to bond market will result in a balance $x + (1-\mu)y$; while if $y < 0$, the balance will be $x + y/(1-\lambda)$. The proof is completed by replacing x with these two terms in respective situations.

Q.E.D.

Proposition 6.2 *For $(t,x,y) \in Q$, we have the following upper bound for V,*

$$V(t,x,y) \leq \frac{1}{p}f_\nu(t)^{1-p}(x+y)^p, \tag{6.5}$$

where

$$f_\nu(t) = \frac{1-p}{\nu p}(e^{\frac{\nu p}{1-p}(T-t)} - 1), \qquad \nu = r + \frac{(\alpha - r)^2}{2\sigma^2(1-p)}. \tag{6.6}$$

Proof: Suppose we place the investor in a Merton's frictionless market – a market with the same market coefficients as in Section 5.1 but free of transaction cost, then the value function of Merton's model must be an upper bound for our value function V. (To make this statement rigorous, we can check that our admissible control set $\mathcal{A}(t,x,y)$ as defined in (5.3) is a subset of the admissible control set of Merton's model.)

The calculation of the value function V(6.5) and (6.6) is standard, it could be found in [12] p160-161.

Q.E.D.

Proposition 6.3 *The value function V has the following properties.*

1. *(Concave property) For each fixed $t \in [0,T]$, $V(t,\cdot,\cdot)$ are concave functions on $\overline{S}$. In particular, $V(t,\cdot,\cdot)$ are continuous on $\overline{S}$.*

2. *(Homothetic property) For $(t,x,y) \in \overline{Q}$, $\gamma \geq 0$, $V(t,\gamma x,\gamma y) = \gamma^p V(t, x, y)$.*

3. *(Monotone property) For fixed $(x,y) \in \overline{S}$, $V(\cdot,x,y)$ are decreasing functions.*

The proof of the first two statements in the above proposition are essentially similar to that of [28], the third statement could be verified directly.

In Merton's model, if we change the underlying probability P to the so called neutral probability $\tilde{P}$, then we can obtain a super-martingale property for the sum of the discounted wealth and consumption processes. This property still holds in our model.

Let $\kappa = \sigma^{-1}(\alpha - r)$, and for $t \in [0,T]$ define

$$\theta_s^t = exp\{-\kappa(W(s) - W(t)) - \frac{1}{2}\kappa^2(s-t)\}, \quad s \geq t, \tag{6.7}$$

$$\tilde{P}(A) = E(\theta_T^t I_A), \qquad A \in \mathcal{F}_T. \tag{6.8}$$

By Gisanov's theorem, $\tilde{P}$ is a probability measure on $(\Omega, \mathcal{F}_T)$ and under this measure the process

$$\tilde{W}(s) = W(s) - W(t) + \kappa(s-t), \;\; \mathcal{F}_s, \;\; s \geq t,$$

is a Brownian motion(e.g., [24] III.5).

If we denote $Z(s) = X(s) + Y(s)$, then it will evolve according to the following equation:

$$dZ(s) = (rZ(s) - c(s))ds + \sigma Y(s)d\tilde{W}(s) - \lambda dL(s) - \mu dM(s), \qquad s \geq t.$$

Thus, if the control processes $(c(s), L(s), M(s)) \in \mathcal{A}(t, x, y)$, then the process

$$e^{r(s-t)}Z(s) + \int_t^s e^{-r(u-t)}c(u)du + \int_t^s e^{-r(u-t)}(\lambda dL(u) + \mu dM(u))$$
$$= z_t + \int_t^s e^{-r(u-t)}\sigma Y(u)d\tilde{W}(u), \tag{6.9}$$

is a nonnegative super-martingale with respect to the probability measure $\tilde{P}$.

Proposition 6.4 *The value function V is continuous on $\overline{Q}$.*

Proof: From Proposition 6.3 we know for fixed $t \in [0, T]$, $V(t, \cdot, \cdot)$ are continuous functions; and for fixed $(x, y) \in \mathcal{S}$, $V(\cdot, x, y)$ are decreasing functions. By Dini's theorem, we only have to show for each fixed $(x, y) \in \overline{\mathcal{S}}$, $V(\cdot, x, y)$ is continuous with respect to t.

First, we show for any $t_0 \in [0, T)$, $\lim_{t_n \downarrow t_0} V(t, x, y) = V(t_0, x, y)$.

For an arbitrary $\epsilon > 0$, we choose $(c(s), L(s), M(s)) \in \mathcal{A}(t_0, x, y)$ such that

$$V(t_0, x, y) \leq E\int_{t_0}^T \frac{c(s)^p}{p} ds + \epsilon.$$

Then $\{c(s), L(s), M(s); \quad t_0 \leq s \leq T - (t_n - t_0)\} \in \mathcal{A}(t_n, x, y)$, thus

$$E\int_{t_n}^T \frac{c(s)^p}{p} ds \leq V(t_n, x, y).$$

By monotone theorem,

$$\lim_{t_n \downarrow t_0} E\int_{t_n}^T \frac{c(s)^p}{p} ds = E\int_{t_0}^T \frac{c(s)^p}{p} ds.$$

The right continuity of $V(\cdot, x, y)$ follows immediately.

Next we show for any $t_0 \in (0, T]$, $\lim_{t_n \uparrow t_0} V(t, x, y) = V(t_0, x, y)$.

We choose for each index n, $(c_n(s), L_n(s), M_n(s);\ \ s \geq t_n) \in \mathcal{A}(t_n, x, y)$ such that

$$V(t_n, x, y) \leq E\int_{t_n}^{T} \frac{c_n(s)^p}{p} ds + \frac{1}{n}.$$

We observe that $\{c_n(s), L_n(s), M_n(s); t_n \leq s \leq T - (t_0 - t_n)\} \in \mathcal{A}(t_0, x, y)$. Thus

$$E\int_{t_n}^{T-(t_0-t_n)} \frac{c_n(s)^p}{p} ds \leq V(t_0, x, y).$$

For $1 < \delta < \frac{1}{p}$, let $\delta' = \delta/(\delta - 1)$. We have

$$\begin{aligned}
E\int_{T-(t_0-t_n)}^{T} \frac{c_n(s)^p}{p} ds &\leq \frac{1}{p}(t_0 - t_n)^{\frac{1}{\delta'}} E\{\int_{T-t_0+t_n}^{T} c_n(s)^{p\delta} ds\}^{\frac{1}{\delta}} \\
&\leq \frac{1}{p}(t_0 - t_n)^{\frac{1}{\delta'}} E\{\int_{t_n}^{T} c_n(s)^{p\delta} ds\}^{\frac{1}{\delta}} \\
&\leq \frac{1}{p}(t_0 - t_n)^{\frac{1}{\delta'}} e^{r(T-t_n)p}\{\tilde{E}[(\theta_T^{t_n})^{-1} \\
&\quad \cdot \int_{t_n}^{T} (e^{-r(s-t_n)} c_n(s))^{p\delta} ds]\}^{\frac{1}{\delta}} \quad \text{(by (6.8))} \\
&\leq \frac{1}{p}(t_0 - t_n)^{\frac{1}{\delta'}} e^{r(T-t_n)p}\{\tilde{E}[\int_{t_n}^{T} (e^{-r(s-t_n)} c_n(s))^{p\delta} ds]^{\frac{1}{p\delta}}\}^{p} \\
&\quad \cdot \{\tilde{E}(\theta_T^{t_n})^{\frac{-1}{1-p\delta}}\}^{\frac{1-p\delta}{\delta}} \text{(by Holder inequality)} \\
&\leq \frac{1}{p}(t_0 - t_n)^{\frac{1}{\delta'}} e^{r(T-t_n)p}(T - t_n)^{\frac{1-p\delta}{\delta}} \\
&\quad \{\tilde{E}\int_{t_n}^{T} e^{-r(s-t_n)} c_n(s) ds\}^p e^{\frac{\kappa^2(T-t_n)}{2\delta}(\frac{1}{1-p\delta}+1)} \\
&\quad \text{(by Holder inequality and (6.7))} \\
&\leq C(t_0 - t_n)^{\frac{1}{\delta'}} (x + y)^p, \text{(by (6.9))}
\end{aligned}$$

where C is a positive constant that does not depend on n.

Combine the above estimates we have

$$\begin{aligned}
V(t_n, x, y) &\leq E\int_{t_n}^{T-(t_0-t_n)} \frac{c_n(s)^p}{p} ds + E\int_{T-(t_0-t_n)}^{T} \frac{c_n(s)^p}{p} ds + \frac{1}{n} \\
&\leq V(t_0, x, y) + C(t_0 - t_n)^{\frac{1}{\delta'}} (x + y)^p + \frac{1}{n}.
\end{aligned}$$

The proof is completed by letting $n \to \infty$.

Q.E.D.

The following lower bound of $\frac{\partial V}{\partial x}(t,x,y)$ will be an important a priori estimate for the development of this paper. It allows us to deal with the nonlinear term $(\frac{\partial V}{\partial x})^{-\frac{p}{1-p}}$ in the equation (5.7).

Proposition 6.5 *Let $(t,x,y) \in Q$, $\varphi(x,y) \in C^2(S)$. If $\varphi \geq V(t,\cdot,\cdot)$ or $\varphi \leq V(t,\cdot,\cdot)$ in a neighborhood of (x,y) and $\varphi(t,x,y) = V(t,x,y)$. Then*

$$\frac{pV(t,x,y)}{x+y/(1-\lambda)} \leq \frac{\partial\varphi}{\partial x}(t,x,y) \leq \frac{pV(t,x,y)}{x+(1-\mu)y}, \qquad y_0 \geq 0,$$
$$\frac{pV(t,x,y)}{x+(1-\mu)y} \leq \frac{\partial\varphi}{\partial x}(t,x,y) \leq \frac{pV(t,x,y)}{x+y/(1-\lambda)} \qquad y_0 < 0.$$

The above lower bounds could be derived from the homothetic property and a direct transaction argument. We refer the reader to [28] Corollary 3.7 for the details.

Definition 6.1 *A function $V \in C(\overline{Q})$ is said to be a viscosity solution of the Hamilton-Jacobi-Bellman equation (5.7) (5.8) if the following are satisfied,*

1. *V is a viscosity sub-solution of (5.7). That is, for any $\varphi \in C^{1,2}(Q)$ and any $(t_0,x_0,y_0) \in Q$ such that (t_0,x_0,y_0) is a local maximum point of $V-\varphi$, we have*

$$min\{\mathcal{L}\varphi - \overline{U}(\frac{\partial\varphi}{\partial x}),\ \frac{\partial\varphi}{\partial y} - (1-\mu)\frac{\partial\varphi}{\partial x},\ \frac{\partial\varphi}{\partial x} - (1-\lambda)\frac{\partial\varphi}{\partial y}\}(t_0,x_0,y_0) \leq 0.$$

2. *V is a viscosity super-solution of (5.7). That is, for any $\varphi \in C^{1,2}(Q)$ and any $(t_0,x_0,y_0) \in Q$ such that (t_0,x_0,y_0) is a local minimum point of $V-\varphi$, we have*

$$min\{\mathcal{L}\varphi - \overline{U}(\frac{\partial\varphi}{\partial x}),\ \frac{\partial\varphi}{\partial y} - (1-\mu)\frac{\partial\varphi}{\partial x},\ \frac{\partial\varphi}{\partial x} - (1-\lambda)\frac{\partial\varphi}{\partial y}\}(t_0,x_0,y_0) \geq 0.$$

The proof of the following theorem is similar to [28] theorem 7.7 and we refer the readers to that paper for the details.

Theorem 6.1 *The value function V defined by (5.4) is a viscosity solution of the Hamilton-Jacobi-Bellman equation (5.7) with boundary condition (5.8).*

Remark 6.1 *An examination on the proof of the theorem shows that the value function possesses viscosity property in a stronger sense. That is, in Definition 6.1, the test function φ can be relaxed to be a local maximum(minimum) only in the half ball $B_r(t_0, x_0, y_0) \cap \{t : t > t_0\}$. This fact will be used in our proof of regularity results.*

6.2 Comparison Result

In order to prove that the equation (5.7) (5.8) has a unique viscosity solution, we have to make some restrictions on the solution space.

Definition 6.2 *A function V on Q is said to be in class $\mathcal{D}_0$ if it satisfies the following:*

1. *V is a nonnegative continuous function on $\overline{Q}$. $V = 0$ on the $\partial^* Q$.*

2. *V possesses the homothetic property, concave property and monotone property as prescribed in Proposition 6.3.*

3. *For $(t, x, y) \in Q$*

$$\begin{aligned} V(t,x,y) &\geq \frac{1}{p} f(t)^{1-p}(x + (1-\mu)y)^p, \qquad y \geq 0, \\ V(t,x,y) &\geq \frac{1}{p} f(t)^{1-p}(x + y/(1-\lambda))^p, \qquad y < 0, \end{aligned}$$

   where $f(t)$ is given by (6.2).

4. *For $(t, x, y) \in Q$*

$$\begin{aligned} \frac{\partial V^{\pm}}{\partial x}(t,x,y) &\geq pV(t,x,y)(x + \frac{y}{1-\lambda})^{-1}, \qquad y \geq 0, \\ \frac{\partial V^{\pm}}{\partial x}(t,x,y) &\geq pV(t,x,y)(x + (1-\mu)y)^{-1}, \qquad y < 0. \end{aligned}$$

The homothetic property allows us to transform the Hamilton-Jacobi-Bellman equation (5.7) into an equivalent equation that has only one space variable and is defined on a bounded domain.

Let us denote $I = (-(1-\lambda)/\lambda, 1/\mu)$, $Q_1 = [0,T) \times I$, and define

$$u(t,z) = V(t, 1-z, z), \qquad (t,z) \in Q_1. \tag{6.10}$$

By homothetic property we have

$$V(t,x,y) = (x+y)^p u(t, y/(x+y)). \tag{6.11}$$

Proposition 6.6 *Assume $u(t,z)$ and $V(t,x,y)$ are related by (6.11). Then $V(t,x,y)$ is a viscosity solution of (5.7) (5.8) if and only if $u(t,z)$ is a viscosity solution of the following equation*

$$\begin{aligned} min \quad & \{-\frac{\partial u}{\partial t} - d_1(z)pu - d_2(z)\frac{\partial u}{\partial z} - d_3(z)\frac{\partial^2 u}{\partial z^2} - \overline{U}(pu - z\frac{\partial u}{\partial z}), \\ & pu + d_4(z)\frac{\partial u}{\partial z}, \qquad pu - d_5(z)\frac{\partial u}{\partial z}\} = 0, \qquad (t,z) \in Q_1, \end{aligned} \tag{6.12}$$

where

$$\begin{aligned} d_1(z) &= r + (\alpha - r)z - \frac{1}{2}\sigma^2(1-p)z^2, \\ d_2(z) &= (\alpha - r)z(1-z) - \sigma^2(1-p)z^2(1-z), \\ d_3(z) &= \frac{1}{2}\sigma^2 z^2(1-z)^2, \\ d_4(z) &= \frac{1}{\mu}(1-\mu z), \\ d_5(z) &= \frac{1}{\lambda}(1-\lambda(1-z)), \end{aligned}$$

with boundary condition

$$u(t,z) = 0, \qquad (t,z) \in \partial^* Q_1. \tag{6.13}$$

We will denote $\mathcal{D}_1 = \{u(t,z) : u(t,z) = V(t, 1-z, z), (t,z) \in Q_1,$ for some $V(t,x,y) \in \mathcal{D}_0\}$.

Now let us transform u by

$$w(t,z) = e^{k_0 t} u(t,z), \quad (t,z) \in Q_1, \tag{6.14}$$

where $k_0 = \max_{z \in I} |pd_1(z)|$.

We will denote $\mathcal{D}_2 = \{w : w(t,z) = e^{k_0 t} u(t,z), (t,z) \in Q_1,$ for some $u \in \mathcal{D}_1\}$.

Similar to Proposition 6.6, we have

Proposition 6.7 *Suppose $w(t,z)$ and $u(t,z)$ are related by (6.14). Then u is a viscosity solution of (6.12) and (6.13) if and only if w is a viscosity solution of the following equation*

$$\begin{aligned} min \quad & \{e^{-k_0 t}[-\frac{\partial w}{\partial t} + (k_0 - d_1(z)p)w - d_2(z)\frac{\partial w}{\partial z} - d_3(z)\frac{\partial^2 w}{\partial z^2}] \\ & -e^{\frac{pk_0 t}{1-p}}\overline{U}(pw - z\frac{\partial w}{\partial z}), \qquad e^{-k_0 t}[pw + d_4(z)\frac{\partial w}{\partial z}], \\ & e^{-k_0 t}[pw - d_5(z)\frac{\partial w}{\partial z}]\} = 0, \quad (t,z) \in Q_1, \end{aligned} \tag{6.15}$$

$$w(t,z) = 0, \qquad (t,z) \in \partial^* Q_1. \tag{6.16}$$

We notice that in (6.15), the three terms inside the minimum are increasing with respect to w, this fact will useful in our proof of comparison result.

To simplify the notations, we will use (q_1, q_2, A) to represent $(\frac{\partial w}{\partial t}, \frac{\partial w}{\partial z}, \frac{\partial^2 w}{\partial z^2})$. We will also use the notations

$$\begin{aligned} H^{(1)}_{k_0}(t,z,w,q_1,q_2,A) &= e^{-k_0 t}[-q_1 + (k_0 - d_1(z)p)w - d_2(z)q_2 - d_3(z)A] \\ &\quad -e^{\frac{pk_0 t}{1-p}}\frac{1-p}{p}(pw - zq_2)^{-\frac{p}{1-p}}, \text{ for } pw - zq_2 > 0, \\ H^{(2)}_{k_0}(t,z,w,q_2) &= e^{-k_0 t}(pw + d_4(z)q_2), \\ H^{(3)}_{k_0}(t,z,w,q_2) &= e^{-k_0 t}(pw - d_5(z)q_2). \end{aligned}$$

$$\begin{aligned} H_{k_0}(t,z,w,q_1,q_2,A) = \min\{&H^{(1)}_{k_0}(t,z,w,q_1,q_2,A), \quad H^{(2)}_{k_0}(t,z,w,q_2), \\ &H^{(3)}_{k_0}(t,z,w,q_2)\}. \end{aligned} \tag{6.17}$$

Definition 6.3 *Let W be a continuous function on $\overline{Q_1}$,*

1. *The set of second order (parabolic) super-differentials of W at $(t,z) \in Q_1$ is*

$$D^{+(1,2)}W(t,z) = \{(q_1,q_2,A) \in R^3 : \limsup_{(h_1,h_2)\to(0,0)} \frac{W(t+h_1, z+h_2) - W(t,z) - q_1 h_1 - q_2 h_2 - \frac{1}{2}Ah_2^2}{|h_1| + h_2^2} \le 0\}.$$

2. *The set of second order (parabolic) sub-differentials of W at $(t,z) \in Q_1$ is*

$$D^{-(1,2)}W(t,z) = \{(q_1,q_2,A) \in R^3 : \liminf_{(h_1,h_2)\to(0,0)} \frac{W(t+h_1,z+h_2) - W(t,z) - q_1h_1 - q_2h_2 - \frac{1}{2}Ah_2^2}{|h_1| + h_2^2} \geq 0\}.$$

Definition 6.4 *Let W be a continuous function on $\overline{Q_1}$,*

1. *A triplet (q_1,q_2,A) belongs to $cD^{+(1,2)}W(t,z)$ if there exists a sequence $(t_m,z_m) \to (t,z)$, and another sequence $(q_1^m,q_2^m,A^m) \to (q_1,q_2,A)$ such that*
$$(q_1^m,q_2^m,A^m) \in D^{+(1,2)}W(t,z).$$

2. *A triplet (q_1,q_2,A) belongs to $cD^{-(1,2)}W(t,z)$ if there exists a sequence $(t_m,z_m) \to (t,z)$, and another sequence $(q_1^m,q_2^m,A^m) \to (q_1,q_2,A)$ such that*
$$(q_1^m,q_2^m,A^m) \in D^{-(1,2)}W(t,z).$$

Definition 6.5 *A continuous function W on $\overline{Q}_1$ is called semi-convex if there exists a constant $K > 0$, such that the function $W(t,z) + K(t^2+z^2)$ is convex on $\overline{Q}_1$. W is called semi-concave if $-W$ is semi-convex.*

The following two lemmas are standard results in viscosity solution theory. We state them for completeness of our proof. For a detailed exposition of viscosity solution theory, we refer the readers to [13].

The first lemma combines Alexandrov's result on a.s. twice differentiability of convex functions, Jensen's maximal principle and Ishii's procedure.

Lemma 6.1 *Let W,V be semi-convex and semi-concave respectively on $\overline{Q_1}$. Suppose that ϕ is twice continuously differentiable on $Q_1 \times Q_1$ and $\Phi(t,z,s,w) = W(t,z) - V(s,w) - \phi(t,z,s,w)$ attains an interior maximum $(\bar{t},\bar{z}),(\bar{s},\bar{w}) \in (0,T) \times I$ satisfying*

$$\Phi(\bar{t},\bar{z},\bar{s},\bar{w}) > \sup_{\partial Q_1 \times \partial Q_1} \Phi.$$

Then there are A and B satisfying

$$(\frac{\partial}{\partial t}\phi(\bar{t},\bar{z},\bar{s},\bar{w}), D_z(\bar{t},\bar{z},\bar{s},\bar{w}), A) \in cD^{+(1,2)}W(\bar{t},\bar{z}),$$

$$(-\frac{\partial}{\partial s}\phi(\bar{t},\bar{z},\bar{s},\bar{w}), -D_w\phi(\bar{t},\bar{z},\bar{s},\bar{w}), B) \in cD^{-(1,2)}V(\bar{s},\bar{w}),$$

and

$$-KI_2 \le \begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix} \le D^2_{z,w}\phi(\bar{t},\bar{z},\bar{s},\bar{w}).$$

Where K is a suitable constant depending on W and V but not on ϕ.

The second lemma states that any continuous function can be suitably approximated by semi-convex (semi-concave) functions.

Lemma 6.2 *Let $W, V \in C(\overline{Q_1})$. Let us define*

$$k_1 = (max\{1 + 4 \parallel W \parallel, 1 + 4 \parallel V \parallel\})^{\frac{1}{2}}, \tag{6.18}$$

$$Q_1^\gamma = [\gamma, T-\gamma) \times \{z \in I : dist(z, \partial I) > \gamma\}, \tag{6.19}$$

1. *For each $\epsilon > 0$, there exists semi-convex functions W^ϵ such that $W^\epsilon \to W$ uniformly on $\overline{Q}_1$ as $\epsilon \to 0$.*

   Moreover, if $(\bar{t},\bar{z}) \in {Q_1}^{\epsilon k_1}$ and $(q_1, q_2, A) \in cD^{+(1,2)}W^\epsilon(\bar{t},\bar{z})$, then we can find $(\bar{s},\bar{w})$ such that
   $|\bar{s}-\bar{t}|^2 + |\bar{w}-\bar{z}|^2 \le \epsilon^2 {k_1}^2$ *and* $(q_1, q_2, A) \in cD^{+(1,2)}W^\epsilon(\bar{s},\bar{w})$.

2. *For each $\epsilon > 0$, there exists semi-concave functions V_ϵ such that $V_\epsilon \to V$ uniformly on $\overline{Q}_1$ as $\epsilon \to 0$.*

   Moreover, if $(\bar{t},\bar{z}) \in {Q_1}^{\epsilon k_1}$ and $(q_1, q_2, A) \in cD^{+(1,2)}V_\epsilon(\bar{t},\bar{z})$, then we can find $(\bar{s},\bar{w})$ such that
   $|\bar{s}-\bar{t}|^2 + |\bar{w}-\bar{z}|^2 \le \epsilon^2 {k_1}^2$ *and* $(q_1, q_2, A) \in cD^{+(1,2)}V_\epsilon(\bar{s},\bar{w})$.

Theorem 6.2 *Suppose W, V be two functions in class $\mathcal{D}_2$. If W, V are viscosity sub-solution and super-solution respectively of equation (6.15), $W = V$ on $\partial^* Q_1$. Then*

$$\sup_{(t,z)\in Q_1} (W-V)(t,z) \le 0. \tag{6.20}$$

Proof: Suppose to the contrary, i.e.,

$$\sup_{(t,z)\in Q_1} (W - V)(t,z) = M > 0. \tag{6.21}$$

Let us define for $\gamma > 0$,

$$(\bar{t},\bar{z}) = \text{argmax}\{(W - V)(t,z), (t,z) \in Q_1\},$$

$$Q_1[\gamma] = \{(t,z) \in Q_1 : (W - V)(t,z) \geq \gamma\},$$

$$\delta(\gamma) = dist\{Q_1[\gamma], \partial^* Q_1\}.$$

Clearly $(\bar{t},\bar{z}) \in Q_1[M]$,

Step 1:

Let k_1, Q_1^γ be defined as in (6.18) and (6.19). For $\rho, \epsilon > 0$, define

$$W^{\epsilon,\rho} = W^\epsilon - \frac{\rho}{t - \epsilon k_1}, \qquad (t,z) \in Q_1^{2\epsilon k_1},$$

$$(\bar{t}_\epsilon, \bar{z}_\epsilon) = \text{argmax}(W^{\epsilon,\rho} - V_\epsilon)(t,z), (t,z) \in Q_1^{2\epsilon k_1}\}.$$

It is easy to see $W^{\epsilon,\rho}$ is still semi-convex in $Q_1^{2\epsilon k_1}$. Moreover, since W^ϵ, V_ϵ converges uniformly to W, V respectively on $\overline{Q_1}$, we can find sufficiently small $\epsilon_0, \rho_0 > 0$, such that for all $\epsilon < \epsilon_0$, with ρ_0 fixed,

$$(\bar{t}_\epsilon, \bar{z}_\epsilon) \in Q_1(\frac{3M}{4}) \cap \{(t,z) \in Q_1 : t > \gamma_0(\rho_0)\}, \tag{6.22}$$

where $\gamma_0(\rho_0)$ is a constant that depends only on ρ_0. Notice also that (6.22) mandates

$$dist((\bar{t}_\epsilon, \bar{z}_\epsilon), \partial^* Q_1) > \delta(\frac{3M}{4}) > 0.$$

Later in the proof, we may require ϵ_0, ρ_0 to be reduced further(c.f.(6.25) second inequality). It is understood that the statement will be true for all $\epsilon < \epsilon_0$ with ρ_0 fixed.

Now for $\beta, \alpha > 0$, consider the auxiliary function

$$\Phi(t,z,s,w) = (W^{\epsilon,\rho}(t,z) - V_\epsilon(s,w)) - \frac{1}{2\alpha}((t-s)^2 + (z-w)^2) + \beta(t - T).$$

Let

$$(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}, \bar{s}_{\epsilon,\alpha,\beta}, \bar{w}_{\epsilon,\alpha,\beta}) = \text{argmax}\{\Phi(t,z,s,w) : (t,z,s,w) \in \overline{Q_1^{2\epsilon k_1}} \times \overline{Q_1^{2\epsilon k_1}}\}.$$

We claim there exist $\epsilon_0, \alpha_0, \beta_0 > 0$, such that for $\epsilon < \epsilon_0$, $\alpha < \alpha_0$, $\beta < \beta_0$,

$$(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}), (\bar{s}_{\epsilon,\alpha,\beta}, \bar{w}_{\epsilon,\alpha,\beta}) \in Q_1(\frac{M}{4}) \cap \{(t,z) : t > \frac{\gamma^0}{2}\}. \tag{6.23}$$

Set $M_{V_\epsilon}(d) = \sup\{|V_\epsilon(t,z) - V_\epsilon(s,w)| : dist\{(t,z),(s,w)\} \leq d)\}$.

From the inequality

$$\Phi(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}, \bar{s}_{\epsilon,\alpha,\beta}, \bar{w}_{\epsilon,\alpha,\beta}) \geq \Phi(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}, \bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}),$$

we have

$$\begin{aligned}\frac{1}{2\alpha}((\bar{t}_{\epsilon,\alpha,\beta} - \bar{s}_{\epsilon,\alpha,\beta})^2 + (\bar{z}_{\epsilon,\alpha,\beta} - \bar{w}_{\epsilon,\alpha,\beta})^2) &\leq V_\epsilon(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}) - V_\epsilon(\bar{s}_{\epsilon,\alpha,\beta}, \bar{w}_{\epsilon,\alpha,\beta}) \\ &\leq M_{V_\epsilon}\{((\bar{t}_{\epsilon,\alpha,\beta} - \bar{s}_{\epsilon,\alpha,\beta})^2 \\ &\quad +(\bar{z}_{\epsilon,\alpha,\beta} - \bar{w}_{\epsilon,\alpha,\beta})^2)^{\frac{1}{2}}\} \\ &\leq k_2,\end{aligned}$$

where k_2 is the upper bound for the function $M_{V_\epsilon}(d)$ (k_2 does not depend on ϵ).

Thus

$$(\bar{t}_{\epsilon,\alpha,\beta} - \bar{s}_{\epsilon,\alpha,\beta})^2 + (\bar{z}_{\epsilon,\alpha,\beta} - \bar{w}_{\epsilon,\alpha,\beta})^2 \leq 2\alpha k_2,$$

which, by the uniform continuity of V_ϵ, further implies

$$\frac{1}{2\alpha}((\bar{t}_{\epsilon,\alpha,\beta} - \bar{s}_{\epsilon,\alpha,\beta})^2 + (\bar{z}_{\epsilon,\alpha,\beta} - \bar{w}_{\epsilon,\alpha,\beta})^2) \to 0 \quad \text{as } \alpha \downarrow 0. \tag{6.24}$$

Suppose $(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}) \notin Q_1(\frac{M}{4})$.

We have on one hand for sufficiently small $\alpha < \alpha_0$

$$\begin{aligned}
\Phi(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}, \bar{s}_{\epsilon,\alpha,\beta}, \bar{w}_{\epsilon,\alpha,\beta}) &\leq W^{\epsilon,\rho}(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}) - V_\epsilon(\bar{s}_{\epsilon,\alpha,\beta}, \bar{w}_{\epsilon,\alpha,\beta}) \\
&\leq W(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}) - V(\bar{s}_{\epsilon,\alpha,\beta}, \bar{w}_{\epsilon,\alpha,\beta}) + \frac{M}{8} \\
&\quad (W^\epsilon \text{ converges to W uniformly on } \overline{Q}_1) \\
&\leq V(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}) - V(\bar{s}_{\epsilon,\alpha,\beta}, \bar{w}_{\epsilon,\alpha,\beta}) + \frac{M}{4} + \frac{M}{8} \\
&\leq M_{V_\epsilon}\{((\bar{t}_{\epsilon,\alpha,\beta} - \bar{s}_{\epsilon,\alpha,\beta})^2 + (\bar{z}_{\epsilon,\alpha,\beta} - \bar{w}_{\epsilon,\alpha,\beta})^2)^{\frac{1}{2}}\} \\
&\quad + \frac{M}{4} + \frac{M}{8} \\
&\leq \frac{M}{2}. \text{ (for } \alpha \text{ sufficiently small)} \qquad (6.25)
\end{aligned}$$

On the other hand, if $\beta < \frac{M}{8T}$,

$$\begin{aligned}
\Phi(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}, \bar{s}_{\epsilon,\alpha,\beta}, \bar{w}_{\epsilon,\alpha,\beta}) &\geq W^{\epsilon,\rho}(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}) - V_\epsilon(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}) \\
&\quad + \beta(\bar{t}_{\epsilon,\alpha,\beta} - T) - \frac{M}{8} \\
&\quad \text{(by (6.24), we can let } \alpha_0 \text{ be small)} \\
&\geq \frac{3M}{4} + \beta(\bar{t}_{\epsilon,\alpha,\beta} - T) \qquad \text{(by (6.22))} \\
&> \frac{M}{2} \qquad (6.26)
\end{aligned}$$

The contradiction of (6.25) and (6.26) imply that $(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}) \in Q_1(\frac{M}{4})$. The claim $(\bar{s}_{\epsilon,\alpha,\beta}, \bar{w}_{\epsilon,\alpha,\beta}) \in Q_1(\frac{M}{4})$ can be proved similarly.

The claim that

$$(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}), (\bar{s}_{\epsilon,\alpha,\beta}, \bar{w}_{\epsilon,\alpha,\beta}) \in \{(t, z) : t > \frac{\gamma^0}{2}\}$$

is clear from the form of $W^{\epsilon,\rho}$ and the fact that our ρ_0 are fixed in the above procedure.

Step 2:

In step 1, we showed that the maximum points of Φ are uniformly bounded away from ∂Q_1 with respect to the parameters $\epsilon < \epsilon_0$, $\alpha < \alpha_0$, $\beta < \beta_0$ and with fixed ρ_0. We can now apply Ishii's lemma and obtain $\overline{A}_{\epsilon,\alpha,\beta}$, $\overline{B}_{\epsilon,\alpha,\beta}$ and

$k_{\epsilon,\alpha,\beta} > 0$ such that

$$\{\frac{\bar{t}_{\epsilon,\alpha,\beta} - \bar{s}_{\epsilon,\alpha,\beta}}{\alpha} - \beta - \frac{\rho_0}{(\bar{t}_{\epsilon,\alpha,\beta} - \epsilon k_1)^2}, \frac{\bar{z}_{\epsilon,\alpha,\beta} - \bar{w}_{\epsilon,\alpha,\beta}}{\alpha}, \overline{A}_{\epsilon,\alpha,\beta}\} \in cD^{+(1,2)}W^{\epsilon}(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}), \tag{6.27}$$

$$\{\frac{\bar{t}_{\epsilon,\alpha,\beta} - \bar{s}_{\epsilon,\alpha,\beta}}{\alpha}, \frac{\bar{z}_{\epsilon,\alpha,\beta} - \bar{w}_{\epsilon,\alpha,\beta}}{\alpha}, \overline{B}_{\epsilon,\alpha,\beta}\} \in cD^{-(1,2)}V_{\epsilon}(\bar{s}_{\epsilon,\alpha,\beta}, \bar{w}_{\epsilon,\alpha,\beta}), \tag{6.28}$$

$$-k_{\epsilon,\beta} \le \overline{A}_{\epsilon,\alpha,\beta} \le \overline{B}_{\epsilon,\alpha,\beta} \le k_{\epsilon,\beta}, \tag{6.29}$$

where $k_{\epsilon,\beta}$ is a constant that depends on ϵ and β but not on α.

From Lemma 6.2 we have

$$cD^{+(1,2)}W^{\epsilon}(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}) \subseteq cD^{+(1,2)}W(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}) \tag{6.30}$$

where $(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta})$ are points in Q_1 such that

$$dist\{(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta}), (t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta})\} \le \epsilon k_1, \tag{6.31}$$

we have

$$H_{k_0}\{t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}, W(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}), \frac{\bar{t}_{\epsilon,\alpha,\beta} - \bar{s}_{\epsilon,\alpha,\beta}}{\alpha} - \beta - \frac{\rho_0}{(\bar{t}_{\epsilon,\alpha,\beta} - \epsilon k_1)^2}, \frac{\bar{z}_{\epsilon,\alpha,\beta} - \bar{w}_{\epsilon,\alpha,\beta}}{\alpha}, \overline{A}_{\epsilon,\alpha,\beta}\} \le 0. \tag{6.32}$$

Similarly still by Lemma 6.2,

$$cD^{-(1,2)}V_{\epsilon}(\bar{s}_{\epsilon,\alpha,\beta}, \bar{w}_{\epsilon,\alpha,\beta}) \subseteq cD^{-(1,2)}V(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}), \tag{6.33}$$

where $(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta})$ are points in Q_1 such that

$$dist\{(\bar{s}_{\epsilon,\alpha,\beta}, \bar{w}_{\epsilon,\alpha,\beta}), (s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta})\} \le \epsilon^2 {k_1}^2, \tag{6.34}$$

we have

$$H_{k_0}\{s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}, V(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}), \frac{\bar{t}_{\epsilon,\alpha,\beta} - \bar{s}_{\epsilon,\alpha,\beta}}{\alpha}, \frac{\bar{z}_{\epsilon,\alpha,\beta} - \bar{w}_{\epsilon,\alpha,\beta}}{\alpha}, \overline{B}_{\epsilon,\alpha,\beta}\} \ge 0. \tag{6.35}$$

From (6.32) and (6.35) we arrive at the conclusion that at least one of following three inequalities must hold. That is, either

$$
\begin{aligned}
&H_{k_0}^{(1)}(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}, W(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}), \frac{\overline{t}_{\epsilon,\alpha,\beta} - \overline{s}_{\epsilon,\alpha,\beta}}{\alpha} - \beta, \frac{\overline{z}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta}}{\alpha}, \\
&\qquad\qquad \overline{A}_{\epsilon,\alpha,\beta}) \\
&-H_{k_0}^{(1)}(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}, V(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta})\frac{\overline{t}_{\epsilon,\alpha,\beta} - \overline{s}_{\epsilon,\alpha,\beta}}{\alpha}, \frac{\overline{z}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta}}{\alpha}, \overline{B}_{\epsilon,\alpha,\beta}) \\
&\le 0, \qquad (6.36)
\end{aligned}
$$

or

$$
\begin{aligned}
&H_{k_0}^{(2)}(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}, W(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}), \frac{\overline{z}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta}}{\alpha},) \\
&-H_{k_0}^{(2)}(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}, V(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}), \frac{\overline{z}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta}}{\alpha}) \le 0, \qquad (6.37)
\end{aligned}
$$

or

$$
\begin{aligned}
&H_{k_0}^{(3)}(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}, W(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}), \frac{\overline{z}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta}}{\alpha},) \\
&-H_{k_0}^{(3)}(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}, V(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}), \frac{\overline{z}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta}}{\alpha}) \le 0. \qquad (6.38)
\end{aligned}
$$

From (6.23), (6.24),(6.31) and (6.34) we know for sufficiently small ϵ_0, α_0,

$$
W(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}) \ge V(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}) + \frac{M}{8}. \qquad (6.39)
$$

Suppose (6.37) holds, we have by (6.39),

$$
\begin{aligned}
&H_{k_0}^{(2)}(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}, W(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}), \frac{\overline{z}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta}}{\alpha},) \\
&-H_{k_0}^{(2)}(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}, V(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}), \frac{\overline{z}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta}}{\alpha}), \\
\ge\ & e^{-k_0 t_{\epsilon,\alpha,\beta}}\frac{pM}{8} - e^{-k_0 t_{\epsilon,\alpha,\beta}} \sup_{z\in I}\{d_4'(z)\}\frac{(\overline{z}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta})^2}{\alpha} \\
&-e^{-k_0(t_{\epsilon,\alpha,\beta}\wedge s_{\epsilon,\alpha,\beta})} pV(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}|t_{\epsilon,\alpha,\beta} - s_{\epsilon,\alpha,\beta}| \\
&-e^{-k_0(t_{\epsilon,\alpha,\beta}\wedge s_{\epsilon,\alpha,\beta})}|\frac{\overline{z}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta}}{\alpha}||t_{\epsilon,\alpha,\beta} - s_{\epsilon,\alpha,\beta}| \\
>\ & 0, \qquad (6.40)
\end{aligned}
$$

where the last inequality is obtained in the following way: first let $\epsilon \to 0$, so that

$$
|t_{\epsilon,\alpha,\beta} - s_{\epsilon,\alpha,\beta}| \to |\overline{t}_{\epsilon,\alpha,\beta} - \overline{s}_{\epsilon,\alpha,\beta}|, \quad \text{(by (6.34))}
$$

and let $\alpha \to 0$ to get

$$\frac{(\overline{z}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta})^2}{\alpha} + \frac{(\overline{t}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta})^2}{\alpha} \to 0 \quad \text{(by (6.24))},$$

finally, we use Schwartz inequality to show that the last term goes to 0.

We conclude that the inequality (6.37) can not hold.

Similar as above, we can prove inequality (6.38) can not hold either.

Now we suppose (6.36) hold.

By claim (6.23), (6.31) and (6.34) we know for sufficiently small $\epsilon < \epsilon_0$, $\alpha < \alpha_0$ and $\beta < \beta_0$, there exists positive constant $C(\epsilon_0, \alpha_0, \beta_0) > 0$, such that

$$dist\{(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}), \partial^* Q\} \geq C(\epsilon_0, \alpha_0, \beta_0),$$

$$dist\{(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}), \partial^* Q\} \geq C(\epsilon_0, \alpha_0, \beta_0),$$

if we further take into consideration (6.27), (6.28), (6.30),(6.33) and our hypothesis that $W, V \in \mathcal{D}_2$, we know there exists $C > 0$ such that

$$pW(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}) - z_{\epsilon,\alpha,\beta}(\frac{\overline{z}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta}}{\alpha}) \geq C,$$

$$pV(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}) - w_{\epsilon,\alpha,\beta}(\frac{\overline{z}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta}}{\alpha}) \geq C.$$

Moreover, $(\overline{z}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta})/\alpha$ is bounded.

We conclude for $\epsilon < \epsilon_0$ sufficiently small,

$$pV(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}) - z_{\epsilon,\alpha,\beta}(\frac{\overline{z}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta}}{\alpha}) \geq \frac{C}{2}. \tag{6.41}$$

Now we try to obtain a contradiction to (6.36). First notice that the operator $H^{(1)}_{k_0}$ is increasing with respect to w and decreasing with respect to A. we have

$$H^{(1)}_{k_0}\{t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}, W(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}), \frac{\overline{t}_{\epsilon,\alpha,\beta} - \overline{s}_{\epsilon,\alpha,\beta}}{\alpha} - \beta, \frac{\overline{z}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta}}{\alpha}, \overline{A}_{\epsilon,\alpha,\beta}\}$$

$$\geq H^{(1)}_{k_0}\{t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}, V(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}), \frac{\overline{t}_{\epsilon,\alpha,\beta} - \overline{s}_{\epsilon,\alpha,\beta}}{\alpha} - \beta, \frac{\overline{z}_{\epsilon,\alpha,\beta} - \overline{w}_{\epsilon,\alpha,\beta}}{\alpha}, \overline{B}_{\epsilon,\alpha,\beta}\} \tag{6.42}$$

We conclude that if (6.36) holds, then

$$\begin{aligned}\beta e^{k_0 t_{\epsilon,\alpha,\beta}} \le\ & H_{k_0}^{(1)}\{s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}, V(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}), \\ & \qquad \frac{\bar{t}_{\epsilon,\alpha,\beta} - \bar{s}_{\epsilon,\alpha,\beta}}{\alpha}, \frac{\bar{z}_{\epsilon,\alpha,\beta} - \bar{w}_{\epsilon,\alpha,\beta}}{\alpha}, \overline{B}_{\epsilon,\alpha,\beta}\} \\ - & H_{k_0}^{(1)}\{t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}, V(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}), \\ & \qquad \frac{\bar{t}_{\epsilon,\alpha,\beta} - \bar{s}_{\epsilon,\alpha,\beta}}{\alpha}, \frac{\bar{z}_{\epsilon,\alpha,\beta} - \bar{w}_{\epsilon,\alpha,\beta}}{\alpha}, \overline{B}_{\epsilon,\alpha,\beta}\} \end{aligned} \tag{6.43}$$

Notice that the operator $H_{k_0}^{(1)}$ is continuously differentiable if its arguments are as in the right hand side of inequality (6.43), we can repeat the same procedure as we did in (6.40) to obtain a contradiction to (6.43).

Q.E.D.

Remark 6.2 *Although we proved the above comparison result in in the cylinder domain Q_1, a review on the steps shows the result holds true for arbitrary sub-domains of Q_1 with smooth boundaries. Indeed, the difficulty lies in the lower boundedness of $\frac{\partial V}{\partial x}$, and the case Q_1 is the most difficult situation.*

CHAPTER 7

THE EXISTENCE OF THE FREE BOUNDARY

The fact that the value function V is concave on $\overline{S}$ could be used, in combination with its viscosity solution property, to partition the domain Q into three sub-domains corresponding to the three terms in the Hamilton-Jacobi-Bellman equation (5.7).

We define for $(t, x, y) \in Q$, the sub-differential with respect to (x, y) by

$$\partial v(t,x,y) = \{(\delta_x, \delta_y) \in R^2 : \\ v(t,\xi,\eta) \le v(t,x,y) + \delta_x(\xi - x) + \delta_y(\eta - y), \forall(\xi,\eta) \in R^2\}.$$

Proposition 7.1-Proposition 7.3 are finite horizon version of the convex analysis results of section 6 [28].

Proposition 7.1 *For all $(t,x,y) \in Q$, $(\delta_x, \delta_y) \in \partial v(t,x,y)$, we have the following properties:*

1. $\delta_x - (1-\lambda)\delta_y \ge 0, \quad \delta_y - (1-\mu)\delta_x \ge 0.$
2. $\delta_x > 0, \quad \delta_y > 0.$
3. $x\delta_x + y\delta_y = pv(t,x,y).$
4. $\gamma^{p-1}\partial v(t,x,y) = \partial v(t,\gamma x,\gamma y), \ \forall \gamma > 0.$

5. Let $(\overline{\delta}_x, \overline{\delta}_y) \in \partial v(t, \overline{x}, \overline{y})$*, then* $(\delta_x - \overline{\delta}_x)(x - \overline{x}) + (\delta_y - \overline{\delta}_y)(y - \overline{y}) \leq 0$.

Define for $(t, x, y) \in Q$,

$$\theta^+(t,x,y) = \max\{\delta_y - (1-\mu)\delta_x : (\delta_x, \delta_y) \in \partial v(t,x,y)\},$$

$$\theta^-(t,x,y) = \min\{\delta_y - (1-\mu)\delta_x : (\delta_x, \delta_y) \in \partial v(t,x,y)\}.$$

Clearly, $\theta^+(t,x,y) \geq \theta^-(t,x,y) \geq 0$, and the above maxima and minima are attained because $\partial v(t,x,y)$ is compact.

For fixed $t \in [0,T]$, $\rho \geq 0$, define

$$(x(\rho), y(\rho)) = (1 - (1-\mu)\rho, -(1-\lambda) + \rho), \tag{7.1}$$

$$\rho_0(t) = \inf\{\rho > 0; \theta^-(t,x,y) = 0\}. \tag{7.2}$$

Proposition 7.2 *Under the above setting, for* $0 \leq \rho < \overline{\rho} < \infty$*, we have*

$$\theta^+(t, x(\overline{\rho}), y(\overline{\rho})) \leq \theta^-(t, x(\rho), y(\rho)).$$

Moreover, if $\rho_0(t) < \infty$*, then* $\theta^-(t, x(\rho_0(t)), y(\rho_0(t))) = 0$*, and*

$$\theta^+(t, x(\overline{\rho}), y(\overline{\rho})) = 0, \quad \forall \overline{\rho} > \rho_0.$$

For fixed $t \in [0,T]$, if $\rho_0(t) < \infty$, we can partition the sector set

$$Q_t = \{(t,x,y) \in Q\}$$

into two parts,

$$SS(t) = \{(t,x,y) : (\gamma x, \gamma y) = (x(\overline{\rho}), y(\overline{\rho})), \text{ for some } \gamma > 0, \text{and some } \overline{\rho} > \rho_0(t)\},$$

$$Q_t \setminus \overline{SS(t)} = \{(t,x,y) : (\gamma x, \gamma y) = (x(\rho), y(\rho)) \text{ for some } \gamma > 0, \text{and some } \rho < \rho_0(t)\}.$$

Similarly we can define for fixed $t \in [0,T]$, $\rho \geq 0$,

$$(\tilde{x}(\rho), \tilde{y}(\rho)) = (-(1-\mu) + \rho, 1 - (1-\lambda)\rho), \tag{7.3}$$

$$\tilde{\rho}_0(t) = \inf\{\rho > 0 : \delta_x - (1-\lambda)\delta_y = 0 \text{ for some } (\delta_x, \delta_y) \in \partial v(t,x,y)\}. \tag{7.4}$$

If $\tilde{\rho}_0(t) < \infty$, we can also partition Q_t into

$$SMM(t) = \{(t,x,y) : (\gamma x, \gamma y) = (x(\overline{\rho}), y(\overline{\rho})), \text{ for some } \gamma > 0, \text{and some } \overline{\rho} > \tilde{\rho}_0(t)\},$$

$$Q_t \setminus \overline{SMM(t)} = \{(t,x,y) : (\gamma x, \gamma y) = (x(\rho), y(\rho)), \text{ for some } \gamma > 0, \text{and some } \rho < \tilde{\rho}_0(t)\}.$$

Proposition 7.3 *For any* $t \in [0,T]$, $(\delta_x, \delta_y) \in \partial v(t,x,y)$,

1. $\delta_y - (1-\mu)\delta_x = 0$, *if* $(t,x,y) \in SS(t)$; $\delta_y - (1-\mu)\delta_x > 0$, *if* $(t,x,y) \in Q_t \setminus \overline{SS(t)}$.
2. $\delta_x - (1-\lambda)\delta_y = 0$, *if* $(t,x,y) \in SMM(t)$; $\delta_x - (1-\lambda)\delta_y > 0$, *if* $(t,x,y) \in Q_t \setminus \overline{SMM(t)}$.
3. $SS(t) \cap SMM(t) = \emptyset$.

Proposition 7.4 *For* $t \in [0,T)$,

$$\liminf_{s \to t} \rho_0(s) \geq \rho_0(t), \tag{7.5}$$

$$\liminf_{s \to t} \tilde{\rho}_0(s) \geq \tilde{\rho}_0(t). \tag{7.6}$$

Proof: To prove (7.5), let $(x(\rho), y(\rho))$ be defined by (7.1), ρ_0 be defined by (7.2). We observe that $V(s, x(\rho), y(\rho))$, as a function of ρ, is strictly increasing if $\rho < \rho_0(s)$; is constant if $\rho > \rho_0(s)$. Moreover, $\rho_0(s) > 0$, for all $s \in [0,T)$.

Suppose there exists a sequence $\{s_n\}$, such that $\lim_{s_n \to t} \rho_0(s_n) < \rho_0(t) - 2\epsilon$ for some $\epsilon > 0$, then we have on the on hand,

$$\begin{aligned} V(s_n, x(\rho_0(t)-\epsilon), y(\rho_0(t)-\epsilon)) &= V(s_n, x(\rho_0(t)), y(\rho_0(t))) \\ &\to V(t, x(\rho_0(t)), y(\rho_0(t))). \end{aligned}$$

On the other hand,

$$V(s_n, x(\rho_0(t) - \epsilon), y(\rho_0(t) - \epsilon)) \to V(t, x(\rho_0(t) - \epsilon), y(\rho_0(t) - \epsilon)).$$

So $V(t, x(\rho_0(t) - \epsilon), y(\rho_0(t) - \epsilon)) = V(t, x(\rho_0(t)), y(\rho_0(t)))$, which is a contradiction.

The proof of (7.6) is similar.

Q.E.D.

For each fixed $t \in [0,T)$, we will denote $NT(t) = Q(t) \setminus \{\overline{SS(t)} \cup \overline{SMM(t)}\}$, $SS = \cup_{t\in[0,T]} SS(t)$, $SMM = \cup_{t\in[0,T]} SMM(t)$ and $NT = \cup_{t\in[0,T]} NT(t)$.

Proposition 7.5 *For each fixed $t \in [0,T)$, $NT(t) \neq \emptyset$. Moreover, NT is an open subset in Q.*

Proof: Suppose $NT(t) = \emptyset$ for some $t \in [0,T)$, then $\{\overline{SS(t)} \cup \overline{SMM(t)} = \overline{S}$. By Proposition 4.3, either $SS(t) = S$ or $SMM(t) = S$ which are both impossible.

That NT is an open set in Q can be directly verified by (7.5) and (7.6).

Q.E.D.

Proposition 7.6 *Suppose we have that $SS(t) \neq \emptyset$ for some $t \in [0,T)$, then there exists $\delta > 0$, such that for all $s \in [0,t]$, $SS(s)$ contains the wedge*

$$\{(x,y): \quad x + (1-\mu)y \geq 0, \quad x + (1-\mu-\delta)y \leq 0\}.$$

Proof: For $t \in [0,T)$, let $\delta > 0$ be a small number to be determined later. We define

$$Q(\delta,t) = \{(s,x,y) \in Q : 0 \leq s \leq t, \quad x + (1-\mu)y \geq 0, \quad x + (1-\mu-\delta)y \leq 0\}.$$

For $(s,x,y) \in Q(\delta,t)$, we define

$$\bar{V}(s,x,y) = \frac{1}{p} A(s)^{1-p} (x + (1-u)y)^p,$$

where

$$A(s) = \{\frac{pV(s, -(1-\mu)+\delta, 1)}{\delta^p}\}^{\frac{1}{1-p}}.$$

From the hypothesis $SS(t) \neq \emptyset$ and the homothetic property of V and $\tilde{V}$, we have for $\delta > 0$ sufficiently small,

$$\tilde{V}(s,x,y) = V(s,x,y), \quad (s,x,y) \in \partial^* Q(\delta,t).$$

We claim that $\tilde{V}$ is a viscosity solution of (5.7) in the domain $Q(\delta,t)$.

That $\tilde{V}$ is a viscosity sub-solution can be seen from the fact that $\tilde{V}(t,\cdot,\cdot)$ are continuously differentiable functions and

$$\frac{\partial \tilde{V}}{\partial y} - (1-\mu)\frac{\partial \tilde{V}}{\partial x} = 0. \tag{7.7}$$

We prove $\tilde{V}$ is a viscosity super-solution of (5.7).

Suppose $(s,x,y) \in Q(\delta,t)$, $\varphi \in C^{1,2}(Q(\delta,t))$ and $\tilde{V}-\varphi$ has a local minimum at (s,x,y). We have,

$$\begin{aligned}
\{\mathcal{L}\varphi - \overline{U}(\frac{\partial\varphi}{\partial x})\}(s,x,y) \geq\; & -\frac{\partial\varphi}{\partial s}(s,x,y) + A(s)^{-p}(x+(1-\mu)y)^p \\
& \cdot\{\frac{(1-p)\sigma^2}{2}(\frac{(1-\mu)y}{x+(1-\mu)y})^2 A(s) \\
& -\alpha\frac{(1-\mu)y}{x+(1-\mu)y}A(s) - r\frac{x}{x+(1-\mu)y}A(s) \\
& -\frac{1-p}{p}\} \\
\geq\; & A(s)^{-p}(x+(1-\mu)y)^p \\
& \cdot\{\frac{1}{2(1-p)}(\frac{\sigma(1-p)(1-\mu)}{\delta} - \frac{\alpha-r}{\sigma})^2 A(s) \\
& -(r+\frac{(\alpha-r)^2}{2(1-p)\sigma^2})A(s) - \frac{1-p}{p}\} \\
\geq\; & A(s)^{-p}(x+(1-\mu)y)^p \\
& \cdot\{\frac{1}{2(1-p)}(\sigma(1-p)(1-\mu) - \frac{\alpha-r}{\sigma}\delta)^2\frac{f(s)}{\delta^2} \\
& -(r+\frac{(\alpha-r)^2}{2(1-p)\sigma^2})A(0) - \frac{1-p}{p}\} \\
\geq\; & 0,
\end{aligned} \tag{7.8}$$

where the the last step is achieved by making the constant δ sufficiently small.

We also have

$$\begin{aligned}\{\frac{\partial\varphi}{\partial x}-(1-\lambda)\frac{\partial\varphi}{\partial y}\}(s,x,y) &= \{\frac{\partial\tilde{V}}{\partial x}-(1-\lambda)\frac{\partial\tilde{V}}{\partial y}\}(s,x,y)\\ &= A(s)^{1-p}(x+(1-\mu)y)^{p-1}(1-(1-\lambda)(1-\mu))\\ &\geq 0. \end{aligned}\tag{7.9}$$

(7.7),(7.8) and (7.9) showed that $\tilde{V}$ is a viscosity super-solution of (5.7) on $Q(\delta,t)$.

Now we can invoke Theorem 6.2 and Remark 6.2 to obtain that $V\equiv\overline{V}$ on $Q(\delta,t)$.

Q.E.D.

Proposition 7.7 *Suppose we have that $SMM(t)\neq\emptyset$ for some $t\in[0,T)$, then there exists $\delta>0$, such that for all $s\in[0,t]$, $SMM(s)$ contains the wedge $\{(x,y):\ x+y/(1-\lambda)\geq 0,\ \ x+y/(1-\lambda-\delta)\leq 0\}$.*

Proposition 7.8 *Let Ω be a sub-domain of Q with smooth boundary. Suppose the value function V satisfies*

$$\frac{\partial V}{\partial y}-(1-\mu)\frac{\partial V}{\partial x}>0,\qquad (t,x,y)\in\Omega. \tag{7.10}$$

then V is also the value function of a new control problem which has all the characteristics as described in Section 5.1 except that it does not has the option of making transactions from stock to bond.

Similarly, suppose V satisfies

$$\frac{\partial V}{\partial x}-(1-\lambda)\frac{\partial V}{\partial y}>0,\qquad (t,x,y)\in\Omega, \tag{7.11}$$

then V is also the value function of a new control problem which has all the characteristics as described in Section 5.1 except that it does not has the the option of making transactions from bond to stock.

Proof: We only give a sketch proof of (7.10).

Following the procedure from Theorem 6.1 to Theorem 6.2, we can show that both V and the value function of the new control problem are the unique viscosity solution of the equation

$$\min\{\mathcal{L}\tilde{V} - \overline{U}(\frac{\partial \tilde{V}}{\partial x}),\ \frac{\partial \tilde{V}}{\partial x} - (1-\lambda)\frac{\partial \tilde{V}}{\partial y}\} = 0, \quad (t,x,y) \in \Omega.$$

$$\tilde{V}(t,x,y) = V(t,x,y), \quad (t,x,y) \in \partial^{*}\Omega.$$

Thus they must be the same.

Q.E.D.

Proposition 7.9 *For each $t \in [0,T)$, $SS(t) \neq \emptyset$.*

Proof: Suppose for some $t \in [0,T)$,

$$SS(t) = \emptyset. \tag{7.12}$$

By Proposition 7.6, we know $SS(s) = \emptyset$ for all $s \in [t,T]$. In particular, for $s \geq t$, $\mathcal{S} = NT(s) \cup SMM(s)$.

For $\delta \geq 0$, we have by Proposition 6.1,

$$V(t, -(1-\mu)+\delta, 1) \geq \frac{1}{p} f(t)^{1-p}\delta^{p}. \tag{7.13}$$

In the following, we will estimate the increasing rate of $V(t, -(1-\mu)+\delta, 1)$ w.r.t. δ under hypothesis (7.12) and derive a contradiction against (7.13).

From hypothesis (7.12) and Proposition 7.8, we can assume the state processes (X,Y) satisfy for $s \geq t$,

$$\begin{aligned} dX(s) &= (rX(s) - c(s))ds - dL(s), \quad X(t-) = -(1-\mu)+\delta, \\ dY(s) &= \alpha Y(s)ds + \sigma Y(s)dW(s) + (1-\lambda)dL(s), \quad Y(t-) = 1. \end{aligned}$$

Let us denote $Z(s) = X(s) + (1-\mu)Y(s)$, similar to the derivation of (6.9) we have that

$$\begin{aligned} e^{r(s-t)}Z(s) &+ \int_t^s e^{-r(u-t)}c(u)du + \int_t^s e^{-r(u-t)}(1-(1-\lambda)(1-\mu))dL(u) \\ &= \delta + \int_t^s e^{-r(u-t)}\sigma(1-\mu)Y(u)d\tilde{W}(u) \end{aligned}$$

is a nonnegative super-martingale w.r.t. the probability $\tilde{P}$.

Define

$$\tau = \inf\{s \geq t : X(s) + (1-\mu)Y(s) = 0\} \wedge T.$$

We have for any admissible control $(c(s), L(s))$ and $1 < q < \frac{1}{p}$,

$$\begin{aligned}
E\int_t^T \frac{c(s)^p}{p} ds &= E\int_t^\tau \frac{c(s)^p}{p} ds \\
&\leq \frac{1}{p}\{E(\tau - t)\}^{\frac{1}{q'}} E\{\int_t^\tau c(s)^{pq} ds\}^{\frac{1}{q}} \\
&= \frac{1}{p}\{E(\tau - t)\}^{\frac{1}{q'}} \tilde{E}\{(\theta_T^{t_n})^{-1} \int_t^\tau c(s)^{pq} ds\}^{\frac{1}{q}} \\
&\leq \frac{1}{p}\{E(\tau - t)\}^{\frac{1}{q'}} e^{r(T-t)p} \\
&\quad \cdot\{\tilde{E}[\int_t^T (e^{-r(s-t)} c_n(s))^{pq} ds]^{\frac{1}{pq}}\}^p \{\tilde{E}(\theta_T^t)^{\frac{-1}{1-pq}}\}^{\frac{1-pq}{q}} \\
&\leq \frac{1}{p}\{E(\tau - t)\}^{\frac{1}{q'}} e^{r(T-t)p} (T-t)^{\frac{1-pq}{q}} \\
&\quad \cdot\{\tilde{E}\int_t^T e^{-r(s-t)} c(s) ds\}^p e^{\frac{\kappa^2(T-t)}{2q}(\frac{1}{1-pq}+1)} \\
&\leq C\{E(\tau - t)\}^{\frac{1}{q'}} \delta^p. \qquad (7.14)
\end{aligned}$$

We claim that $\lim_{\delta \to 0} E(\tau - t) = 0$. Thus (7.14) is a contradiction against (7.13), and the proof will be finished.

Since $(X(s), Y(s))$ have the following explicit solution:

$$\begin{aligned}
X(s) &= [-(1-\mu) + \delta] e^{r(s-t)} - \int_t^s e^{r(s-u)} c^\epsilon(u) du - \int_t^s e^{r(s-u)} dL(u), \\
Y(s) &= \exp\{(\alpha - \frac{\sigma^2}{2})(s-t) + \sigma(W(s) - W(t))\} \\
&\quad + \int_t^s \exp\{(\alpha - \frac{\sigma^2}{2})(s-u) + \sigma W(s-u)\}(1-\lambda) dL(u),
\end{aligned}$$

we have

$$\begin{aligned}X(s)+(1-\mu)Y(s) &= [-(1-\mu)+\delta]e^{r(s-t)}\\ &+(1-\mu)\exp\{(\alpha-\frac{\sigma^2}{2})(s-t)+\sigma W(s-t)\}\\ &+\int_t^s\{-e^{r(s-u)}+(1-\mu)(1-\lambda)\\ &\cdot\exp\{(\alpha-\frac{\sigma^2}{2})(s-u)+\sigma W(s-u)\}dL(u)\\ &-\int_t^s -e^{r(s-u)}c(u)du. \qquad (7.15)\end{aligned}$$

To estimate the first term in (7.15), let us define

$$\begin{aligned}\tau_1 &= \inf\{s\geq t:[-(1-\mu)+\delta]e^{r(s-t)}\\ &\quad+(1-\mu)\exp\{(\alpha-\frac{\sigma^2}{2})(s-t)+\sigma W(s-t)\}=0\}\\ &= \inf\{s\geq t:(\alpha-\frac{\sigma^2}{2}-r)(s-t)+\sigma W(s-t))=\ln(1-\frac{\delta}{1-\mu})\}.\end{aligned}$$

Let us denote $\nu=\frac{1}{\sigma}(\alpha-\frac{\sigma^2}{2}-r)$, then $\nu(s-t)+(W(s)-W(t))$ is a Brownian motion with drift ν, and τ_1 is its passage time to hit $b(\delta)=\frac{1}{\sigma}\ln(1-\frac{\delta}{1-\mu})$. The density function of τ_1 is given by (e.g. [24] p197)

$$P\{\tau_1-t\in d(s-t)\}=\frac{|b|}{\sqrt{2\pi(s-t)^3}}\exp\{-\frac{(b-\nu(s-t))^2}{2(s-t)}\}.$$

Clearly we have

$$E[(\tau_1-t)\wedge T]=O(\delta). \qquad (7.16)$$

To estimate the second term in (7.15), let us define

$$W^*(s-t)=\sup_{t\leq u\leq s}|\nu(u-t)+(W(u)-W(t))|,$$

$$\begin{aligned}\tau_2 &= \inf\{s\geq t:W^*(s-t)=\frac{-1}{2\sigma}\log[(1-\mu)(1-\lambda)]\\ &\leq \inf\{s\geq t:-e^{r(s-u)}+(1-\mu)(1-\lambda)\}\\ &\quad\cdot\exp\{(\alpha-\frac{\sigma^2}{2})(s-u)+\sigma(W(s)-W(u))=0,\\ &\quad\text{for some } 0\leq u\leq s\}.\end{aligned}$$

Clearly, $P\{\tau_2 > 0\} = 1$, so (7.16) implies

$$P\{\tau 1 > \tau 2\} \to 0, \qquad \text{as } \delta \to 0.$$

On the set $(\tau_1 \leq \tau_2)$, we have by (7.15) $(\tau \leq \tau_1)$. Thus

$$\begin{aligned} E\{(\tau - t)\} &= E\{(\tau - t)I_{(\tau_1 \leq \tau_2)}\} + E\{(\tau - t)I_{(\tau_1 > \tau_2)}\} \\ &\leq E\{(\tau_1 - t)I_{(\tau_1 \leq \tau_2)}\} + (T - t)P\{\tau_1 > \tau_2\} \\ &\leq E\{\tau_1 - t\} + (T - t)P\{\tau_1 > \tau_2\} \\ &\to 0 \quad (\text{ as } \delta \to 0) \end{aligned}$$

Q.E.D.

Similar to the above proposition we also have

Proposition 7.10 *For each* $t \in [0, T)$, $SMM(t) \neq \emptyset$.

CHAPTER 8

THE RUGULARITY OF THE VALUE FUNCTION

8.1 C^0 Continuity of the Value Function

Recall that we used (6.10) to transform the value function V to u. Then all the properties of of u can be mapped V by this transformation, and vice versa. The domains SMM, SS, NT are also transformed into disjoint sub-domains of Q_1, which for notational simplicity we will still denote by SMM, SS and NT. The particular meaning of them depends on the context.

Proposition 8.1 *Let $\Omega = [t_0, t_1) \times (a, b)$ be a subset of NT in Q_1. Then on this subset Ω, the function u as defined by (6.10) is the unique viscosity solution of the equation*

$$-\frac{\partial \bar{u}}{\partial t} - d_1(z)p\bar{u} - d_2(z)\frac{\partial \bar{u}}{\partial z} - d_3(z)\frac{\partial^2 \bar{u}}{\partial z^2} - \overline{U}(p\bar{u} - z\frac{\partial \bar{u}}{\partial z}) = 0, \qquad \forall (t,z) \in \Omega, \quad (8.1)$$

$$\bar{u}(t,z) = u(t,z) \qquad \forall (t,z) \in \partial^* \Omega. \qquad (8.2)$$

Proof: Same as Proposition 7.8.

Q.E.D.

Proposition 8.2 *Let $\Omega = [t_0, t_1) \times (a, b)$ with $0, 1 \notin (a, b)$. Then on Ω, $\frac{\partial u}{\partial z}(t, z)$ exists everywhere and is continuous w.r.t. both (t, z).*

Proof: We first show that $u(t,\cdot)$ is continuously differentiable on (a,b).

Let $(t_0, z_0) \in \Omega$, and suppose we have,

$$\frac{\partial u^+}{\partial z}(t_0, z_0) < \frac{\partial u^-}{\partial z}(t_0, z_0). \tag{8.3}$$

For $\epsilon > 0$ we construct a test function φ by

$$\varphi(t,z) = u(t_0,z_0) + \delta(z - z_0) - \frac{1}{2\epsilon}(z-z_0)^2 + (t-t_0)^2,$$

$$\forall (t,z) \in B_r(z_0) \times (t : t \geq t_0, t - t_0 \leq r),$$

where $\delta = \frac{1}{2}(\frac{\partial u^+}{\partial z}(t_0,z_0) + \frac{\partial u^-}{\partial z}(t_0,z_0))$ and $r > 0$ is a small positive number to be determined later.

Fix an $\epsilon > 0$, from (8.3) it is easy to see that there exists $r > 0$ such that

$$\varphi(t_0, z) \geq u(t_0, z) \quad \forall z \in B_r(z_0). \tag{8.4}$$

Since $u(\cdot, z_0)$ is a decreasing function, we also have the strict inequality

$$\varphi(t, z_0) > u(t, z_0) \quad \forall t > t_0, t - t_0 < r.$$

Thus (8.3) implies that for each fixed $t > t_0, t - t_0 < r$, there exists $r(t) > 0$, such that

$$\varphi(t,z) \geq u(t,z) \quad \forall |z - z_0| < r(t).$$

We take $r(t) = inf\{r : \varphi(t,z) > u(t,z), \forall |z - z_0| < r\}$, and without lose of generality we assume $\varphi(t, r(t)) = u(t, r(t))$.

We claim

$$\liminf_{t \Downarrow t_0} r(t) > 0. \tag{8.5}$$

Suppose (8.5) is not true, then we can find a sequence $t_n \downarrow t_0$, such that $\lim_{n\to\infty} r(t_n) = 0$.

By the concave property of the function $u(t,\cdot)$, we can find $\triangle > 0$, such that

$$u(t_0, r(t)) \leq u(t_0, z_0) + \delta(r(t) - z_0) - \triangle |r(t) - z_0|,$$

Thus

$$\begin{aligned} u(t_n, r(t_n)) &= \varphi(t_n, r(t_n)) \\ &= u(t_0, z_0) + \delta(r(t_n) - z_0) - \frac{1}{2\epsilon}(r(t_n) - z_0)^2 + (t_n - t_0)^2 \\ &\geq u(t_0, r(t_n)) + \triangle|r(t_n) - z_0| - \frac{1}{2\epsilon}(r(t_n) - z_0)^2 \\ &\geq u(t_n, r(t_n)) + \triangle|r(t_n) - z_0| - \frac{1}{2\epsilon}(r(t_n) - z_0)^2 \end{aligned}$$

So

$$\triangle|r(t_n) - z_0| - \frac{1}{2\epsilon}(r(t_n) - z_0)^2 \leq 0, \quad \forall t_n.$$

This inequality can not hold for small $r(t_n)$. So claim (8.5) is true.

(8.4) and (8.5) imply that there exists a $r > 0$ such that

$$\varphi \geq u \quad \forall (t, z) \in B_r(z_0) \times (t : t \geq t_0, t - t_0 \leq r),$$

Since u is a viscosity solution of (8.1) (c.f. Remark 3.3), from sub-solution property we have

$$-d_1(z_0)pu(t_0, z_0) - d_2(z_0)\delta + \frac{1}{\epsilon}d_3(z_0) - \overline{U}(pu(t_0, z_0) - z_0\delta) \leq 0.$$

Since $d_3(z_0) > 0$ when $z_0 \neq 0, 1$, the above inequality can not hold for small ϵ. We conclude (8.3) is not true. This says $u(t, \cdot)$ is differentiable pointwise, which further implies $u(t, \cdot)$ is continuously differentiable.

Next, we show on the line $z = z_0$

$$\lim_{t \to t_0} \frac{\partial u}{\partial z}(t, z_0) = \frac{\partial u}{\partial z}(t_0, z_0). \tag{8.6}$$

Suppose (8.6) is not true, without lose of generality, we assume there exists $\epsilon > 0$ and a sequence $t_n \to t_0$, such that

$$\frac{\partial u}{\partial z}(t_n, z_0) \geq \frac{\partial u}{\partial z}(t_0, z_0) + \epsilon. \tag{8.7}$$

We pick z such that $z < z_0$. Then

$$\begin{aligned} u(t_n, z) &\leq u(t_n, z_0) + \frac{\partial u}{\partial z}(t_n, z_0)(z - z_0) \\ &\leq u(t_0, z_0) + \frac{\partial u}{\partial z}(t_0, z_0)(z - z_0) + \epsilon(z - z_0) + O(t_n - t_0) \\ &= u(t_0, z) + \{\frac{\partial u}{\partial z}(t_0, z_0) - \frac{\partial u}{\partial z}(t_0, z_\theta)\}(z - z_0) \\ &\quad + \epsilon(z - z_0) + O(t_n - t_0) \end{aligned} \tag{8.8}$$

Let $t_n \to t_0$ and choose z close to z_0, we have by the continuity of $\frac{\partial u}{\partial z}(t_0, \cdot)$

$$|u(t_n, z) - u(t_0, z)| \geq \frac{\epsilon}{2}|z_0 - z)|, \quad \forall t_n,$$

So (8.7) can not hold and this imply (8.6) must be true.

Finally, to show $\frac{\partial u}{\partial z}$ is jointly continuous w.r.t. (t, z), we claim that the convergence of the following limits are uniform w.r.t. t,

$$\lim_{z \downarrow z_0} \frac{\partial u}{\partial z}(t, z) = \frac{\partial u}{\partial z}(t, z_0), \quad \lim_{z \uparrow z_0} \frac{\partial u}{\partial z}(t, z) = \frac{\partial u}{\partial z}(t, z_0).$$

But this can be readily derived from the concavity of $u(t, \cdot)$, (8.6) and Dini's theorem.

Q.E.D.

8.2 The Bootstrap Method

Proposition 8.3 *Let Ω be defined as in Proposition 8.2. Then the function u as defined by (6.10) is the unique viscosity solution of the following linear parabolic differential equation*

$$-\frac{\partial \tilde{u}}{\partial t} - d_3(z)\frac{\partial^2 \tilde{u}}{\partial z^2} + h(t, z) = 0, \quad \forall (t, z) \in \Omega, \tag{8.9}$$

$$\tilde{u}(t, z) = u(t, z), \quad \forall (t, z) \in \partial^* \Omega, \tag{8.10}$$

where $h(t, z)$ is given by

$$h(t, z) = -d_0(z)pu(t, z) - d_2(z)\frac{\partial u}{\partial z}(t, z) - \overline{U}(pu(t, z) - z\frac{\partial u}{\partial z}(t, z)). \tag{8.11}$$

Proof: That u is a viscosity solution of (8.9)(8.10) can be proved by direct verification. We only need to show the uniqueness result.

We first observe from Proposition 8.2 that $h(t, z) \in C(\overline{\Omega})$. This fact entitles us to use the method described in Lemma 6.1, Lemma 6.2 and Theorem 6.2 to prove similarly a comparison result. We recapitulate the essential point in the following.

A careful examination of the comparison proof shows that all the steps that lead to (6.32) (6.35) can be directly modified to provide us the following result:

Suppose there are two distinct viscosity sub and super solutions to the equation (8.9) and (8.10), then there exists $\epsilon_0, \alpha_0, \beta_0, \rho_0 > 0$, such that for fixed ρ_0 and for all $\epsilon < \epsilon_0$, $\alpha < \alpha_0$, $\beta < \beta_0$, we can find $(\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta})$, $(\bar{s}_{\epsilon,\alpha,\beta}, \bar{w}_{\epsilon,\alpha,\beta})$ bounded away from $\partial\Omega$, and uniformly bounded $\overline{A}_{\epsilon,\alpha,\beta} \leq \overline{B}_{\epsilon,\alpha,\beta}$, satisfying

$$\beta + \frac{\rho_0}{(\bar{t}_{\epsilon,\alpha,\beta} - \epsilon k_1)^2} - \frac{\bar{t}_{\epsilon,\alpha,\beta} - \bar{s}_{\epsilon,\alpha,\beta}}{\alpha} - d_3(z_{\epsilon,\alpha,\beta})\overline{A}_{\epsilon,\alpha,\beta} + h(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}) \leq 0, \quad (8.12)$$

$$-\frac{\bar{t}_{\epsilon,\alpha,\beta} - \bar{s}_{\epsilon,\alpha,\beta}}{\alpha} - d_3(w_{\epsilon,\alpha,\beta})\overline{B}_{\epsilon,\alpha,\beta} + h(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}) \geq 0, \quad (8.13)$$

where $(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta})$ and $(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta})$ are points in the interior of Ω such that

$$dist\{(t_{\epsilon,\alpha,\beta}, z_{\epsilon,\alpha,\beta}), (\bar{t}_{\epsilon,\alpha,\beta}, \bar{z}_{\epsilon,\alpha,\beta})\} \leq \epsilon k_1,$$

$$dist\{(s_{\epsilon,\alpha,\beta}, w_{\epsilon,\alpha,\beta}), (\bar{s}_{\epsilon,\alpha,\beta}, \bar{w}_{\epsilon,\alpha,\beta})\} \leq \epsilon k_1.$$

Moreover, we have:

$$\frac{1}{2\alpha}((\bar{t}_{\epsilon,\alpha,\beta} - \bar{s}_{\epsilon,\alpha,\beta})^2 + (\bar{z}_{\epsilon,\alpha,\beta} - \bar{w}_{\epsilon,\alpha,\beta})^2) \rightarrow 0.$$

Subtract (8.13) from (8.12), let $\epsilon \rightarrow 0$ first and then let $\alpha \rightarrow 0$, we will obtain a contradiction w.r.t. $\beta > 0$.

Thus the uniqueness result must hold.

Q.E.D.

To continue our process of upgrading the regularity of u, we need to invoke some results from the theory of second order partial differential equations of parabolic type. We refer the readers to [25] for the detailed information on this theory.

We will need the following divergence form of (8.9) (8.10).

$$\frac{\partial \tilde{u}}{\partial t} + \frac{\partial}{\partial z}(d_3(z)\frac{\partial \tilde{u}}{\partial z}) - d_3'(z)\frac{\partial \tilde{u}}{\partial z} = h(t,z) \qquad \forall (t,z) \in \Omega \qquad (8.14)$$

$$\tilde{u}(t,z) = u(t,z) \qquad \forall (t,z) \in \partial^*\Omega \qquad (8.15)$$

Definition 8.1 *A function $\bar{u} \in C(\Omega)$ is called a continuous weak solution of (8.14) (8.15) if $u \in W_p^{1,2}(\Omega)$ for some $p \geq 1$ and for all $\varphi \in C_0^{1,2}(\Omega)$,*

$$\int_\Omega \{-\frac{\partial \bar{u}}{\partial t}\varphi + d_3(z)\frac{\partial \bar{u}}{\partial z}\frac{\partial \varphi}{\partial z} + d_3'(z)\frac{\partial \bar{u}}{\partial z}\varphi + h\varphi\}dxdt = 0,$$

and moreover, $\bar{u}(t,z) = u(t,z)$ for all $(t,z) \in \partial^*\Omega$.

Definition 8.2 *Let $X = (s,z) \in \Omega$, the parabolic cylinder is defined by*

$$Q(X,r) = \{Y \in R^2 : |Y - X| < r, t > s\}, \text{where} \quad |X| = \max\{|z|, s^{\frac{1}{2}}\}.$$

The parabolic Holder norms are defined by:

$$[u]_{1+\alpha}^0 = \sup\{d(X,Y)^{1+\alpha}\frac{|D_z u(X) - D_z u(Y)|}{|X-Y|^\alpha} : X \neq Y in\ \Omega\},$$

$$\langle u\rangle_{1+\alpha}^0 = \sup\{d(X,Y)^{1+\alpha}\frac{|u(X) - u(Y)|}{|X-Y|^{1+\alpha}}\},$$

$$|u|_{1+\alpha}^* = |u|_0 + diam(\Omega)|D_z u|_0 + [u]_{1+\alpha}^0 + \langle u\rangle_{1+\alpha}^0,$$

where $d(X) = dist\{X, \partial^*\Omega \cap (t > s)\}$, $d(X,Y) = \min\{d(X), d(Y)\}$ and $0 < \alpha < 1$.

u is said to be in $H_{1+\alpha}^*$ iff $|u|_{1+\alpha}^* < \infty$. Clearly $H_{1+\alpha}^*$ is a Banach space.

We will also need the following weighted Morrey space norm:

$$|u|_{1,2+\alpha}^{(2)} = \sup_{\substack{X \in \Omega \\ 0 \leq r \leq d(X)/2}} r^{-2-\alpha}d(X)^{1+\alpha}\int_{Q(X,r)} |u|dY.$$

u is said to be in $M_{1,2+\alpha}^{(2)}$ iff $|u|_{1,2+\alpha}^{(2)} < \infty$.

It is easy to verify that if u is a continuous function, then

$$|u|_{1,2+\alpha}^{(2)} \leq C(\Omega)|u|_0$$

.

The following a priori Holder estimate for $H_{1+\alpha}^*$ weak solutions is taken from [25] Theorem 4.8.

Lemma 8.1 *Let Ω be as in Proposition 8.3. Suppose $h \in M^{(2)}_{1,2+\alpha}$ and $u \in H^*_{1+\alpha}$ is a weak solution of (8.14) (8.15). Then there exists a constant C depending only on $d_3(z)$, α and Ω, such that,*

$$|u|^*_{1+\alpha} \leq C(|u|_0 + |h|^{(2)}_{1,2+\alpha}). \tag{8.16}$$

The following stability property on viscosity solution will be needed also. (e.g. [13] chapter 2 Lemma 6.2)

Lemma 8.2 *Let $h_\epsilon \in C^\infty(\overline{\Omega})$, and $h_\epsilon \to h$ uniformly on $\overline{\Omega}$ as $\epsilon \to 0$. Suppose for each ϵ, u_ϵ is a viscosity solution of the linear equation*

$$\frac{\partial \tilde{u}_\epsilon}{\partial t} + d_3(z)\frac{\partial^2 \tilde{u}_\epsilon}{\partial z^2} = h_\epsilon(t,z), \quad \forall (t,z) \in \Omega, \tag{8.17}$$

$$\tilde{u}_\epsilon(t,z) = u(t,z) \quad \forall (t,z) \in \partial^* \Omega. \tag{8.18}$$

Moreover, $\tilde{u}_\epsilon \to u_0$ uniformly on $\overline{\Omega}$.

Then u_0 must be the viscosity solution of the equation (8.9) (8.10). In particular, $u_0 = u$.

Theorem 8.1 *Let Ω be defined as in Proposition 5.2. Then the function u as defined in (6.10) is C^∞ on Ω.*

Proof: Let us define

$$\phi(t,z) = \begin{cases} K \exp\{1/(t^2+z^2-1)\} & \text{if } t^2+z^2 < 1, \\ 0 & \text{if } t^2+z^2 \geq 1, \end{cases}$$

and let

$$h_\epsilon(t,z) = h * \phi_\epsilon(t,z),$$

where $\phi_\epsilon = 1/\epsilon^2 \phi(t/\epsilon, z/\epsilon)$.

Then $h_\epsilon \in C^\infty(\overline{\Omega})$ and

$$h_\epsilon(t,z) \to h(t,z) \quad \text{uniformly on } \overline{\Omega} \text{ as } \epsilon \to 0. \tag{8.19}$$

By standard result in linear parabolic equation theory(e.g. [25] Theorem 5.9, Theorem 5.10), we know for each $\epsilon > 0$, there exists an $u_\epsilon \in C^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ which is the classical unique solution of the equation (8.17) (8.18).

Clearly, the classical solution u_ϵ is also a viscosity solution of (8.17) (8.18), and moreover, a $H^*_{1+\alpha}$ weak solution of (8.14) (8.15) as well.

Because u_ϵ is a $H^*_{1+\alpha}$ weak solution of (8.14) (8.15), we can invoke Lemma 8.1 and obtain the estimate

$$\begin{aligned} |u_{\epsilon_m} - u_{\epsilon_n}|^*_{1+\alpha} &\leq C(|u_{\epsilon_m} - u_{\epsilon_n}|_0 + |h_{\epsilon_m} - h_{\epsilon_n}|^{(2)}_{1,2+\alpha}) \\ &\leq C(|h_{\epsilon_m} - h_{\epsilon_n}|_0), \end{aligned} \tag{8.20}$$

where the second inequality is obtained by the standard maximum principle(e.g. [25] Theorem 2.11).

Thus $(u_{\epsilon_m})_{m=1}^{\infty}$ is a Cauchy sequence in the Banach space $H^*_{1+\alpha}$, and so there exists $u_0 \in H^*_{1+\alpha}$ such that $|u_{\epsilon_m} - u_0|^*_{1+\alpha} \to 0$. In particular, we have

$$|u_{\epsilon_m} - u_0|_0 \to 0, \qquad \text{as } \epsilon \to 0, \tag{8.21}$$

$$\frac{\partial u_0}{\partial z} \in H_\alpha(\Omega_0), \qquad \Omega_0 \subset\subset \Omega. \tag{8.22}$$

On the other hand, because u_ϵ is a viscosity solution of (8.17) (8.18), and also because of (8.19) (8.21) we can invoke Lemma 4.2 and conclude that u_0 is actually the unique viscosity solution of (8.9) (8.10). Moreover, by the uniqueness of viscosity solution, we know $u_0 = u$.

We conclude from (8.22) that the function u is actually parabolic Holder continuous, that is,

$$\frac{\partial u}{\partial z} \in H_\alpha(\Omega_0), \qquad \Omega_0 \subset\subset \Omega. \tag{8.23}$$

Now from (8.11) (8.23), we know that $h \in H_\alpha(\Omega_0)$. So we can invoke the standard theory (e.g. [25] Theorem 5.9, Theorem 5.10) and obtain $u \in H^*_{2+\alpha}(\Omega_0)$.

We can continue this bootstrap process (e.g. using the method similar to [25] Theorem 6.6) to get

$$u \in C^\infty(\Omega).$$

Q.E.D.

CHAPTER 9

ESTIMATES ON THE LOCATION OF THE FREE BOUNDARY

9.1 Upper Bound of the Free Boundary

In accordance with our intuition, we have that it is never optimal to be short in stock market if $\alpha > r$.

Proposition 9.1

$$\{(t, x, y) \in Q : y \leq 0\} \cap NT = \emptyset.$$

Proof: Suppose the above statement is not true, then on the x-axis there exists a point $(t_0, x_0, 0) \in NT$. Because $NT(t_0)$ is of wedge shape, we have$\{(t_0, x, 0) : \forall x > 0\} \subset NT$.

Moreover, since NT is open, if we define

$$T_0 = \sup\{t : t > t_0, (t, x, 0) \in NT, \forall x > 0\},$$

then we have $t_0 < T_0$ and $\{(t, x, 0) : \forall x > 0\} \subset NT$ hold true for all $t_0 \leq t < T_0$.

Clearly we have $\{(T_0, x, 0) : x > 0\} \subset \overline{SMM}$. Thus,

$$V(T_0, x, y) = V(T_0, x + y/(1-\lambda), 0), \quad \forall (x, y) \in \mathcal{S} \cap (y < 0). \tag{9.1}$$

Using the method similar to that of Proposition 8.3, we can show for $(s, x) \in (t_0, T_0) \times (x : x \geq 0)$, the function $\tilde{V}(s, x) = V(s, x, 0)$ is actually the value function of the following control problem:

$$\sup_{c \in \mathcal{A}_t} E\{\int_t^{T_0} \frac{c(s)^p}{p} ds + V(T_0, X(T_0), 0)\},$$

$$dX(s) = (rX(s) - c(s))ds, \quad X(t) = x,$$

where the admissible control set is $\mathcal{A}_t = \{c(s) \geq 0 : \text{such that } X(s) \geq 0, \forall s \geq t\}$.

The dynamical programming equation of this problem is

$$\frac{\partial v}{\partial s} + \sup_{c>0}\{rx\frac{\partial v}{\partial x} - c\frac{\partial v}{\partial x} + \frac{c^p}{p}\} = 0, \qquad (s, x) \in (t_0, T_0) \times (x : x \geq 0),$$

$$v(T_0, x) = V(T_0, x, 0), \forall x \geq 0; \qquad v(s, 0) = 0, \forall s \in [t_0, T_0].$$

Notice that $V(T_0, x, 0) = V(T_0, 1, 0)x^p$, we can use the same method as in Proposition 6.1 and conclude

$$v(s, x) = \frac{1}{p} A(s) x^p,$$

where $A(s) = \{\frac{1-p}{rp})(e^{\frac{rp}{1-p}(T-s)} - 1) + (pV(T_0, 1, 0))^{\frac{1}{1-p}} e^{\frac{rp}{1-p}(T-s)}\}^{1-p}$, and it satisfies the equation

$$\frac{1-p}{p} A'(s) + rA(s) + \frac{1-p}{p} = 0, \quad A(T_0) = (pV(T_0, 1, 0))^{\frac{1}{1-p}}. \tag{9.2}$$

Hence

$$V(s, x, 0) = \frac{1}{p} A(s) x^p, \quad \forall (t, x) \in [t_0, T_0] \times (x : x \geq 0). \tag{9.3}$$

Let $\Omega = (t_0, T_0) \times \{(x, y) : x + y/(1-\lambda) \geq 0\} \cap \{y \leq 0\}$, and define,

$$\tilde{V}(t, x, y) = \frac{1}{p} A(s)^{1-p} (x + \frac{y}{1-\lambda})^p, \qquad (t, x, y) \in \Omega. \tag{9.4}$$

We claim that $\tilde{V} = V$ on Ω. This will contradict with our supposition that $(t_0, x, 0) \in NT$, and thus complete our proof.

To prove the claim, we suffice to show(c.f. Remark 3.5) $\tilde{V}$ is the viscosity solution of the equation (5.7) with boundary condition

$$\tilde{V}(t,x,y) = V(t,x,y), \qquad (t,x,y) \in \partial^* \Omega.$$

From (9.1) (9.3) and the definition of $\tilde{V}$ (9.4) we see readily that the above boundary condition is satisfied.

That $\tilde{V}$ is a viscosity sub-solution is also clear from the definition of $\tilde{V}$.

That $\tilde{V}$ is a viscosity super-solution of (5.7) could be seen from the following computation similar to that of (7.8).

Let $\varphi \in C^{1,2}(\Omega)$, and suppose $\tilde{V} - \varphi$ attains a local minimum at $(t,x,y) \in \Omega$, then

$$\begin{aligned} \mathcal{L}\varphi(t,x,y) - \overline{U}(\frac{\partial \varphi}{\partial x}) &\geq A(t)^{-p}(x + \frac{y}{1-\lambda})^p \\ &\cdot \{\frac{1}{2(1-p)}(\frac{\sigma(1-p)y}{(1-\lambda)x+y} - \frac{\alpha - r}{\sigma})^2 A(t) \\ &- \frac{1-p}{p}A'(t) - (r + \frac{(\alpha-r)^2}{2(1-p)\sigma^2})A(t) - \frac{1-p}{p})\} \\ &\geq A(t)^{-p}(x + \frac{y}{1-\lambda})^p\{-\frac{1-p}{p}A'(t) - rA(t) - \frac{1-p}{p}\} \\ &\quad \text{(Here we use the standing assumption } \alpha > r) \\ &= 0. \text{(by (9.2))} \end{aligned}$$

Q.E.D.

9.2 Lower Bound of the Free Boundary

In the following of this section, we will use $x + (1-\mu)y = 1$ as the reference line to reduce the value function $V(t,x,y)$ to the two dimension value function $u(t,z)$. We denote

$$Q_1 = [0,T) \times (-(1-\lambda)/(\mu + \lambda - \lambda\mu), \infty),$$

and define

$$u(t,z) = V(t, 1-(1-\mu)z, z), \qquad (t,z) \in Q_1. \tag{9.5}$$

Similar to Proposition 6.6, we know $u(t,z)$ is the unique viscosity solution of the following equation for $(t,z) \in Q_1$,

$$\begin{aligned} \min \quad & \{-\frac{\partial u}{\partial t} - d_1(z)pu - d_2(z)\frac{\partial u}{\partial z} - d_3(z)\frac{\partial^2 u}{\partial z^2} - \overline{U}(pu - z\frac{\partial u}{\partial z}), \\ & \frac{\partial u}{\partial z}, \quad \mu pu - [z + (1-\lambda)(1-(1-\mu)z]\frac{\partial u}{\partial z} \ \} = 0, \end{aligned} \tag{9.6}$$

where

$$\begin{aligned} d_1(z) &= r + (\alpha - r)(1-\mu)z - \frac{1}{2}\sigma^2(1-p)(1-\mu)^2 z^2, \\ d_2(z) &= (\alpha - r)z(1-(1-\mu)z) - \sigma^2(1-p)(1-\mu)z^2(1-(1-\mu)z), \\ d_3(z) &= \frac{1}{2}\sigma^2 z^2(1-(1-\mu)z)^2, \end{aligned}$$

with boundary condition

$$u(t,z) = 0, \qquad (t,z) \in \partial^* Q_1. \tag{9.7}$$

We will denote on the line $x + (1-\mu)y = 1$ the location of the free boundary between NT and SS by $(1-(1-\mu)z^*(t), z^*(t))$. From Proposition 4.4 and 4.6, we know $z^*(t)$ is lower semi-continuous and is locally bounded.

The proof of Proposition 8.2 could also be extended to provide us the following result:

Proposition 9.2 *$\frac{\partial u}{\partial z}$ is continuous on $Q_1 \setminus \{z = 1/1-\mu\}$. In particular, for any $t \in [0,T)$ with $z^*(t) \neq 1/(1-\mu)$, we have*

$$\lim_{(s,z)\to(t,z^*(t))} \frac{\partial u}{\partial z}(s,z) = \frac{\partial u}{\partial z}(t, z^*(t)) = 0.$$

Since NT is an open set, for a fixed point $(t,x,y) \in \overline{NT} \setminus \{x=0\}$, we can find a open set O, such that $(t,x,y) \in O$ and O is bounded away from the close set $SMM \cup \{x=0\}$. In the following, we will always restrict the state processes in such an open set. So we can assume

$$\begin{aligned} dX(s) &= (rX(s) - c(s))ds + (1-\mu)dM(s), X(t) = x, s \in [t, \tau_0], && (9.8) \\ dY(s) &= \alpha Y(s)ds + \sigma Y(s)dW(s) - dM(s), Y(t) = y, s \in [t, \tau_0], && (9.9) \end{aligned}$$

where $\tau_0 = \inf\{s \geq t : (X(s), Y(s)) \notin O)\}$.

For any admissible strategy $(c(s), M(s)) \in \mathcal{A}(t, x, y)$, (9.8) and (9.9) have the following explicit solution for $s \in [t, \tau_0]$:

$$\begin{aligned} X(s) &= xe^{r(s-t)} - \int_t^s e^{r(s-u)}c(u)du + (1-\mu)\int_t^s e^{r(s-u)}dM(u), \\ Y(s) &= y\eta(s) - \int_t^s \eta(s)\eta(u)^{-1}dM(u), \end{aligned}$$

where

$$\eta(s) = \exp\{(\alpha - \frac{\sigma^2}{2})(s-t) + \sigma W_{s-t}\}.$$

For $\epsilon > 0$, let us define

$$NT(\epsilon) = \{(t, x, y) \in NT : \frac{y}{x + (1-\mu)y} < z^*(t) - \epsilon\}.$$

Clearly, $(s, X(s), Y(s)) \in \overline{NT(\epsilon)}$ holds for all $s \in [t, \tau_0]$ if and only if for all $s \in [t, \tau_0]$,

$$\begin{aligned} &\int_t^s \{\eta(s)\eta(u)^{-1} + (1-\mu)(z^*(s) - \epsilon)e^{r(s-u)} \\ &\qquad - (1-\mu)(z^*(s)-\epsilon)\eta(s)\eta(u)^{-1}\}dM(u) \\ &\geq y\eta(s) - \{xe^{r(s-t)} - \int_t^s e^{r(s-u)}c(u)du \\ &\qquad + (1-\mu)y\eta(s)\}(z^*(s) - \epsilon), \end{aligned} \tag{9.10}$$

We notice that the strict inequality holds if and only if $(s, X(s), Y(s)) \in NT(\epsilon)$.

To make (9.10) amenable to handling, let us define for small $\delta > 0$,

$$\tau_\delta = \inf\{s \geq t : |W(s-t)| \geq \delta^{\frac{1}{4}}\} \wedge \tau_0 \wedge (t + \delta), \tag{9.11}$$

and we will consider the time that predates τ_δ.

By making $\delta > 0$ small, we can assume for $s \in [t, \tau_\delta]$ and all sufficiently small $\epsilon > 0$,

$$xe^{r(s-t)} - \int_t^s e^{r(s-u)}c(u)du + (1-\mu)y\eta(s) \approx x + (1-\mu)y > 0,$$

$$\eta(s)\eta(u)^{-1} + (1-\mu)(z^*(s) - \epsilon)e^{r(s-u)} - (1-\mu)(z^*(s) - \epsilon)\eta(s)\eta(u)^{-1} \approx 1 > 0.$$

Now let us define for $s \in [t, \tau_\delta]$,

$$a_{\delta,\epsilon}(s) = y\eta(s) - \{xe^{r(s-t)} - \int_t^s e^{r(s-u)}c(u)du + (1-\mu)y\eta(s)\}(z^*(s) - \epsilon), \quad (9.12)$$

$$m_{\delta,\epsilon}(s) = \sup\{a_{\delta,\epsilon}(u) : u \in [t, s]\}. \quad (9.13)$$

From the fact that $z^*(s)$ is lower semi-continuous and locally bounded, it is easy to see that $m_{\delta,\epsilon}(s)$ is right continuous, increasing and uniformly bounded w.r.t. ϵ. In particular, $m_{\delta,\epsilon}(s)$ induces a measure on $[t, \tau_\delta]$.

The transaction process can now be constructed as the following:

$$b_\epsilon(s) = \eta(s)\eta(u)^{-1} + (1-\mu)(z^*(s) - \epsilon)e^{r(s-u)} - (1-\mu)(z^*(s) - \epsilon)\eta(s)\eta(u)^{-1}\}^{-1}, \quad (9.14)$$

$$M_{\delta,\epsilon}(s) = \int_t^s b_\epsilon(s) dm_{\delta,\epsilon}(u). \quad (9.15)$$

Clearly, $M(s)$ is right continuous and uniformly bounded with respect to ϵ on $[t, \tau_\delta]$.

Proposition 9.3 *For $(t, x, y) \in \overline{NT}$, suppose we choose the open set O, the stopping time τ_δ, and the transaction process $M_{\delta,\epsilon}$ according to the procedure described above, with the consumption strategy specified by:*

$$c_{\delta,\epsilon}(s, X(s), Y(s)) = \{\frac{\partial V}{\partial x}(s, X(s), Y(s))\}^{\frac{-1}{1-p}}, \quad s \in [t, \tau_\delta). \quad (9.16)$$

Then

$$V(t-, x, y) \le E\{\int_t^{\tau_\delta} \frac{c_{\delta,\epsilon}(s)^p}{p} + V(\tau_\delta, X(\tau_\delta), Y(\tau_\delta))\} + C(\delta, O)\epsilon, \quad (9.17)$$

where $C(\delta, O)$ is a constant that depends only on δ and the open set O, but not on ϵ.

Proof: Since $\{(X(s), Y(s)), s \in [t, \tau_\delta]\}$ are restricted in $\overline{NT(\epsilon)} \subset NT \cap \{x = 0\}$, and $V \in C^{1,2}(NT \cap \{x=0\})$. We can apply Ito's formulae to get

$$\begin{aligned} V(\tau_\delta, X(\tau_\delta), Y(\tau_\delta)) &= V(t-, x, y) + \{V(t, X(t), Y(t)) - V(t-, x, y)\} \\ &+ \int_t^{\tau_\delta} \{-\mathcal{L}V(s, X(s), Y(s)) \\ &\qquad - c_{\delta,\epsilon}(s)\frac{\partial V}{\partial x}(s, X(s), Y(s))\}ds \\ &+ \int_t^{\tau_\delta} \{-\frac{\partial V}{\partial y}(s, X(s), Y(s)) \\ &\qquad + (1-\mu)\frac{\partial V}{\partial x}(s, X(s), Y(s))\}dM_{\delta,\epsilon}(s) \\ &+ \int_t^{\tau_\delta} \sigma Y(s)\frac{\partial V}{\partial y}(s, X(s), Y(s))dW(s). \end{aligned} \tag{9.18}$$

From our choice of $c_{\delta,\epsilon}(s)$ and the smoothness of V in NT, we have

$$-\mathcal{L}V(s, X(s), Y(s)) - c_{\delta,\epsilon}(s)\frac{\partial V}{\partial x}(s, X(s), Y(s)) = -\frac{c_{\delta,\epsilon(s)}{}^p}{p}.$$

To estimate the third term in (9.18), we first observe that

$$\int_t^{\tau_\delta} I_{\{(X(s),Y(s))\in NT(\epsilon)\}}dM_{\delta,\epsilon}(s) = 0,$$

and moreover, from Proposition 6.1 and (9.6), we have

$$\begin{aligned} -\frac{\partial V}{\partial y}(s,x,y) + (1-\mu)\frac{\partial V}{\partial y}(s,x,y) &= -(x+(1-\mu)y)^{p-1} \\ &\quad \frac{\partial u}{\partial z}(s, \frac{y}{x+(1-\mu)y}) \\ &= -(x+(1-\mu)y)^{p-1} \\ &\quad \cdot \frac{\partial^2 u}{\partial z^2}(s, \theta(\frac{y}{x+(1-\mu)y} - z^*(s)))\epsilon \\ &\leq C(\delta, O)\epsilon. \end{aligned}$$

Since $M_{\delta,\epsilon}$ is uniformly bounded with respect to ϵ, we obtain:

$$\int_t^{\tau_\delta} \{-\frac{\partial V}{\partial y}(s, X(s), Y(s)) + (1-\mu)\frac{\partial V}{\partial x}(s, X(s), Y(s))\}dM_{\delta,\epsilon}(s) \leq C(\delta, O)\epsilon.$$

$V(t, X(t), Y(t)) - V(t-, x, y)$ can be estimated similarly.

(9.17) now follows forms (9.18) by taking expectations on both sides.

Q.E.D.

The following result on Brownian motion is standard (e.g. [24] p95).

Proposition 9.4 *Let* $\{W(s), \mathcal{F}_s, s \geq 0\}$ *be a Brownian motion on* $(\Omega, \mathcal{F}, P)$, $b > 0$ *be a fixed number. Define*

$$W^*(s) = \max_{0 \leq u \leq s} |W(u)|, \qquad \tau = \inf\{s \geq 0 : W^*(s) = b\}.$$

Then

$$P\{\tau \leq s\} \leq C \exp\{-\frac{b^2}{2s}\}.$$

Theorem 9.1 *Let* $z_0 = (\alpha - r)/(1-\mu)(1-p)\sigma^2$.

1. *If* $z_0 \leq 1/(1-\mu)$, *then for all* $t \in [0, T)$, *we must have*

$$z^*(t) \geq z_0 = \frac{\alpha - r}{(1-\mu)(1-p)\sigma^2}.$$

2. *If* $z_0 > 1/(1-\mu)$, *then either*

$$z^*(t) \geq z_0, \quad \forall t \in [0, T),$$

or

$$z^*(t) \equiv \frac{1}{1-\mu} \quad \forall t \in [0, T).$$

Proof:

Let us first consider the case $z_0 \leq \frac{1}{1-\mu}$.

Suppose there exists a $t \in [0, T)$ such that

$$z^*(t) < z_0., \tag{9.19}$$

We try to derive a contradiction.

We pick a point z_1, such that $z^*(t) < z_1 < z_0$. and then choose an open set $O \subset Q$ such that O contains the point $(t, 1-(1-\mu)z^*(t), z^*(t))$ and satisfies

$$O \cap \{(t, x, y) : \frac{y}{x + (1-\mu)y} \geq z_{\frac{1}{2}} = \frac{z^*(t) + z_1}{2}\} = \emptyset.$$

For the point $(t, 1-(1-\mu)z^*(t), z^*(t))$, and with arbitrary $\epsilon > 0$, and $\delta > 0$, we invoke Proposition 9.3 with the control $(c_{\delta,\epsilon}, M_{\delta,\epsilon})$ as described therein. We obtain:

$$\begin{aligned} V(t,1-(1-\mu)z^*(t),z^*(t)) &\le E\{\int_t^{\tau_\delta} \frac{c_{\delta,\epsilon}(s)^p}{p}ds + V(\tau_\delta, X(\tau_\delta), Y(\tau_\delta))\} \\ &\quad +C(\delta,O)\epsilon \\ &\le E\int_t^{\tau_\delta} \frac{c_{\delta,\epsilon}(s)^p}{p}ds + C(\delta,O)\epsilon \\ &\quad +E\{(X(\tau_\delta)+(1-\mu)Y(\tau_\delta))^p \\ &\quad \cdot V(\tau_\delta, 1-(1-\mu)z_{\frac{1}{2}}, z_{\frac{1}{2}})\}, \end{aligned} \tag{9.20}$$

where the second inequality came from the fact that $V(t,1-(1-\mu)z,z)$ is increasing w.r.t. z.

Observe the definition of stopping time τ_δ (9.11), we have from Proposition 9.4 that for sufficiently small $\delta > 0$,

$$P\{\tau_\delta < t+\delta\} \le C\exp\{-\frac{1}{2\sqrt{\delta}}\}.$$

Thus (9.20) can be relaxed to:

$$\begin{aligned} V(t,1-(1-\mu)z^*(t),z^*(t)) &\le E\int_t^{t+\delta} \frac{c_{\delta,\epsilon}(s)^p}{p}ds + C(\delta,O)\epsilon \\ &\quad +E(X(t+\delta)+(1-\mu)Y(t+\delta))^p \\ &\quad \cdot u(t+\delta, z_{\frac{1}{2}}) + C\exp\{-\frac{1}{2\sqrt{\delta}}\} \end{aligned} \tag{9.21}$$

Now we use (9.8) and (9.9) to compute the term $E(X(t+\delta)+(1-\mu)Y(t+\delta))$ in (9.21). We have:

$$\begin{aligned} E(X(t+\delta)+(1-\mu)Y(t+\delta))^p &= 1 - E\int_t^{t+\delta} p(X(s)+(1-\mu)Y(s))^{p-1} \\ &\qquad \cdot (r - c_{\delta,\epsilon}(s))ds \\ &\quad +E\int_t^{t+\delta} p(X(s)+(1-\mu)Y(s))^p \\ &\qquad \cdot \{(1-\mu)(\alpha-r)z(s) \\ &\quad -\frac{1-p}{2}(1-\mu)^2\sigma^2 z(s)^2\}ds, \end{aligned} \tag{9.22}$$

where $z(s) = Y(s)/(X(s)+(1-\mu)Y(s))$.

Let us denote

$$\gamma_1(s) = Ep(X(s)+(1-\mu)Y(s))^{p-1}(r - c_{\delta,\epsilon}(s)),$$

$$\gamma_2(s) = Ep(X(s) + (1-\mu)Y(s))^p\{(1-\mu)(\alpha - r)z(s) - \frac{1-p}{2}(1-\mu)^2\sigma^2 z(s)^2.$$

Since $M_{\delta,\epsilon}(s)$ is a right continuous process, so are $(X(s), Y(s))$. Thus by dominated convergence theorem $\gamma_1(s), \gamma_2(s)$ are right continuous functions. We conclude that $E(X(t+\delta) + (1-\mu)Y(t+\delta))^p$ is equal to

$$\begin{aligned} & 1 - \gamma_1(t)\delta + \gamma_2(t)\delta + o(\delta) \\ = \; & 1 - p\{r - [pu(t, z^*(t))]^{\frac{-1}{1-p}}\}\delta + p\{(1-\mu)(\alpha - r)z^*(t) \\ & -\frac{1-p}{2}(1-\mu)^2\sigma^2 z^*(t)^2\}\delta + o(\delta) \end{aligned} \tag{9.23}$$

where $o(\delta)/\delta \to 0$ as $\delta \to 0$.

Substitute (9.23) into (9.21) we obtain that $V(t, 1-(1-\mu)z^*(t), z^*(t))$ is bounded by

$$\begin{aligned} & E\int_t^{t+\delta} \frac{c_{\delta,\epsilon}(s)^p}{p} ds + C(\delta, O)\epsilon + C\exp\{-\frac{1}{2\sqrt{\delta}}\} + o(\delta) \\ & + u(t+\delta, z_{\frac{1}{2}})\{1 - p\{r - [pu(t, z^*(t))]^{\frac{-1}{1-p}}\}\}\delta \\ & + pu(t+\delta, z_{\frac{1}{2}})\{(1-\mu)(\alpha - r)z^*(t) - \frac{1-p}{2}(1-\mu)^2\sigma^2 z^*(t)^2\}\delta. \end{aligned} \tag{9.24}$$

We now construct a consumption/transaction strategy for the point $(t, 1-(1-\mu)z_1, z_1)$. we first choose an open set O_1 such that O_1 contains the point $(t, 1-(1-\mu)z_1, z_1)$ and satisfies

$$O_1 \cap \{(t, x, y) : \frac{y}{x + (1-\mu)y} \le z_{\frac{1}{2}}, \frac{y}{x + (1-\mu)y} \ge z_0\} = \emptyset.$$

In the following, we will restrict the state processes in this open set. In particular, the stopping time τ_0 in the equations (9.8) (9.9) will be $\tau_0 = \inf\{s \ge t : (X(s), Y(s)) \notin O_1\}$.

In this open set O_1, we define τ_δ the same way as in (9.11), define consumption strategy to be exactly the same as $c_{\delta,\epsilon}(s)$ (for $(t, 1-(1-\mu)z^*(t), z^*(t))$), and we do not make any transactions. We will have the following estimates:

$$\begin{aligned} V(t, 1-(1-\mu)z_1, z_1) &\geq E\int_t^{\tau_\delta} \frac{c_{\delta,\epsilon}(s)^p}{p} ds + EV(\tau_\delta, X(\tau_\delta), Y(\tau_\delta)) \\ &\geq E\int_t^{t+\delta} \frac{c_{\delta,\epsilon}(s)^p}{p} ds - C\exp\{-\frac{1}{2\sqrt{\delta}}\} \\ &\quad + E(X(t+\delta) + (1-\mu)Y(t+\delta))^p \\ &\quad \cdot u(t+\delta, z_{\frac{1}{2}}). \end{aligned} \tag{9.25}$$

To compute $E(X(t+\delta)+(1-\mu)Y(t+\delta))^p$, we notice that the differences are that $(X(s), Y(s))$ start at $(t, 1-(1-\mu)z_1, z_1)$ and $dM(s) = 0$. We can repeat the computation from (9.22) to (9.23) to get

$$\begin{aligned} E(X(t+\delta)+(1-\mu)Y(t+\delta))^p &= 1 - p\{r - [pu(t, z^*(t))]^{\frac{-1}{1-p}}\}\delta \\ &\quad + p\{(1-\mu)(\alpha - r)z_1 \\ &\quad - \frac{1-p}{2}(1-\mu)^2\sigma^2 z_1^2\}\delta + o(\delta) \end{aligned} \tag{9.26}$$

Thus,

$$\begin{aligned} V(t, 1-(1-\mu)z_1, z_1) &\geq E\int_t^{\tau_\delta} \frac{c_{\delta,\epsilon}(s)^p}{p} ds - C\exp\{-\frac{1}{2\sqrt{\delta}}\} + o(\delta) \\ &\quad + u(t+\delta, z_{\frac{1}{2}})\{1 - p\{r - [pu(t, z^*(t))]^{\frac{-1}{1-p}}\}\}\delta \\ &\quad + pu(t+\delta, z_{\frac{1}{2}})\{(1-\mu)(\alpha - r)z_1 \\ &\quad - \frac{1-p}{2}(1-\mu)^2\sigma^2 {z_1}^2\}\delta. \end{aligned} \tag{9.27}$$

Comparing (9.24) and (9.27), if we let $\epsilon \to 0$, we will get

$$V(t, 1-(1-\mu)z_1, z_1) > V(t, 1-(1-\mu)z^*(t), z^*(t)).$$

Contradiction.

Thus if $z_0 \leq 1/(1-mu)$, then $z^*(t) \geq z_0$.

Now let us consider the case $z_0 > 1/(1-mu)$.

We first notice that it is impossible to find a $t \in [0, T)$ such that

$$\frac{1}{1-\mu} < z^*(t) < z_0, \quad \text{or} \quad z^*(t) < \frac{1}{1-\mu}.$$

Because then $z^*(t)$ is locally bounded away from the degenerate line and we still have the regularity of u_z near $(t, z^*(t))$, so we can repeat the proof for the case $z_0 \leq 1/(1-mu)$ and derive a contradiction.

If for some $t \in [0,T)$, $z^*(t) = 1/(1-\mu)$, then because $z^*(s)$ is lower semi-continuous, we must have $z^*(t) \equiv 1/(1-\mu)$. Otherwise we can reduce the case to the above situation.

Q.E.D.

Similar result holds for the free boundary between NT and SMM.

We use the reference line $x + y/(1-\lambda) = 1$ and denote the free boundary between NT and SMM at time t by $(1 - y^*(t)/(1-\lambda), y^*(t))$.

Theorem 9.2 *For all $t \in [0,T)$, we have*

$$y^*(t) \leq \frac{(1-\lambda)(\alpha - r)}{(1-p)\sigma^2}.$$

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APPENDIX

This appendix contains some elementary compuation on the discrete Fourier transformation and discrete Cauchy distribution.

For a function $f(x)$ defined on Z^d satisfying $\sum_{x\in Z^d}|f(x)|<\infty$, we define its discrete Fourier Transform:

$$\hat{f}(k)=\sum_{x\in Z^d}\exp\{ik\cdot x\}f(x),\quad k\in[-\pi,\pi]^d.$$

Clearly $\hat{f}(k)$ is a bounded function on $[-\pi,\pi]^d$.

The inverse Fourier transform relation also holds true:

$$f(x)=\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\exp\{-ik\cdot x\}\hat{f}(k)dk,\quad x\in Z^d.$$

Proposition: Suppose

$$\sum_{x\in Z^d}|f(x)|<\infty,$$

then the following Pasavel equality holds:

$$\sum_{x\in Z^d}f(x)^2=\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\hat{f}(k)^2dk.$$

Proof: By hypothesis on f, clearly we have $sum_{x\in Z^d}|f(x)|^2<\infty$. Thus

$$\begin{aligned}\int_{[-\pi,\pi]^d}\hat{f}(k)^2dk &= \int_{[-\pi,\pi]^d}\{\sum_{x\in Z^d}\exp\{ik\cdot x\}f(x)\}^2dk\\ &= \int_{[-\pi,\pi]^d}\{\sum_{x,y\in Z^d}\exp ik\cdot(x+y)f(x)f(y)dk\\ &= (2\pi)^d\sum_{x\in Z^d}f(x)^2.\end{aligned}$$

Q.E.D.

By Fourier analysis we have:

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}, \qquad x \in [-\pi, \pi],$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos(nx)}{n^2}, \qquad x \in [-\pi, \pi].$$

Combine the above two expression we obtain:

$$\frac{1}{4}(|x|-\pi)^2 - \frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2}.$$

In the above, if we let $x = 0$, we will have

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

For a random variable X with discrete Cauchy distribution, i.e. $P\{X = \pm n\} = C/n^2, n \in N$, we know from the above equality $C = 3/\pi^2$. Moreover, the characteristic function is given by

$$E\exp\{itX\} = 2C\sum_{n=1}^{\infty}\frac{\cos(nt)}{n^2} = 1 - \frac{3}{\pi}|t| + \frac{3}{2\pi^2}t^2.$$

For a d-dimension random vector X satisfying discrete Cauchy distribution, we have

$$\hat{D}(k) = E\exp\{ik\cdot X\} = 1 - \frac{1}{d}\sum_{i=1}^{d}\{\frac{3}{\pi}|k_i| - \frac{3}{2\pi^2}|k_i|^2\}, \qquad k \in [-\pi, \pi]^d.$$

The following estimate on $1 - \hat{D}(k)$ will be critical in proving the convergence of lace expansion:

$$\frac{3|k|}{2\pi d} \le 1 - \hat{D}(k) \le \frac{3|k|}{2\pi\sqrt{d}}.$$