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ON SOME DEGENERATE DEFORMATIONS OF COMMUTATIVE POLYNOMIAL ALGEBRAS

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in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

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by
Melanie B. Butler
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ABSTRACT

ON SOME DEGENERATE DEFORMATIONS OF COMMUTATIVE POLYNOMIAL ALGEBRAS

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DOCTOR OF PHILOSOPHY

Temple University, August, 2004

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We examine the prime spectra of algebras of the form

$$A = K\{x_1, x_2, \cdots, x_n\}/\langle x_i x_j - \alpha_{ij} x_j x_i - \beta_{ij}, i < j\rangle,$$

where α_{ij} and β_{ij} are elements of the algebraically closed field K. When $\beta_{ij} = 0$ for all i and j, we give a complete classification of the prime ideals, primitive ideals, and irreducible representations of A. We also completely describe spec$A$ when $n = 3$. These classifications prove that, under the Zariski topology, the topological dimension of spec$A$ is not greater than its Gelfand-Kirillov dimension, when $n = 3$ or when $\beta_{ij} = 0$ for all i and j.

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Finally, I am very grateful to Adam Berliner for allowing me to include his unpublished result (see Proposition 2.2.13).

To Freddy,

with all my love.

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CHAPTER 1

INTRODUCTION

Noncommutative polynomial equations have been the subject of extensive study since Dirac's formulation of Heisenberg's principles of quantum mechanics (see [3]). The solutions to noncommutative polynomial equations are *representations.* A deeper study of the topology of representations leads to the study of prime and primitive ideals. Broadly speaking, these are the topics discussed in this dissertation.

For example, we might want to study the solutions to the noncommutative polynomial equation $xy = -yx$, in noncommuting indeterminants x and y. The numerical solutions to this equation are not interesting. There are, however, many linear operators satisfying this equation. The representations of the algebra defined by $K\{x, y\}/\langle xy + yx\rangle$, where K is an algebraically closed field, are the solutions to this equation.

Systematic investigations of the representation theory of noncommutative algebras have been long standing (see, e.g., [10] and [21]). More recently the study of the representation theory of noncommutative algebras has been approached using noncommutative algebraic geometry (see, e.g., [27]). This dissertation focuses on examples not covered by these previous studies. The specific constructions that we will discuss are skew and skew-Laurent polynomial rings. In this chapter, we discuss some of the history of skew polynomial rings and then describe the main results of the dissertation.

1.1 History

Skew polynomial rings in several variables with coefficients from a field K were introduced by Noether and Schmeidler in 1920 ([25]). Manipulations with relations of the form $pq - qp = ih$ arising from quantum mechanics occurred in the work of Dirac in 1926 (see [8]) and Weyl in 1928 ([28]). In the 1930's, Jacobson and Ore began to study iterated Ore extensions (see [19] and [26]). Ore produced a systematic investigation of skew polynomial rings in one variable over a division ring in 1933 ([26]). Dixmier introduced the terminology *Weyl algebra* in 1968 ([9]).

More specifically, in the 1930's, Jacobson began to explore cases of algebras in noncommuting variables over a field, modulo an ideal generated by relations of degree less than or equal to two ([19]). Algebras of this type are particularly interesting because, as filtered vector spaces, they are very close to commutative algebras. Thus these algebras can be viewed as natural generalizations of commutative polynomial algebras.

Since Jacobson studied algebras of this type, there have been numerous other successful studies of the prime ideals and representations of iterated Ore extensions and of other finitely generated noetherian algebras. For instance, the prime and primitive ideal theory of quantum groups, enveloping algebras, and noetherian group algebras is reasonably well understood (see, e.g., [24] and [16]). More specifically, Irving, in the 1970's, studied the prime ideal structure of arbitrary Ore extensions of commutative noetherian rings (see [17] and [18]). Gerritzen also played an important role by classifying the irreducible representations of $K\{x,y\}/\langle yx-1\rangle$, where K is an algebraically closed field (see [11]). More recently there have been studies of iterated Ore extensions by Goodearl and Letzter (see [13], [14], and [15]) and Cauchon (see [1], [4], [5], [6], and [7]). Many of the algebras discussed in this dissertation are iterated

Ore extensions, however, the prime and primitive ideal theory is not covered by these previous studies.

1.2 Summary of Main Results and Statement of Main Theorem

Inspired by the work discussed in Section 1.1, this dissertation will study the prime and primitive ideals of algebras of the form

$$K\{x_1, x_2, \cdots, x_n\}/\langle x_i x_j - \alpha_{ij} x_j x_i - \beta_{ij}, i < j\rangle,$$

where K is an algebraically closed field and α_{ij} and β_{ij} are elements of K. Until now the studies appearing in the research literature have primarily been of noetherian domains (except for Irving and Gerritzen). In this dissertation, however, we are concerned with the less well-behaved deformations of these algebras. For instance, well-known work has shown that

$$R = K\{x, y, z\}/\langle xy - 0yx - 0, xz - 0zx - 1, yz - 0zy - 0\rangle$$

$$= K\{x, y, z\}/\langle xy, xz - 1, yz\rangle$$

has many "bad" properties, such as non-Goldie prime factors and infinite Krull dimension. However, the Gelfand-Kirillov dimension of R is 3.

We begin by studying the prime and primitive ideal theory of algebras of the form

$$K\{x, y, z\}/\langle xy - \alpha_1 yx - \beta_1, xz - \alpha_2 zx - \beta_2, yz - \alpha_3 zy - \beta_3\rangle.$$

Remark 1.2.1. Note that for any algebra of the form

$$K\{x, y, z\}/\langle xy - \alpha_1 yx - \beta_1, xz - \alpha_2 zx - \beta_2, yz - \alpha_3 zy - \beta_3\rangle,$$

we can make a change of variable, replacing x_i with αx_i for any $1 \leq i \leq m$ and any $\alpha \in K$, without changing the isomorphism class of the algebra.

First, we consider the 64 cases that occur when the scalars α_1, α_2, α_3, β_1, β_2, and β_3 of

$$K\{x,y,z\}/\langle xy - \alpha_1 yx - \beta_1, xz - \alpha_2 zx - \beta_2, yz - \alpha_3 zy - \beta_3\rangle$$

are either zero or nonzero. These algebras are degenerate, isomorphic to known cases, or noetherian, except in two cases (up to isomorphism). The two cases (up to isomorphism) that are nondegenerate, not previously studied, and nonnoetherian are as follows.

- $K\{x,y,z\}/\langle xy - \beta_1, xz - \alpha_2 zx, yz - \alpha_3 zy\rangle$, where α_2, α_3, and β_1 are nonzero, and
- $K\{x,y,z\}/\langle xy, xz - \beta_2, yz\rangle$, where β_2 is nonzero.

In the next chapter, the prime spectra of a case that reduces to known cases, a degenerate case, and a noetherian case will be completely described. The prime and primitive spectra of the first nonnoetherian case are discussed in Chapter 4. The prime and primitive ideals of

$$K\{x_1,\ldots,x_n\}/\langle x_i x_j - \alpha_{ij}, i < j\rangle,$$

for K an algebraically closed field, $n \geq 3$, and $\alpha_{ij} \in K$ are classified in Chapter 5 by reducing the study of the n-variable algebra to the study of $K\{x,y,z\}/\langle xy, xz-1, yz\rangle$. The co-finite dimensional primitive ideals are classified by Theorem 2.2.13 ([2]). A classification of the prime ideals, primitive ideals, and irreducible representations of $K\{x,y,z\}/\langle xy, yz, xz-1\rangle$ is given in Chapter 5.

A complete list of the 64 cases and classifications of the prime ideals in each case (or appropriate references) are included in Appendix A. These classifications prove the following.

Theorem 1.2.2. *Let $S_1 = K\{x,y,z\}/\langle xy - \alpha_1 yx - \beta_1, xz - \alpha_2 zx - \beta_2, yz - \alpha_3 zy - \beta_3\rangle$, where K is an algebraically closed field and α_1, α_2, α_3, β_1, β_2, and*

β_3 are elements of K. Let $S_2 = K\{x_1, \ldots, x_n\}/\langle x_i x_j - \alpha_{ij}, i < j\rangle$, for K an algebraically closed field, $n \geq 3$, and $\alpha_{ij} \in K$. Under the Zariski topology, the topological dimensions of specS_1 and specS_2 are no greater than the Gelfand-Kirillov dimensions of S_1 and S_2.

CHAPTER 2

PRELIMINARIES

In this chapter, we discuss the notation and definitions that will be used throughout. Also, we recall some known results that will be used in later chapters.

2.1 Notation and Definitions

We will use the following notation and definitions. These and other definitions can be found in [24] or [16]. Throughout the dissertation, K will stand for an algebraically closed field.

2.1.1 K-Algebras

Definition 2.1.1. A *K-algebra* is a ring R (with one), together with a ring homomorphism ϕ from K to R such that $\phi(K)$ is contained in the center of R. We will often refer to K-algebras as algebras.

Definition 2.1.2. *Algebra homomorphisms* are ring homomorphisms that restrict to the identity map on K. An algebra automorphism ϕ of a K-algebra R is an *inner automorphism* if there exists an invertible $a \in R$ such that $\phi(r) = a^{-1}ra$ for all $r \in R$.

Definition 2.1.3. Let R be a K-algebra and let a and b be elements of R. The element a is a *factor* of b if $b = ras$ for some elements r and s in R.

Definition 2.1.4. A *regular element* in a K-algebra R is any element that is not a zero-divisor.

Notation 2.1.5. Let R be a K-algebra and $r_1, \cdots, r_n \in R$. Then $K\langle r_1, \cdots r_n \rangle$ will refer to the K-subalgebra of R generated by $r_1, \cdots, r_n$.

Notation 2.1.6. The notation $K[a_1, \cdots, a_n]$ is used for the commutative polynomial algebra in variables $a_1, \cdots, a_n$ over K and the notation $K\{a_1, \cdots, a_n\}$ for the free algebra in the noncommuting variables $a_1, \cdots, a_n$ over K.

Remark 2.1.7. Let R be a K-algebra. If y is an element of R that is algebraically independent over K, we will identify the K-subalgebra of R generated by y, $K\langle y \rangle$, with the K-algebra $K[y]$.

2.1.2 Ideals

Notation 2.1.8. If R is a ring and $r_1, \cdots, r_n \in R$, then $\langle r_1, \cdots, r_n \rangle$ will denote the (two-sided) ideal of R generated by $r_1, \cdots, r_n$.

Definition 2.1.9. A ring R is *prime* if the product of any two nonzero ideals of R is nonzero. An ideal P of R is *prime* if R/P is a prime ring.

Definition 2.1.10. A *minimal prime ideal* in a ring R is any prime ideal of R which does not properly contain any other prime ideals.

Definition 2.1.11. Let R be a subring of a ring S, and let P and Q be prime ideals of S and R, respectively. We say that P *lies over* Q if Q is minimal over $P \cap R$.

Definition 2.1.12. Let R be a ring and ϕ an automorphism of R. Then an ideal I of R is *ϕ-stable* if $\phi(I) \subseteq I$. The ideal I of R is *ϕ-prime* if I is ϕ-stable and $AB \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$ for all ϕ-stable ideals A and B of R. An ideal I of R is called *ϕ-cyclic* if $I = \cap \phi^k(J)$ for a prime ideal J of R with $\phi^m(J) = J$ for some m.

2.1.3 Modules and Primitive Ideals

Notation 2.1.13. All modules will be assumed to be left modules unless otherwise noted.

Definition 2.1.14. A module is *simple* if it has no nonzero proper submodules.

Definition 2.1.15. A left or right module M is called *noetherian* if every submodule of M is finitely generated. A ring R is called *noetherian* if R is noetherian as a left and right module over itself.

Definition 2.1.16. An ideal P of a ring R is *primitive* if P is the annihilator of a simple (left) R-module. A ring R is *primitive* if $\langle 0 \rangle$ is a primitive ideal of R.

Definition 2.1.17. A *representation* of a K-algebra R is a K-algebra homomorphism from R to $End_K(V)$, for some K-vector space V. Note that we can view V as an R-module. Given a representation from a K-algebra R to $End_K(V)$, for a K-vector space V, we say that the representation is *irreducible* if V is simple as an R-module. The representation is called *finite dimensional (infinite dimensional)* if V is finite dimensional (infinite dimensional) over K.

2.1.4 Ore Extensions

Definition 2.1.18. Let R be a ring and ϕ an endomorphism of R. A *left ϕ-derivation* of R is an additive map δ from R to itself such that $\delta(rs) = \phi(r)\delta(s) + \delta(r)s$ for all r and s in R. A *right ϕ-derivation* of R is an additive map δ from R to itself such that $\delta(rs) = \delta(r)\alpha(s) + r\delta(s)$ for all r and s in R. By a *ϕ-derivation* of R, we mean a left ϕ-derivation of R, unless otherwise noted.

Definition 2.1.19. Let R be a ring, ϕ an endomorphism of R, δ a ϕ-derivation of R, and θ an indeterminate. Let S be a ring, containing R as a subring, and a free left R-module with basis of the form $1, \theta, \theta^2, \ldots$ and $\theta r = \phi(r)\theta + \delta(r)$ for all $r \in R$. The ring S is called a *left Ore extension* of R (or *left skew* *polynomial extension* of R) and is denoted $S = R[\theta;\phi,\delta]$. The existence of such constructions is assured by [16, Proposition 1.10]. If ϕ is the identity map on R, then we write $S = R[\theta;\delta]$. If δ is the zero map on R, then we write $S = R[\theta;\phi]$. One can similarly define a *right Ore extension* of R. Unless otherwise noted, an *Ore extension* of R will mean a left Ore extension of R.

Definition 2.1.20. Let R be a ring, θ_i an indeterminate, ϕ_i an endomorphism of R, and δ_i a ϕ_i-derivation, for $1 \le i \le n$. We will refer to constructions of the form

$$R[\theta_1;\phi_1,\delta_1][\theta_2;\phi_2,\delta_2]\cdots[\theta_n;\phi_n,\delta_n]$$

as *iterated Ore extensions* of R.

Definition 2.1.21. Let R be a ring and ϕ an automorphism of R. Let T be a ring and a free left R-module containing R as a subring with an invertible element $\theta \in T$ and basis $1, \theta, \theta^{-1}, \theta^2, \theta^{-2}, \ldots$ such that $\theta r = \phi(r)\theta$ for all $r \in R$. Then T is called a *skew-Laurent extension* of R and is denoted $T = R[\theta, \theta^{-1};\phi]$.

2.1.5 Zariski Topology

Notation 2.1.22. Let R be a ring. The set of prime ideals of R will be denoted *specR*, the set of maximal ideals *maxR*, and the set of primitive ideals *primR*.

Definition 2.1.23. Let R be a ring. The set spec$R$ is a topological space under the *Jacobson (Zariski) topology* when the closed sets of spec$R$ are taken to be $V(I) = \{P \in spec(R) : I \subseteq P\}$ for ideals I of R.

Remark 2.1.24. The Zariski topology is the only topology that we will consider on prime spectra in this dissertation.

2.1.6 Localizations and Quotient Rings

We recall some definitions regarding Ore localizations and symmetric quotient rings. For more background, see [16, Chapter 9] and [24, Chapter 10], respectively.

Definition 2.1.25. A *multiplicative set* in a ring R is any subset $X \subseteq R$ such that $1 \in X$ and X is closed under multiplication. A *right ring of fractions* (or *right Ore quotient* or *right Ore localization*) for R with respect to X is a ring homomorphism $\phi : R \rightarrow S$ such that

1. $\phi(x)$ is a unit of S for all $x \in X$.

2. Each element of S has the form $\phi(r)\phi(x)^{-1}$ for some $r \in R$ and some $x \in X$.

3. $\ker(\phi) = \{r \in R : rx = 0 \text{ for some } x \in X\}$.

A *left ring of fractions* for R with respect to X is defined symmetrically.

Definition 2.1.26. Let X be a multiplicative set in a ring R. Then X satisfies the *right Ore condition* if and only if $rX \cap xR$ is nonempty for all $r \in R$ and $x \in X$. The set X is *right reversible* if and only if whenever there exists an $r \in R$ and $x \in X$ such that $xr = 0$ then there exists an $x' \in X$ such that $rx' = 0$. A *right Ore set* is any multiplicative set satisfying the right Ore condition and a *right denominator set* is any right reversible right Ore set.

Remark 2.1.27. Given a right denominator set X in a ring R, by [16, Chapter 9], there exists a unique right ring of fractions $\phi : R \rightarrow S$ for R with respect to X.

Notation 2.1.28. Given a right denominator set X in a ring R, we will denote the right ring of fractions of R with respect to X by RX^{-1}.

Definition 2.1.29. A collection F of right ideals of a ring R is called a *right localization set* if for any I_1 and I_2 in F and any $\phi \in Hom(I_2, R)$, there exists an I_3 and I_4 in F such that

1. $I_3 \subseteq I_1 \cap I_2$ and

2. $I_4 \subseteq I_2$ and $\phi(I_4) \subseteq I_1$.

Remark 2.1.30. Note that the set of nonzero ideals of a prime ring R is a right localization set for R.

Definition 2.1.31. Given a right localization set F, the *localization* R_F of R with respect to F is defined to be the set $\cup\{Hom(I,R) : I \in F\}$ modulo the equivalence relation given by $\phi_1 \sim \phi_2$ if $\phi_1 : I_1 \to R, \phi_2 : I_2 \to R$ and $\phi_1 = \phi_2$ when restricted to some $I_3 \in F$ with $I_3 \subseteq I_1 \cap I_2$.

Definition 2.1.32. Let F be the set of nonzero ideals of a prime ring R. Then we call R_F the *Martindale right quotient ring* of R.

Definition 2.1.33. The *symmetric quotient ring*, T, of a prime ring R, is the subring of the Martindale right quotient ring, R_F, of R consisting of elements $r \in R_F$ such that $rI \subset R$ for some nonzero ideal I of R depending on r.

2.2 Background

2.2.1 Ore Extensions and Skew-Laurent Extensions

We recall and collect some known results about Ore extensions. Specific references are given and more background can be found in [24, 1.2.1]. Similar results hold for right Ore extensions.

Remark 2.2.1. Let R be a ring, ϕ an endomorphism of R, δ a ϕ-derivation of R, and $S = R[x; \phi, \delta]$. Recall, from [24, 1.2.3], that every element of S can be written in the form $\sum a_i x^i$ for some $a_i \in R$ and this expression is unique. In the iterated Ore extensions that we discuss in Chapters 3 and 4, we will use several applications of this argument.

The next several results make connections betweens ideals in a ring R and ideals in an extension ring of R.

Theorem 2.2.2. *[16, Theorem 1.12 and Theorem 1.17] Let R be a ring, ϕ an automorphism of R, and δ a ϕ-derivation of R. If R is noetherian, then* *the Ore extension $S = R[\theta; \phi, \delta]$ is also noetherian. Also, the skew-Laurent extension $T = R[\theta, \theta^{-1}; \phi]$ is noetherian.*

Theorem 2.2.3. *[24, Proposition 1.2.9] Let R be a prime ring, ϕ an automorphism of R, and δ a ϕ-derivation of R. If $S = R[\theta; \phi, \delta]$ is an Ore extension of R, then S is prime .*

Lemma 2.2.4. *[24, Proposition 1.2.9] Let R be a ring, ϕ an automorphism of R, and $T = R[\theta, \theta^{-1}; \phi]$. If A is a prime ideal of T, then $A \cap R$ is a ϕ-prime ideal of R.*

Theorem 2.2.5. *[16, Corollary 7.28] Let R be a noetherian ring and ϕ an automorphism of R. If $T = R[\theta, \theta^{-1}; \phi]$ and P is a prime ideal of T, then there exists a prime ideal Q of R and a positive integer m such that $P \cap R = Q \cap \phi(Q) \cap \cdots \cap \phi^{m-1}(Q)$ and $\phi^m(Q) = Q$.*

Theorem 2.2.6. *[20, Theorem 1] Let R be a nontrivial ring with identity and ϕ an automorphism of R. Let S be the ring $S = R[\theta, \theta^{-1}; \phi]$. Then the ring S is simple if and only if:*

1. *the only ϕ-ideals of R are $\langle 0 \rangle$ and R; and*

2. *there is no positive integer n for which ϕ^n is inner.*

The next two results relate ϕ-cyclic and ϕ-prime ideals.

Lemma 2.2.7. *[24, Lemma 10.6.11] Let R be a ring and ϕ an automorphism of R. A ϕ-cyclic ideal of R is ϕ-prime.*

Proposition 2.2.8. *[24, Proposition 10.6.14] Let R be a ring and ϕ an automorphism of R. If some power of ϕ is inner and P is a ϕ-prime ideal of R, then P is ϕ-cyclic.*

For the next two results, let A denote a prime ring, C the symmetric quotient of A, μ an automorphism of A, Z the center of $C[t; \mu]$, and D the ring of all central elements in C which are μ-invariant.

Proposition 2.2.9. *[23, Proposition 1.3] There exists an invertible λ in C and an $\ell \geq 0$ such that $Z = D[u]$, where $u = \lambda t^\ell$. Moreover, $Z \neq D$ (i.e., $\ell \neq 0$) if and only if a nonzero power of μ is an inner automorphism of C.*

Theorem 2.2.10. *[23, Theorem 2.10] Suppose that P is a prime ideal of $A[t;\mu]$ such that $P \cap A = \langle 0 \rangle$. Then $P = f(t)C[t;\mu] \cap A[t;\mu]$ where:*

1. *$f(t)$ is either equal to t or*
2. *the center Z of $C[t;\mu]$ is not equal to D and there is an invertible $\beta \in C$ such that $\beta f(t) \in Z = D[u]$ is a monic irreducible polynomial (as a polynomial in u) different from u.*

2.2.2 Quantized Weyl Algebras

For later discussions, we now review some known results about the prime spectra of algebras of the form

$$R = K\{x, y\}/\langle yx - qxy - \lambda \rangle,$$

where K is an algebraically closed field and q and λ are nonzero elements of K. More background can be found in [13, Section 13]. First, note that, by a standard change of variables, we can assume $\lambda = 1$. Algebras of this form are referred to as quantized Weyl algebras and will be denoted $A_1(K, q)$.

Case 1: Suppose that q is not a root of unity. In [13, 2.9], the prime spectra of quantized Weyl algebras over noetherian rings are described when q is not a root of unity. In this case, [13, 2.9] states that

$$spec(R) = \{uR + QR : Q \in spec(K[x]), x \notin Q\},$$

where $u = (\alpha - 1)xz + 1$. Since K is algebraically closed, the nonzero primes of $K[x]$ are of the form $\langle x - \gamma \rangle$ for $\gamma \in K$. If $x = 0$ in R, then $\lambda = 0$, contradicting the choice of nonzero λ. If $x = \gamma$, for γ nonzero, then, by the relations in R, the element $y = 1/(\gamma - \gamma q)$. Hence, the prime ideals of R when q is not a root of unity are as follows.

1. $\langle 0 \rangle$ and

2. $\langle x - \gamma, y - 1/(\gamma - \gamma q) \rangle$, for γ a nonzero element of K.

Case 2: Suppose that q is a primitive ℓth root of unity.

Remark 2.2.11. Recall that if K has positive characteristic than ℓ is invertible modulo the characteristic of K (see [13, 13.6]).

We will use the following result.

Theorem 2.2.12. *[13, Theorem 13.6] Let $S = A_1(T, q)$ where T is a noetherian algebra over a field K and q is a primitive ℓth root of unity in K for some integer $\ell > 1$ which is invertible in K. Set $u = yx - xy$.*

1. *There is a homeomorphism from the set $\{P \in specS : u \in P\}$ onto the set $\{Q \in spec(T(x)) : x \notin Q\}$ given by the rule $P \mapsto P \cap T[x]$.*

2. *There is a homeomorphism from the set $\{P \in specS : u \notin P\}$ onto the set $\{I \in spec(T[x^\ell, y^\ell]) : 1 - (1 - q)^\ell x^\ell y^\ell \notin I\}$ given by the rule $P \mapsto P \cap T[x^\ell, y^\ell]$.*

Thus, in the case when q is a primitive ℓth root of unity, spec(R) is a disjoint union of two subsets homeomorphic respectively to spec$(K[x, x^{-1}])$ and spec$(K[x^\ell, y^\ell, (1 - (1 - q)^\ell x^\ell z^\ell)^{-1}])$. Hence the prime ideals of R when q is a primitive ℓth root of unity are as follows.

1. $\langle 0 \rangle$,

2. $\langle x - \lambda, y - 1/(\lambda - q^{-1}\lambda) \rangle$, where λ is a nonzero element of K, and

3. $\langle x^\ell - \lambda_1, y^\ell - \lambda_2 \rangle$, where λ_1 and λ_2 are nonzero elements of K with $\lambda_1 \lambda_2 \neq 1/(i - q)^\ell$.

2.2.3 Prime Ideals, Primitive Ideals, and Related Results

We now collect some known results that affect our study of prime ideals and irreducible representations. The following result, by Adam Berliner, classifies all of the finite dimensional irreducible representations of

$$R = K\{x, y, z\}/\langle xy, xz - 1, yz\rangle$$

(rings of this type will be discussed in detail in Chapter 4). We thank Berliner for allowing us to use this unpublished result.

Theorem 2.2.13. *[2] Let $X_1, X_2, \cdots, X_m$, for $m \geq 2$, be linear operators on an n dimensional vector space over an algebraically closed field K and let $\alpha_{ij} \in K$ for all i and j. Let α_{ij} also denote the corresponding scalar operator. Suppose that $X_iX_j = \alpha_{ij}$ for all i, j where $1 \leq j < i \leq m$. Then $X_1, X_2, \cdots, X_m$ have a common eigenvector.*

The following classification by Irving will be used in discussing the two nonnoetherian cases.

Theorem 2.2.14. *[18, Theorem 7.1] The following is a complete list (organized by families) of the prime ideals of $K\{x, y\}/\langle xy - 1\rangle$.*

1. $\langle 0\rangle$,
2. $\langle yx - 1\rangle$, *and*
3. $\langle x - \lambda, y - \lambda^{-1}\rangle$, *where λ is a nonzero element of K.*

The following is a complete list (organized by families) of the primitive ideals of $K\{x, y\}/\langle xy - 1\rangle$.

1. $\langle 0\rangle$ *and*
2. $\langle x - \lambda, y - \lambda^{-1}\rangle$, *where λ is a nonzero element of K.*

Remark 2.2.15. Let $R = K\{x, y\}/\langle xy - 1\rangle$. Note that $R/\langle yx - 1\rangle \cong K[x, x^{-1}]$.

Remark 2.2.16. In [11, Section 3], Gerritzen gives a complete classification of the irreducible representations of $K\{x, y\}/\langle xy - 1\rangle$.

2.2.4 Gelfand-Kirillov Dimension

Gelfand-Kirillov dimension, or GK dimension, is one of the important dimension functions in noncommutative algebra. GK dimension can be thought of as a generalization of the Krull dimension of commutative finitely generated rings. Thus, we can think of GK dimension as the dimension of the noncommutative spaces associated to these noncommutative rings. We will discuss some properties of GK dimension, but omit the details. Please see [22] for more background.

More specifically, GK dimension is a measure of the rate of growth of an algebra in terms of any generating set. For example, finite dimensional algebras have GK dimension zero and, if A is a finitely generated commutative domain, then the GK dimension of A is equal to the transcendence degree of A over K. Thus the GK dimension of A is equal to the number of indeterminants in the largest possible polynomial algebra contained in A. If KG is a group algebra, then the GK dimension of KG measures the rate of growth of the group G. Also, the GK dimension of a free algebra on two generators will be infinity. Thus, we can also think of GK dimension as a measure of how far an algebra is from being finite dimensional.

For the work in this dissertation, note that algebras of the form

$$A = K\{x_1, x_2, \cdots, x_n\}/\langle x_i x_j - \alpha_{ij} x_j x_i - \beta_{ij}, i < j\rangle$$

will have GK dimension less than or equal to n. If A is nontrivial and if $x_i \neq \alpha x_j$, for any $1 \leq i, j \leq n$ with $i \neq j$, and for any $\alpha \in K$, then A will have the same standard filtration as a commutative polynomial algebra in n indeterminants. Hence, the GK dimension of A is n (see [22, Chapter 3]).

CHAPTER 3

EXAMPLES OF ELEMENTARY CASES

Before discussing the main results of this dissertation, we carefully consider three illustrative examples of algebras of the form

$$K\{x,y,z\}/\langle xy-\alpha_1 yx-\beta_1, xz-\alpha_2 zx-\beta_2, yz-\alpha_3 zy-\beta_3\rangle$$

where the prime ideal theory is elementary or reduces to known work. The three examples are meant to demonstrate some of the possibilities for prime spectra of algebras of this form. Appendix A contains a description of the prime spectra of the 64 possible cases (or appropriate references).

3.1 Example One

In this section, we consider an example where the study of the prime ideal theory reduces to known cases. We consider algebras of the form

$$S_R = K\{x,y,z\}/\langle xy, xz, yz-\alpha_3 zy\rangle,$$

where α_3 is nonzero. Let x, y, and z stand for their images in S_R. Let ϕ_1 be the K-algebra endomorphism of $K[x]$ sending x to zero. Also, let ϕ_2 be the

K-algebra endomorphism of the right Ore extension $K[x][y;\phi_1]$ sending x to zero and y to $\alpha_3 y$. Note that $S_R = K[x][y;\phi_1][z;\phi_2]$ is an iterated right Ore extension of $K[x]$. Thus, by Remark 2.2.1, elements of S_R can be written in the form $\sum_{\ell=0}^{m} \lambda_\ell z^{i_\ell} y^{j_\ell} x^{k_\ell}$, where $\lambda_\ell \in K$ are nonzero and the (i_ℓ, j_ℓ, k_ℓ) are distinct for distinct ℓ.

Note that $r_1 x \lambda z^i y^j x^k y r_2 = 0$ for any $\lambda \in K$ and any r_1 and r_2 in S_R. Thus, $\langle x \rangle \langle y \rangle = 0$. Hence, every prime ideal of S_R contains x or y. Similarly, every prime ideal of S_R contains x or z. Thus the study of the prime ideal theory of S_R reduces to the known prime ideal theory of $K\{y,z\}/\langle yz - \alpha_3 zy \rangle$ (see [17, Section 8]) and $K[x]$.

3.2 Example Two

In this section, we consider a seemingly interesting example that is actually trivial. We consider algebras of the form

$$K\{x,y,z\}/\langle xy - \alpha_1 yx - \beta_1, xz - \alpha_2 zx - \beta_2, yz - 0zy - \beta_3 \rangle$$

$$= K\{x,y,z\}/\langle xy - \alpha_1 yx - \beta_1, xz - \alpha_2 zx - \beta_2, yz - \beta_3 \rangle,$$

where α_1, α_2, β_1, β_2, and β_3 are nonzero elements of K. Note that, by changes of variables, we may assume that β_2 and β_3 are equal to one. Thus, for the remainder of this section, let

$$S_D = K\{x,y,z\}/\langle xy - \alpha_1 yx - \beta_1, xz - \alpha_2 zx - 1, yz - 1 \rangle.$$

Let x, y, and z also stand for their images in S_D.

Proposition 3.2.1. *The algebra S_D is trivial or isomorphic to K.*

Proof. The equation $xy - \alpha_1 yx = \beta_1$ holds in S_D. Multiplying through by z on the right yields $xyz - \alpha_1 yxz = \beta_1 z$. Simplifying using the other relations implies that the equation $(1 - \alpha_1\alpha_2)x - \alpha_1 y - \beta_1 z = 0$ holds in S_D. Thus, it is natural to divide our investigation into two cases based on whether or not $\alpha_1\alpha_2 = 1$.

Case 1: Suppose $\alpha_1\alpha_2 = 1$. Then $\alpha_1 y + \beta_1 z = 0$. Hence $y = -\beta_1/\alpha_1 z$. Then, again using the relation $yz = 1$, the equation

$$z = y^{-1} = (-\alpha_1/\beta_1)^{1/2}$$

holds in S_D. The relation $xy - \alpha_1 yx = \beta_1$ implies that

$$x = \beta_1/((-\alpha_1/\beta_1)^{1/2} - \alpha_1(-\alpha_1/\beta_1)^{1/2}).$$

Note that

$$(-\alpha_1/\beta_1)^{1/2} - \alpha_1(-\alpha_1/\beta_1)^{1/2} \neq 0$$

since $\beta_1 \neq 0$. Hence, if $\alpha_1\alpha_2 = 1$, then S_D is isomorphic to K.

Case 2: Suppose $\alpha_1\alpha_2 \neq 1$. Since $(1 - \alpha_1\alpha_2)x - \alpha_1 y - \beta_1 z = 0$, the equation

$$x = (\alpha_1 y + \beta_1 z)/(1 - \alpha_1\alpha_2)$$

holds in S_D. Replace x in the relation $xy - \alpha_1 yx = \beta_1$ with

$$x = (\alpha_1 y + \beta_1 z)/(1 - \alpha_1\alpha_2).$$

Further manipulation of the variables implies that the equality

$$zy = 1 - \alpha_1\alpha_2 + \alpha_1 - (\alpha_1 - \alpha_1^2)/(\beta_1)y^2$$

holds in S_D. Multiplying through by z on the left yields

$$(\alpha_1\alpha_2 - \alpha_1)z = (\alpha_1^2 - \alpha_1)/(\beta_1)y.$$

Note that $\alpha_1\alpha_2 - \alpha_1 = 0$ if and only if $\alpha_2 = 1$, and that $\alpha_1^2 - \alpha_1 = 0$ if and only if $\alpha_1 = 1$. This leads us naturally into the following four subcases.

Subcase 1: Suppose $\alpha_1 = 1$ and $\alpha_2 = 1$. This contradicts the assumption that $\alpha_1\alpha_2 \neq 1$.

Subcase 2: Suppose $\alpha_1 \neq 1$ and $\alpha_2 \neq 1$. Then the above relations imply that $y = cz$ for some constant $c \in K$. Thus, as in Case 1, x, y, and z are all equal to constants and hence S_D is isomorphic to K.

Subcase 3: Suppose $\alpha_1 \neq 1$ and $\alpha_2 = 1$. By the above relation, this implies that $y = 0$, contradicting that $yz = 1$. Hence, $S_D = 0$.

Subcase 4: Suppose $\alpha_1 = 1$ and $\alpha_2 \neq 1$. As in Subcase 3, this implies that $z = 0$, contradicting that $yz = 1$. Hence, $S_D = 0$. □

3.3 Example Three

In this section, we consider an example of a noetherian case. Much work has been done on noetherian Ore extensions of polynomial algebras (see, e.g. [13]). We now carefully work out such an example.

We wish to classify the prime and primitive ideals of

$$S = K\{x, y, z\}/\langle xy - \alpha_1 yx - 0, xz - \alpha_2 zx - \beta_2, yz - \alpha_3 zy - 0\rangle$$

$$= K\{x, y, z\}/\langle xy - \alpha_1 yx, xz - \alpha_2 zx - \beta_2, yz - \alpha_3 zy\rangle,$$

where α_1, α_2, α_3, and β_2 are nonzero elements of K. We will use the results about quantized Weyl algebras discussed in Chapter 1 throughout this section.

Proposition 3.3.1. *If $\alpha_1 \neq \alpha_3$, then S is isomorphic to $K\{x, z\}/\langle xz - \alpha_2 zx - \beta_2\rangle$.*

Proof. The proof is shown for $\beta_2 = 1$ and follows analogously for arbitrary nonzero values of β_2. In S, the relation $xy + \alpha_1 yx = 0$ holds. Multiplying both sides of this equation by z on the right, using the other relations in S, and simplifying, the equation $\alpha_1^{-1}\alpha_3 xzy + \alpha_2 yzx = y$ must also hold in S. Note that $xzy + \alpha_2 zxy = y$. Hence

$$\alpha_1^{-1}\alpha_3 xzy + \alpha_2 yzx = xzy + \alpha_2 zxy$$

holds in S and, thus, so does

$$(\alpha_1^{-1}\alpha_3 - 1)xzy = -\alpha_2(yzx - zxy) = -\alpha_2(\alpha_1^{-1}\alpha_3 zxy - zxy).$$

By the relations in S,

$$(\alpha_1^{-1}\alpha_3 - 1)(xz + \alpha_2 zx)y = 0.$$

Thus $\alpha_1^{-1}\alpha_3 - 1 = 0$ or $y = 0$. □

For the remainder of this chapter, let

$$S_N = K\{x, y, z\}/\langle xy - \alpha yx, xz - \alpha_2 zx - 1, yz - \alpha zy\rangle.$$

By Proposition 3.3.1 and a change of variable, a classification of the prime ideals of S_N will complete a classification of the prime ideals of S. Let

$$R = K\{x, z\}/\langle xz - \alpha_2 zx - 1\rangle,$$

where α_2 is a nonzero element of K. Note that $S_N = R[y; \sigma]$, where σ is the automorphism of R sending x to $\alpha^{-1}x$, z to αz, and a to a for all elements $a \in K$. Also, note that S_N is a prime, noetherian domain (see Theorem 2.2.2 and Theorem 2.2.3).

The primes of S_N fall naturally into two categories. The first category is those containing y, which are in one-to-one correspondence with the nonzero primes of R. The second category is those prime ideals not containing y, which are in one-to-one correspondence with the prime ideals of $T = R[y, y^{-1}; \sigma]$. Thus, a classification of the prime ideals of T will complete our classification of the prime ideals of S_N. Note that T is also a prime, noetherian domain (see Theorem 2.2.2 and Theorem 2.2.3).

3.3.1 Scalar Not a Root of Unity Case

In this case, the only finite σ-orbit of prime ideals of R is $A_1 = \{\langle 0\rangle\}$, regardless of whether or not α_2 is a root of unity. Thus, by Theorem 2.2.5, if P is a prime ideal of T then $P \cap R = \langle 0\rangle$.

Proposition 3.3.2. *The following is a complete list of the prime ideals of S_N (organized by families) when α is not a root of unity.*

1. $\langle 0\rangle$,
2. $\langle y\rangle$,
3. $\langle x - \lambda_1, z - \lambda_2, y\rangle$, *where λ_1 and λ_2 are nonzero elements of K, such that $\lambda_2 = 1/(\lambda_1 - \alpha_2\lambda_1)$.*

Proof. Throughout the proof, we use a method adapted from [13, 2.3] and omit some details. We wish to classify the prime ideals of S_N not containing

y that intersect R at $\langle 0 \rangle$. Let C be the set of regular elements of R. Consider the localizations $A = RC^{-1}$ and $B = RC^{-1}[y;\sigma]$ (see [12, 1.3]). Note that A is σ-simple.

Let Y be the set of prime ideals of S_N which lie over minimal primes of R. Since R is prime, Y is in fact the set of prime ideals of S_N which lie over $\langle 0 \rangle$ in R. Note that $\langle 0 \rangle$ is the unique prime ideal of R that is disjoint from C. Hence, Y is equal to the set of prime ideals of S_N which are disjoint from C. Therefore, $Y = \{\gamma^{-1}(P) : P \in \text{spec}B\ \}$, where γ is the natural embedding of S_N into B.

Hence our goal is to describe the prime ideals of B not containing y. These primes will be in one-to-one correspondence with primes of

$$E = RC^{-1}[y, y^{-1}; \sigma].$$

Thus, to finish classifying the prime ideals of S_N in this case, we only need to classify the prime ideals of E.

Since α is not a root of unity, no power of σ will be an inner automorphism. Also, as noted earlier, the only σ-ideals of R are $\langle 0 \rangle$ and R. Thus, E is simple, by Theorem 2.2.6. Therefore, the only prime ideal of S_N not containing y which intersects R at $\langle 0 \rangle$ is the ideal generated by 0 in S_N. □

3.3.2 Scalar Root of Unity Case

For the remainder of this chapter, let α be a primitive ℓth root of unity. It is now necessary to divide our study into cases based on whether or not α_2 is a root of unity.

Subcase 1: Suppose that α_2 is not a root of unity. In this case there will be two finite σ-orbits of prime ideals of R. Let $A_1 = \langle 0 \rangle$ be the finite σ-orbit of $\langle 0 \rangle$ and

$$A_2 = \{\langle x - \lambda_1, z - \lambda_2 \rangle, \langle x - \lambda_1\alpha, z - \lambda_2\alpha^{-1} \rangle, \cdots, \langle x - \lambda_1\alpha^{\ell-1}, z - \lambda_2\alpha^{-\ell+1} \rangle\}$$

be the finite σ-orbit of $\langle x - \lambda_1, z - \lambda_2 \rangle$. We need a few preliminary results.

Lemma 3.3.3. *Let I be a proper ideal of S_N containing $x - \lambda$, for some nonzero $\lambda \in K$. Then $y \in I$ or $\alpha = 1$.*

Proof. Let $S' = S_N/I$ and let x, y, and z also stand for their images in S'. In S', the equations $x = \lambda$ and $xy - \alpha yx = 0$ hold. These equations imply that $\lambda_1 y - \alpha\lambda_1 y = 0$ in S'. Hence $y = 0$ in S' or $\alpha = 1$. □

Recall that $T = R[y, y^{-1}; \sigma]$.

Lemma 3.3.4. *As ideals of T,*

$$\langle x - \lambda_1, z - \lambda_2 \rangle = \langle x - \lambda_1\alpha^r, z - \lambda_2\alpha^{-r} \rangle,$$

for any nonzero λ_1 and λ_2 in K and any nonnegative integer r.

Proof. The proof is shown for $r = 1$ and follows analogously for arbitrary values of r. Let $I = \langle x - \lambda_1\alpha, z - \lambda_2\alpha^{-1} \rangle$. Then

$$(x - \lambda_1\alpha)y = \alpha yx - \lambda_1\alpha y = \alpha y(x - \lambda_1) \in I.$$

Hence,

$$\alpha^{-1}y^{-1}(x - \lambda_1\alpha)y = x - \lambda_1 \in I.$$

Similarly, if an ideal I of T contains $z - \lambda_2\alpha^{-1}$, then I contains $z - \lambda_2$.

To obtain the reverse inclusion, let $J = \langle x - \lambda_1, z - \lambda_2 \rangle$. Then

$$y(x - \lambda_1)y^{-1} = x - \lambda_1\alpha \in J.$$

Similarly $z - \lambda_2\alpha^{-1} \in J$. Thus $I = J$. □

Corollary 3.3.5. *In T, the ideal $\langle x - \lambda_1, z - \lambda_2 \rangle \cap \langle x - \lambda_1\alpha, z - \lambda_2\alpha^{-1} \rangle \cap \cdots \cap \langle x - \lambda_1\alpha^{\ell-1}, z - \lambda_2\alpha^{-\ell+1} \rangle = \langle x - \lambda_1, z - \lambda_2 \rangle$.*

These results allow us to complete the classification.

Proposition 3.3.6. *The following is a complete list of the prime ideals of S_N (organized by families) when α is a primitive ℓth root of unity not equal to one and α_2 is not a root of unity.*

1. $\langle 0 \rangle$,

2. $\langle y \rangle$,

3. $\langle x-\lambda_1, z-\lambda_2, y\rangle$, *where* λ_1 *and* λ_2 *are nonzero elements of* K, *such that* $\lambda_2 = 1/(\lambda_1 - \alpha_2\lambda_1)$.

4. $\langle y^\ell - \lambda \rangle$, *where* $\lambda \in K$.

If $\alpha = 1$ *(ie.,* $\ell = 1$*) and* α_2 *is not a root of unity, then the following prime ideals, in addition to the above list of prime ideals, is a complete list of the prime ideals of* S_N.

1. $\langle x-\lambda_1, z-\lambda_2, y-\lambda_3\rangle$, *where* λ_1, λ_2, *and* λ_3 *are nonzero elements of* K, *such that* $\lambda_2 = 1/(\lambda_1 - \alpha_2\lambda_1)$, *and*

2. $\langle x-\lambda_1, z-\lambda_2\rangle$, *where* λ_1 *and* λ_2 *are nonzero elements of* K, *such that* $\lambda_2 = 1/(\lambda_1 - \alpha_2\lambda_1)$.

Proof. First, we classify the prime ideals P of T such that

$$P \cap R = \langle x-\lambda_1, z-\lambda_2\rangle \cap \langle x-\lambda_1\alpha, z-\lambda_2\alpha^{-1}\rangle \cap \cdots \cap \langle x-\lambda_1\alpha^{\ell-1}, z-\lambda_2\alpha^{-\ell+1}\rangle.$$

By Corollary 3.3.5 and Lemma 3.3.3, $\langle x-\lambda_1, z-\lambda_2\rangle \subseteq P$, and thus $y \in P$ or $\alpha = 1$.

Next, we classify the prime ideals of S_N that intersect R at $\langle 0 \rangle$. Let Y denote the set of prime ideals P of S_N such that $P \cap R = \langle 0 \rangle$. Throughout we follow [13, 2.3] and omit some details. Let C denote the set of regular elements of R. Then C is a denominator set for R and S_N. Let $A = RC^{-1}$ and $B = S_NC^{-1} = A[y;\sigma]$. In order to classify Y, it suffices to describe spec$B$. Let $E = A[y, y^{-1}; \sigma]$. Then, to describe spec$B$ it suffices to describe spec$E$. There are mutually inverse homeomorphisms between spec$E$ and spec$K[y^\ell, y^{-\ell}]$. Hence the nonzero prime ideals of S_N that intersect R at $\langle 0 \rangle$ are of the form $\langle y^\ell - \lambda \rangle$ for $\lambda \in K$. □

Subcase 2: Suppose that α_2 is a primitive tth root of unity. In this case, recall that the prime ideals of R are as follows.

1. $\langle 0 \rangle$,
2. $\langle x - \lambda_1, z - \lambda_2 \rangle$, where λ_1 and λ_2 are nonzero elements of K such that $\lambda_2 = 1/(\lambda_1 - \alpha_2\lambda_1)$, and
3. $\langle x^t - \lambda_1, z^t - \lambda_2 \rangle$, where λ_1 and λ_2 are elements of K with $\lambda_1\lambda_2(1 - \alpha_2)^t$ not equal to 1.

Remark 3.3.7. Note that if $x^t y = \alpha^t y x^t$ and $x^t = \lambda_1$, for some $\lambda_1 \neq 0 \in K$, hold in S_N, then $\alpha^t = 1$ and hence t is a multiple of ℓ.

Proposition 3.3.8. *The following is a complete list of the prime ideals of S_N (organized by families) when α is a primitive ℓth root of unity not equal to one and α_2 is a primitive tth root of unity.*

1. $\langle 0 \rangle$,
2. $\langle y \rangle$,
3. $\langle x - \lambda_1, z - \lambda_2, y \rangle$, *where λ_1 and λ_2 are nonzero elements of K, such that* $\lambda_2 = 1/(\lambda_1 - \alpha_2\lambda_1)$,
4. $\langle x^t - \lambda_1, z^t - \lambda_2, y \rangle$, *where λ_1 and λ_2 are elements of K with $\lambda_1\lambda_2(1 - \alpha_2)^t$ not equal to one,*
5. $\langle x^t - \lambda_1, z^t - \lambda_2 \rangle$, *where λ_1 and λ_2 are elements of K with $\lambda_1\lambda_2(1 - \alpha_2)^t$ not equal to one,*
6. $\langle y^\ell - \lambda \rangle$, *where $\lambda \in K$, and*
7. $\langle x^t - \lambda_1, z^t - \lambda_2, y^\ell - \lambda \rangle$, *where λ_1 and λ_2 are elements of K with $\lambda_1\lambda_2(1 - \alpha_2)^t$ not equal to one and $\lambda \in K$.*

If $\alpha = 1$ (ie., $\ell = 1$) and α_2 is a primitive tth root of unity, then the following prime ideals, in addition to the above list of prime ideals, is a complete list of the prime ideals of S_N.

1. *$\langle x-\lambda_1, z-\lambda_2, y-\lambda_3\rangle$, where λ_1, λ_2, and λ_3 are nonzero elements of K, such that $\lambda_2 = 1/(\lambda_1 - \alpha_2\lambda_1)$, and*

2. *$\langle x-\lambda_1, z-\lambda_2\rangle$, where λ_1 and λ_2 are nonzero elements of K, such that $\lambda_2 = 1/(\lambda_1 - \alpha_2\lambda_1)$.*

Proof. The primes of S_N that intersect R at $\langle 0\rangle$ or

$$\langle x-\lambda_1, z-\lambda_2\rangle \cap \langle x-\lambda_1\alpha, z-\lambda_2\alpha^{-1}\rangle \cap \cdots \cap \langle x-\lambda_1\alpha^{\ell-1}, z-\lambda_2\alpha^{-\ell+1}\rangle,$$

will be the same as in the previous case.

The primes P of T such that

$$P\cap R = \langle x^t-\lambda_1, z^t-\lambda_2\rangle \cap \cdots \cap \langle x^t-\lambda_1\alpha^{r-1}, z^t-\lambda_2\alpha^{-r+1}\rangle,$$

where r is the least common multiple of ℓ and t still need to be classified. As previously, in T,

$$\langle x^t-\lambda_1, z^t-\lambda_2\rangle \cap \cdots \cap \langle x^t-\lambda_1\alpha^{r-1}, z^t-\lambda_2\alpha^{-r+1}\rangle = \langle x^t-\lambda_1, z^t-\lambda_2\rangle,$$

for any positive integer r. Hence P contains $\langle x^t-\lambda_1, z^t-\lambda_2\rangle$. Also, by Remark 3.3.7, the least common multiple of ℓ and t is t. Hence the σ-orbit of $\langle x^t-\lambda_1, z^t-\lambda_2\rangle$ is $\langle x^t-\lambda_1, z^t-\lambda_2\rangle$.

Following [13, 2.3], let $R' = R/\langle x^t-\lambda_1, z^t-\lambda_2\rangle$, $S' = R'[y;\sigma]$, C be the set of regular elements of R', $A = RC^{-1}$, $B = A[y;\sigma]$, and $E = A[y,y^{-1};\sigma]$. By reasoning as before, the nonzero primes of S' intersecting R' at zero will be of the form $\langle y^\ell - \lambda\rangle$, where $\lambda \in K$. □

CHAPTER 4

NONNOETHERIAN CASE ONE

In this chapter, we discuss the prime and primitive ideals of

$$K\{x,y,z\}/\langle xy - 0yx - \beta_1, xz - \alpha_2 zx - 0, yz - \alpha_3 zy - 0\rangle$$

$$= K\{x,y,z\}/\langle xy - \beta_1, xz - \alpha_2 zx, yz - \alpha_3 zy\rangle,$$

where β_1, α_2, and α_3 are nonzero elements of K. Note that by a change of variable, we may assume that $\beta_1 = 1$. Throughout this chapter, let

$$S = K\{x,y,z\}/\langle xy - 1, xz - \alpha_2 zx, yz - \alpha_3 zy\rangle,$$

let $R = K\{x,y\}/\langle xy-1\rangle$ and let $T = R[z, z^{-1}; \sigma]$, where σ is the automorphism of R sending the element x to $\alpha_2 x$, the element y to $\alpha_2^{-1} y$, and elements of K to themselves. When we refer to x, y, or z, we will be referring to their images in R, S, or T. We begin by discussing some preliminary results about these three algebras. We then divide the study into two cases based on whether or not α_2 is a root of unity and classify the prime ideals of S, using the preliminary results. Finally, we discuss the primitive ideals of S, leaving a complete classification open for future work.

4.1 Notation and Preliminary Results

4.1.1 R Preliminaries

Proposition 4.1.1. *The set $\{y^j x^k : j$ and k are integers $\}$ is a K-linear basis for R.*

Proof. Note that $R = K[y][x; \sigma, \delta]$ is an Ore extension of $K[y]$ with $\sigma(y) = 0$ and $\delta(y) = 1$. By Remark 2.2.1, the set $\{y^j x^k : j$ and k are nonnegative integers$\}$ is a K-linear basis for R. □

Thus, when we write a nonzero element of R in the form

$$\sum_\ell \lambda_\ell y^{j_\ell} x^{k_\ell},$$

for $\lambda_\ell \in K$, we will assume that the λ_ℓ are nonzero and that the (j_ℓ, k_ℓ) are distinct for distinct ℓ.

Proposition 4.1.2. *Let a be an element of R. Then $ax = 0$ or $ya = 0$ if and only if $a = 0$. The same proposition holds if $a \in S$.*

Proof. Suppose that $ax = 0$ holds in R. Multiplying through by y on the right yields $axy = 0$. Since $xy = 1$ in R, the element $a = 0$. Proceed similarly if $ya = 0$ or $a \in S$. □

Proposition 4.1.3. *If $ay \in \langle yx - 1\rangle$ or $xa \in \langle yx - 1\rangle$, for $a \in R$, then $a \in \langle yx - 1\rangle$.*

Proof. If $a = 0$, the proposition follows. Suppose that $a = \sum_\ell \lambda_\ell y^{j_\ell} x^{k_\ell}$ is a nonzero element of R with $ay \in \langle yx - 1\rangle$. Note that

$$ay = (\sum_\ell \lambda_\ell y^{j_\ell} x^{k_\ell})y = \sum_{\ell : k_\ell = 0} \lambda_\ell y^{j_\ell + 1} + \sum_{\ell : k_\ell > 0} \lambda_\ell y^{j_\ell} x^{k_\ell - 1}.$$

Hence, letting x and y also stand for their images in $R/\langle yx - 1\rangle$,

$$\sum_{\ell : k_\ell = 0} \lambda_\ell y^{j_\ell + 1} + \sum_{\ell : k_\ell > 0} \lambda_\ell y^{j_\ell} x^{k_\ell - 1}$$

$$= \sum_{\ell:k_\ell=0} \lambda_\ell y^{j_\ell+1} + \sum_{\ell:k_\ell>0, j_\ell\geq k_\ell-1} \lambda_\ell y^{j_\ell-k_\ell+1} + \sum_{\ell:k_\ell>0, k_\ell-1>j_\ell} \lambda_\ell x^{k_\ell-1-j_\ell} = 0,$$

in $R/\langle yx-1\rangle$. Note that the vector space generated by powers of x is orthogonal to the vector space generated by powers of y. Thus, a cannot have any summands with $k_\ell > 0$ and $k_\ell - 1 > j_\ell$ and the equation

$$\sum_{\ell:k_\ell=0} \lambda_\ell y^{j_\ell+1} = -\Big(\sum_{\ell:k_\ell>0, j_\ell\geq k_\ell-1} \lambda_\ell y^{j_\ell-k_\ell+1} \Big)$$

holds in $R/\langle yx-1\rangle$. Hence, if there exists an r such that $k_r = 0$, then there exists an s such that $k_s > 0$, $j_s \geq k_s - 1$, $\lambda_r = \lambda_s$, and $j_r = j_s - k_s$. Similarly for each s such that $k_s > 0$, there exists an r such that $k_r = 0$, $\lambda_r = \lambda_s$, and $j_r = j_s - k_s$. Thus,

$$a = \sum_\ell \lambda_\ell y^{j_\ell-k_\ell} - \lambda_\ell y^{j_\ell} x^{k_\ell} = \sum_\ell \lambda_\ell y^{j_\ell-k_\ell}(1 - y^{k_\ell} x^{k_\ell}),$$

which equals 0 in $R/\langle yx-1\rangle$. Hence, $a \in \langle yx-1\rangle$. Proceed similarly for $xa \in \langle yx-1\rangle$. □

Corollary 4.1.4. *Let a be a nonzero element of R. If $x^r a = 0$ or $ay^r = 0$ for any nonnegative integer r, then $a \in \langle yx-1\rangle$.*

Proof. We proceed by induction on r. If $r = 1$, the corollary holds by Proposition 4.1.3. Let n be an integer greater than 1. Suppose the result is true for $r < n$. The equation

$$ay^n = (ay)y^{n-1} = 0$$

implies that $ay \in \langle yx-1\rangle$ by the induction assumption. Proposition 4.1.3 implies that $a \in \langle yx-1\rangle$. Proceed similarly to prove that if $x^r a = 0$, then $a \in \langle yx-1\rangle$. □

Proposition 4.1.5. *Suppose that I is a nonzero ideal of R and that I is not contained in $\langle yx-1\rangle$. Then I contains a nonzero element that does not have x as a factor of any summand and a nonzero element that does not have y as a factor of any summand.*

Proof. Let $a = \sum_\ell \lambda_\ell y^{j_\ell} x^{k_\ell}$, for $\lambda_\ell \in K$, be a nonzero element of I that is not an element of $\langle yx - 1 \rangle$. Let γ_1 be the highest exponent of x appearing in any summand of a. By Corollary 4.1.4, ay^{γ_1} is nonzero. Also

$$ay^{\gamma_1} = \sum_\ell \lambda_\ell y^{j_\ell} x^{k_\ell} y^{\gamma_1} = \sum_\ell \lambda_\ell y^{j_\ell + \gamma_1 - k_\ell}.$$

Hence, ay^{γ_1} is a nonzero element of I that has no summands with x as a factor.

Let γ_2 be the highest exponent of y appearing in any summand of a. Then $x^{\gamma_2} a$ is an element of I without y as a factor of any summand. By Corollary 4.1.4, $x^{\gamma_2} a$ is nonzero. □

Lemma 4.1.6. *Suppose that I is a nonzero ideal of R that is not contained in $\langle yx - 1 \rangle$. Then I contains a nonzero element with a nonzero constant term and with x not a factor of any summand. Similarly, I contains a nonzero element with a nonzero constant term and with y not a factor of any summand.*

Proof. By Proposition 4.1.5, I contains a nonzero element $a = \sum_r \lambda_r y^r$ without x as a factor of any summand. Suppose that a, written in its unique form, has zero constant term. Let γ be the smallest exponent of y appearing in any nonzero summand of a. Then $x^\gamma a$ is a nonzero element of I that has nonzero constant term λ_γ and x is not a factor of any summand. Proceed similarly to prove that I contains a nonzero element with y not a factor of any summand and a nonzero constant term. □

Proposition 4.1.7. *Suppose that I is a nonzero ideal of R that is contained in $\langle yx - 1 \rangle$. Then I contains nonzero elements of the form $\sum_m \lambda_m y^{j_m}(yx - 1)$ and $\sum_n \lambda_n (yx - 1) x^{k_n}$, where λ_m and λ_n are nonzero elements of K.*

Proof. Let a be a nonzero element of I. Since $a \in \langle yx - 1 \rangle$, the element a is a finite sum of elements of the form

$$\lambda y^{j_r} x^{k_r} (yx - 1) y^{j_s} x^{k_s} = \lambda y^{j_r} x^{k_r} yxy^{j_s} x^{k_s} - \lambda y^{j_r} x^{k_r} y^{j_s} x^{k_s}.$$

Let $\lambda y^{j_m} x^{k_m} yxy^{j_n} x^{k_n} - \lambda y^{j_m} x^{k_m} y^{j_n} x^{k_n}$ be a summand of a. We now divide the proof into cases depending on whether j_m, j_n, k_m, and k_n are zero or nonzero.

Case 1: Suppose that $k_m > 0$. Then

$$\lambda y^{j_m} x^{k_m} yxy^{j_n} x^{k_n} - \lambda y^{j_m} x^{k_m} y^{j_n} x^{k_n} = \lambda y^{j_m} x^{k_m} y^{j_n} x^{k_n} - \lambda y^{j_m} x^{k_m} y^{j_n} x^{k_n} = 0.$$

Case 2: Suppose that $k_m = 0$ and $j_n > 0$. Then

$$\lambda y^{j_m} x^{k_m} yxy^{j_n} x^{k_n} - \lambda y^{j_m} x^{k_m} y^{j_n} x^{k_n} = \lambda y^{j_m+1} y^{j_n-1} x^{k_n} - \lambda y^{j_m+j_n} x^{k_n}$$

$$= \lambda y^{j_m+j_n} x^{k_n} - \lambda y^{j_m+j_n} x^{k_n} = 0.$$

Case 3: Suppose that $k_m = 0$ and $j_n = 0$. Then

$$\lambda y^{j_m} x^{k_m} yxy^{j_n} x^{k_n} - \lambda y^{j_m} x^{k_m} y^{j_n} x^{k_n} = \lambda y^{j_m+1} x^{k_n+1} - \lambda y^{j_m} x^{k_n}$$

$$= \lambda y^{j_m}(yx - 1)x^{k_n}.$$

Thus, if a is a nonzero element of an ideal I contained in $\langle yx - 1 \rangle$, then the only nonzero summands of a are of the form $\lambda y^{j_m+1} x^{k_n+1} - \lambda y^{j_m} x^{k_n}$. Hence a is of the form

$$a = \sum_m \lambda_m y^{j_m}(yx - 1)x^{k_m}.$$

Next, let γ_1 be the highest x-degree and γ_2 the highest y-degree of any summand of a. Then

$$ay^{\gamma_1} = \sum_m \lambda_m y^{j_m}(yx - 1)x^{k_m} y^{\gamma_1} = \sum_m \lambda_m y^{j_m}(yx - 1)y^{\gamma_1 - k_m}$$

$$= \Big(\sum_{m:k_m<\gamma_1} \lambda_m y^{j_m}(yxy^{\gamma_1-k_m} - y^{\gamma_1-k_m}\Big) + \Big(\sum_{m:k_m=\gamma_1} \lambda_m y^{j_m}(yx - 1)\Big)$$

$$= \Big(\sum_{m:k_m<\gamma_1} \lambda_m y^{j_m}(y^{\gamma_1-k_m} - y^{\gamma_1-k_m}\Big) + \Big(\sum_{m:k_m=\gamma_1} \lambda_m y^{j_m}(yx - 1)\Big)$$

$$= \sum_{m:k_m=\gamma_1} \lambda_m y^{j_m}(yx - 1),$$

which is a nonzero element contained in I. Similarly,

$$x^{\gamma_2} a = \sum_{m:j_m=\gamma_2} \lambda_m (yx - 1)x^{k_m}$$

is a nonzero element of I. □

Proposition 4.1.8. *Suppose that I is an ideal of R that is contained in $\langle yx - 1\rangle$. Then $I = \langle yx-1\rangle$.*

Proof. By Proposition 4.1.7, I contains a nonzero element a of the form

$$\sum_m \lambda_m y^{j_m}(yx-1).$$

Let γ be the highest exponent of y appearing in any summand of a. Note that there will be only one summand of a with $j_m = \gamma$. Then

$$x^\gamma\Big(\sum_m \lambda_m y^{j_m}(yx-1)\Big) = \sum_m \lambda_m x^\gamma y^{j_m}(yx-1) = \lambda(yx-1) \in I,$$

for some $\lambda \in K$. Thus, $I = \langle yx-1\rangle$, as desired. □

4.1.2 S Preliminaries

Recall that $S = K\{x,y,z\}/\langle xy-1, xz-\alpha_2 zx, yz-\alpha_3 zy\rangle$.

Lemma 4.1.9. *If $\alpha_3 \neq \alpha_2^{-1}$, then S is isomorphic to R.*

Proof. In S, the equation $yz = \alpha_3 zy$ holds. Multiplying this equation by x on both sides yields $xyz = \alpha_3 xzy$. The equalities $xy = 1$ and $xz = \alpha_2 zx$ imply that $z = \alpha_3\alpha_2 z$. Hence, $\alpha_3 = \alpha_2^{-1}$ or $z = 0$. □

Thus, for the remainder of this chapter, let $\alpha = \alpha_2$ and

$$S = K\{x,y,z\}/\langle xy-1, xz-\alpha zx, yz-\alpha^{-1}zy\rangle.$$

Note that S is not noetherian, since R is not noetherian, and that S is not a domain (for instance, $x(yx-1) = 0$). Also, $S = R[z;\sigma]$ is a right Ore extension of R, where σ is the automorphism of R sending x to αx, y to $\alpha^{-1}y$, and elements of K to themselves.

Corollary 4.1.10. *S is a prime ring.*

Proof. S is an Ore extension of a prime ring where the multiplication is twisted only by an automorphism (see Theorem 2.2.3). □

Corollary 4.1.11. *The ideal $\langle yx-1\rangle$ is prime in S.*

Proof. Note that

$$S/\langle yx-1\rangle \cong K[x,x^{-1}][z;\phi],$$

where ϕ is the K-algebra automorphism of $K[x,x^{-1}]$ sending x to αx. Hence S is an Ore extension of a prime ring, where the multiplication is twisted only by an automorphism. Thus, by Theorem 2.2.3, $S/\langle yx-1\rangle$ is a prime ring. □

Proposition 4.1.12. *The set $\{z^i y^j x^k : i,\ j,$ and k are integers $\}$ is a K-linear basis for S.*

Proof. Let $A = K[y][x;\sigma_1,\delta]$ be an Ore extension of $K[y]$ with $\sigma_1(y) = 0$ and $\delta(y) = 1$. By Remark 2.2.1, the set $\{y^j x^k : j$ and k are nonnegative integers$\}$ is a K-linear basis for A. Then $S = A[z;\sigma_2]$ is a right Ore extension of A with $\sigma_2(y) = \alpha^{-1}y$ and $\sigma_2(x) = \alpha x$. Thus, by Remark 2.2.1, $\{z^i y^j x^k : i,\ j,\ k$ are nonnegative integers$\}$ is a K-linear basis for S. □

Thus, when we write a nonzero element of S in the form

$$\sum_\ell \lambda_\ell z^{i_\ell} y^{j_\ell} x^{k_\ell},$$

for $\lambda_\ell \in K$, we will assume that the λ_ℓ are nonzero and that the (i_ℓ, j_ℓ, k_ℓ) are distinct for distinct ℓ.

Proposition 4.1.13. *Let $Z(S)$ be the center of S.*

1. *If α is not a root of unity, then $Z(S) = K$.*

2. *If α is a primitive ℓth root of unity, then $Z(S) = K\langle z^\ell\rangle$.*

Proof. Certainly K is contained in $Z(S)$, so $Z(S)$ is nonempty. Let

$$a = \sum_r \lambda_r z^{i_r} y^{j_r} x^{k_r} \in Z(S)$$

be a nonzero element. Then a must commute with y, and thus

$$ya = \sum_r \lambda_r \alpha^{-i_r} z^{i_r} y^{j_r+1} x^{k_r}$$

$$= ay = \sum_{r:k_r>0} (\lambda_r z^{i_r} y^{j_r} x^{k_r-1}) + \sum_{r:k_r=0} (\lambda_r z^{i_r} y^{j_r+1}).$$

Suppose that k_r is nonzero for at least one r and that γ is the smallest exponent of y appearing in any nonzero summand of a. Then the smallest y-degree of nonzero summands of ya is $\gamma + 1$ and the smallest y-degree of nonzero summands of ay is γ, contradicting that $ya = ay$. Hence, a must not have x as a factor of any summand (ie., $a = \sum_r \lambda_r z^{i_r} y^{j_r}$). Thus,

$$ya = \sum_r \lambda_r \alpha^{-i_r} z^{i_r} y^{j_r+1} x^{k_r} = ay = \sum_r \lambda_r z^{i_r} y^{j_r+1}.$$

Hence, $\alpha^{-i_r} = 1$ for all r. We now divide the proof into two cases depending on whether or not α is a root of unity.

Case 1: Suppose that α is not a root of unity. Then $i_r = 0$ for all r. This implies that $a = \sum_r \lambda_r y^{j_r}$. Since x commutes with a,

$$ax = \sum_r \lambda_r y^{j_r} x = xa = \sum_{r:j_r>0} (\lambda_r y^{j_r-1}) + \sum_{r:j_r=0} (\lambda_r x).$$

Suppose that j_r is nonzero for at least one r. Then ax has x as a factor of every summand, but xa has at least one summand without x as a factor. Thus, $j_r = 0$ for all r. Hence, $a = \lambda$ for some $\lambda \in K$ and $Z(S) = K$, as desired.

Case 2: Suppose that α is a primitive ℓth root of unity. Then, since $\alpha^{-i_r} = 1$ for all r, the integer i_r is a multiple of ℓ or 0 for each r. The elements x and a commute and thus

$$ax = \sum_r \lambda_r y^{j_r} x = xa = \sum_{r:j_r>0} (\lambda_r y^{j_r-1}) + \sum_{r:j_r=0} (\lambda_r x)$$

and, as before, $j_r = 0$ for all r. Thus, $a = \sum_r \lambda_r z^{i_r}$, where the i_r are multiples of ℓ or are zero for all r. Thus, $a \in K\langle z^\ell \rangle$. An easy check shows that, when α is a primitive ℓth root of unity, $\langle z^\ell \rangle$ is contained in $Z(S)$. Thus, $Z(S) = \langle z^\ell \rangle$ in this case, as desired. □

4.1.3 T Preliminaries

Recall that $T = R[z, z^{-1}; \sigma]$.

Proposition 4.1.14. *Suppose that I is a proper ideal of T containing $z - \lambda$, for some nonzero $\lambda \in K$. Then $\alpha = 1$.*

Proof. Let $T' = T/I$ and let x, y, and z also stand for their images in T'. The equations $xz = \alpha zx$ and $z = \lambda$ hold in T'. These equations imply that $\lambda x = \alpha\lambda x$ in T'. If $x = 0$ in T', then $x \in I$ and thus $xy \in I$. Since $xy = 1$ in T, the element $1 \in I$, contradicting the assumption that I is a proper ideal of T. Hence $\alpha = 1$. □

Proposition 4.1.15. *As an ideal of T,*

$$\langle x - \lambda, y - \lambda^{-1} \rangle = \langle x - \lambda/\alpha^r, y - \alpha^r/\lambda \rangle,$$

for any nonzero λ in K and any nonnegative integer r.

Proof. The proof is shown for $r = 1$ and follows analogously for arbitrary values of r. Let $I = \langle x - \lambda/\alpha, y - \alpha/\lambda \rangle$. Then

$$z(x - \lambda/\alpha) = zx - (\lambda/\alpha)z = \alpha^{-1}xz - (\lambda/\alpha)z = (\alpha^{-1}x - (\lambda/\alpha))z \in I.$$

Hence,

$$\alpha(\alpha^{-1}x - (\lambda/\alpha))zz^{-1} = x - \lambda \in I.$$

Similarly $y - \lambda^{-1} \in I$. Thus, $\langle x - \lambda, y - \lambda^{-1} \rangle \subseteq \langle x - \lambda/\alpha, y - \alpha/\lambda \rangle$.

To obtain the reverse inclusion, let $J = \langle x - \lambda, y - \lambda^{-1} \rangle$. Then

$$(x - \lambda)z = xz - \lambda z = z(\alpha x - \lambda) \in J.$$

Hence

$$\alpha^{-1}z^{-1}(x - \lambda)z = x - \lambda/\alpha \in J.$$

Similarly $y - \alpha/\lambda \in J$. Thus $I = J$. □

Corollary 4.1.16. *As an ideal of T,*

$$\langle x - \lambda, y - \lambda^{-1} \rangle \cap \langle x - \lambda/\alpha, y - \alpha/\lambda \rangle \cap \cdots \cap \langle x - \lambda/\alpha^{n-1}, y - \alpha^{n-1}/\lambda \rangle$$
$$= \langle x - \lambda, y - \lambda^{-1} \rangle.$$

4.2 Prime Ideals of S

Recall that $S = R[z;\sigma]$. Thus, the prime ideals of S that contain z will be in one-to-one correspondence with the prime ideals of R, which are given in 2.2.14. Therefore, it remains to classify the prime ideals of S that do not contain z, which are in one-to-one correspondence with the prime ideals of T. Recall that if P is a prime ideal of T, then $P \cap R$ is a σ-prime ideal of R (see Lemma 2.2.4). Hence a classification of the σ-prime ideals of R will aid in our classification of the prime ideals of T. By the previous results, it makes sense to divide our study into two cases based on whether or not α is a root of unity.

4.2.1 Scalar Not a Root of Unity Case

Throughout this subsection, we will assume that α is not a root of unity. We begin by exploring the σ-prime ideals of R and then use these results to classify the prime ideals of T.

Proposition 4.2.1. *Suppose that I is a proper nonzero ideal of R that is not equal to $\langle yx - 1\rangle$. Then I is not σ-stable.*

Proof. By Proposition 4.1.5, Lemma 4.1.6, and Proposition 4.1.8, I contains a nonzero element with a nonzero constant term and without x as a factor of any summand. Let s be the smallest nonzero exponent of y appearing in any nonzero summand of elements of I of this type. Let $a = \sum_r \lambda_r y^r$ be an element of I with λ_0 not equal to zero and y-degree s.

Next suppose that I is σ-stable. This implies that $\sigma(a) \in I$. Note that

$$\sigma(a) = \sigma(\sum_r \lambda_r y^r) = \sum_r \lambda_r \alpha^{-r} y^r.$$

Then

$$\alpha^s \sigma(a) = \sum_r \lambda_r \alpha^{s-r} y^r,$$

and

$$a - \alpha^s \sigma(a) = \sum_{r:r\neq s} \lambda_r (1 - \alpha^{s-r}) y^r.$$

Note that $a-\alpha^s\sigma(a) \in I$ is nonzero with a nonzero constant term and without x as a factor of any summand. Also, the y-degree of $a-\alpha^s\sigma(a)$ is less than s, contradicting the choice of a. Hence, if I is a nonzero ideal of R that is not equal to $\langle yx-1\rangle$, then I is not σ-stable. □

Corollary 4.2.2. *The only σ-stable ideals of R are $\langle 0\rangle$ and $\langle yx-1\rangle$. Thus, the only σ-prime ideals of R are also $\langle 0\rangle$ and $\langle yx-1\rangle$.*

Proof. Since R is a prime ring (see Theorem 2.2.14), $\langle 0\rangle$ will be a σ-prime ideal of R. Hence, the corollary follows from Proposition 4.2.1 and Proposition 4.1.8. □

Corollary 4.2.3. *Let P be a prime ideal of T. Then $P\cap R$ equals $\langle 0\rangle$ or $\langle yx-1\rangle$.*

Proof. The corollary follows from Corollary 4.2.2 and Lemma 2.2.4. □

Proposition 4.2.4. *If P is a prime ideal of T with $P\cap R=\langle yx-1\rangle$, then $P=\langle yx-1\rangle$.*

Proof. Let $T'=(K[x,y]/\langle yx-1\rangle)[z,z^{-1};\sigma]$ and let P' stand for the natural image of P in T'. Note that P' is a prime ideal of T'. Next, consider $P''=P'\cap K[x,y]/\langle yx-1\rangle$. The ideal P'' will be prime in $K[x,y]/\langle yx-1\rangle$. Hence $P''=\langle 0\rangle$ and $P=\langle yx-1\rangle$, using Proposition 4.1.14. □

To classify the prime ideals of T that intersect R at $\langle 0\rangle$, note that these ideals are in one-to-one correspondence with the prime ideals of S that intersect R at $\langle 0\rangle$ and do not contain z.

Proposition 4.2.5. *Let P be a nonzero prime ideal of S such that $P\cap R=\langle 0\rangle$. Then $P=\langle z\rangle$.*

Proof. Let T' denote the symmetric quotient ring of R, $Z(T')$ the center of T', and D' the ring of all central elements in T' which are σ-invariant. Since α is not a root of unity, no power of σ will be an inner automorphism of T'. Hence, by Proposition 2.2.9, $Z(T')=D'$. Therefore, by Theorem 2.2.10, the only prime ideal of S that intersects R at $\langle 0\rangle$ will be $\langle z\rangle$. □

Corollary 4.2.6. *The following is a complete list of the prime ideals of S (organized by families) in the case when α is not a root of unity.*

1. $\langle 0 \rangle$,
2. $\langle yx - 1 \rangle$,
3. $\langle z \rangle$,
4. $\langle yx - 1, z \rangle$, *and*
5. $\langle x - \lambda, y - \lambda^{-1}, z \rangle$, *where λ is a nonzero element of K.*

4.2.2 Scalar Root of Unity Case

Throughout this subsection, we will assume that α is a primitive ℓth root of unity and hence σ^ℓ is an inner automorphism of R (in fact, σ^ℓ is the identity map on R). Thus, by Lemma 2.2.7 and Proposition 2.2.8, the set of ideals of R that are σ-cyclic is equal to the set of ideals of R that are σ-prime, and the σ-cyclic ideals of R will be one of the following.

1. $\cap \sigma^k(\langle 0 \rangle) = \sigma(\langle 0 \rangle) = \langle 0 \rangle$,
2. $\cap \sigma^k(\langle yx - 1 \rangle) = \sigma(\langle yx - 1 \rangle) = \langle yx - 1 \rangle$, or
3. $\cap \sigma^k(\langle x - \lambda, y - \lambda^{-1} \rangle) = \langle x - \lambda, y - \lambda^{-1} \rangle \cap \cdots \cap \langle x - \lambda/\alpha^{\ell-1}, y - \alpha^{-\ell+1}/\lambda \rangle$, for nonzero $\lambda \in K$.

Thus, if P is a prime ideal of T, then $P \cap R$ will be one of the above σ-cyclic ideals.

Corollary 4.2.7. *If $\alpha \neq 1$ and P is a prime ideal of T not containing z with $P \cap R = \langle yx - 1 \rangle$, then $P = \langle yx - 1 \rangle$. If $\alpha = 1$ and P is a prime ideal of T not containing z with $P \cap R = \langle yx - 1 \rangle$, then P is equal to one of the following.*

1. $\langle yx - 1 \rangle$ *or*

2. $\langle yx-1, z-\lambda\rangle$*, for* λ *a nonzero element of* K*.*

Proof. Let $T' = (K[x,y]/\langle yx-1\rangle)[z, z^{-1}; \sigma]$, let P' stand for the natural image of P in T', and let $P'' = P' \cap K[x,y]/\langle yx-1\rangle$. Note that P'' will be a prime ideal of $K[x,y]/\langle yx-1\rangle$. Using the assumption that $z \notin P$ and Proposition 4.1.14, if $\alpha \neq 1$, the ideal $P' = \langle 0\rangle$. If $\alpha = 1$, the ideal P' equals $\langle 0\rangle$ or $\langle z-\lambda\rangle$, for λ a nonzero element of K. □

Corollary 4.2.8. *Let* P *be a nonzero prime ideal of* S *not containing* z *with* $P \cap R = \langle 0\rangle$*. Then* $P = \langle z^{\ell} - \lambda\rangle$ *for* $\lambda \in K$*.*

Proof. Let T' denote the symmetric quotient ring of R, $Z(T')$ the center of T', and D' the ring of all central elements in T' which are σ-invariant. Since α is an ℓth root of unity, σ^{ℓ} is an inner automorphism of T'. Thus, by Proposition 2.2.9, $Z(T') \neq D'$. Therefore, there exists an invertible λ in T' and an $r \geq 0$ such that $Z(T') = D'[u]$, where $u = \lambda t^{r}$. By Proposition 2.2.9, $u = z^{\ell}$. Hence, again using Theorem 2.2.10, the nonzero prime ideals P of S, not containing z, with $P \cap R = \langle 0\rangle$ are of the form $\langle z^{\ell} - \lambda\rangle$ for $\lambda \in K$. □

Proposition 4.2.9. *Let* P *be a prime ideal of* T *with*

$$P \cap R = \langle x-\lambda, y-\lambda^{-1}\rangle \cap \cdots \cap \langle x-\lambda/\alpha^{\ell-1}, y-\alpha^{-\ell+1}/\lambda\rangle.$$

Then $P = \langle x-\lambda, y-\lambda^{-1}, z-\gamma\rangle$*, with* γ *nonzero only if* $\alpha = 1$*.*

Proof. The proposition follows from Corollary 4.1.16 and Proposition 4.1.14. □

Corollary 4.2.10. *The following is a complete list of the prime ideals of* S *(organized by families) in the case when* α *is a primitive* ℓ*th root of unity.*

1. $\langle 0\rangle$,

2. $\langle yx-1\rangle$,

3. $\langle z\rangle$,

4. $\langle yx - 1, z\rangle$,

5. $\langle x - \lambda, y - \lambda^{-1}\rangle$, *for* λ *a nonzero element of* K,

6. $\langle x - \lambda, y - \lambda^{-1}, z\rangle$, *for* λ *a nonzero element of* K, *and*

7. $\langle z^{\ell} - \gamma\rangle$, *for* γ *a nonzero element of* K.

If $\alpha = 1$ *(i.e.,* $\ell = 1$*), then, in addition to the above prime ideals, the following prime ideals of* S *are a complete list of the prime ideals of* S.

1. $\langle yx - 1, z - \gamma\rangle$, *for* γ *a nonzero element of* K, *and*

2. $\langle x - \lambda, y - \lambda^{-1}, z - \gamma\rangle$, *for* λ *and* γ *nonzero elements of* K.

4.3 Primitive Ideals of S

Since the set of primitive ideals of S is contained in the set of prime ideals of S, we need only consider prime ideals of S to classify the primitive ideals. We leave a complete classification for future work

4.3.1 Scalar Not a Root of Unity Case

In the case when α is not a root of unity, the following are primitive ideals of S because S modulo each of these ideals is known to be a primitive ring.

1. $\langle z\rangle$,
2. $\langle yx - 1, z\rangle$, and
3. $\langle x - \lambda, y - \lambda^{-1}, z\rangle$, where λ is a nonzero element of K.

The primitivity of the following ideals is left open.

1. $\langle 0\rangle$, and
2. $\langle yx - 1\rangle$.

4.3.2 Scalar Root of Unity Case

In the case when α is a primitive ℓth root of unity, the following are primitive ideals of S because S modulo each of these ideals is known to be a primitive ring.

1. $\langle z \rangle$,

2. $\langle yx - 1, z \rangle$, and

3. $\langle x - \lambda, y - \lambda^{-1}, z \rangle$, for λ a nonzero element of K.

If $\alpha = 1$ (i.e., $\ell = 1$), then, in addition to the above primitive ideals, the following are primitive ideals of S.

1. $\langle yx - 1, z - \gamma \rangle$, for γ a nonzero element of K, and

2. $\langle x - \lambda, y - \lambda^{-1}, z - \gamma \rangle$, for λ and γ nonzero elements of K.

The primitivity of the following ideals is left open.

1. $\langle 0 \rangle$,

2. $\langle yx - 1 \rangle$,

3. $\langle x - \lambda, y - \lambda^{-1} \rangle$, for λ a nonzero element of K, and

4. $\langle z^{\ell} - \gamma \rangle$, for γ a nonzero element of K.

CHAPTER 5

NONNOETHERIAN CASE TWO

This chapter will give a classification of the prime ideals, primitive ideals, and irreducible representations of

$$S = K\{x_1, \cdots, x_n\}/\langle x_i x_j - 0x_j x_i - \beta_{ij}, i < j\rangle$$

$$= K\{x_1, \cdots, x_n\}/\langle x_i x_j - \beta_{ij}, i < j\rangle,$$

where K is an algebraically closed field, $n \geq 3$, and $\beta_{ij} \in K$. The prime and primitive spectra of some classes can be immediately classified. For the remaining cases, we reduce our study of the prime and primitive ideal structure of the n-variable case to the study of the prime and primitive ideals of

$$R = K\{x, y, z\}/\langle xy - 0, xz - 1, yz - 0\rangle = K\{x, y, z\}/\langle xy, xz - 1, yz\rangle.$$

After reducing our study of the algebras in n-variables to the study of R, we prove some preliminary results about R. In particular, we note that a more general result by Adam Berliner (Theorem 2.2.13) allows us to completely classify the finite dimensional irreducible representations of R. Next, we explicitly construct an infinite family of infinite dimensional irreducible representations, using a method of Irving. A proof is then given showing that the representations already classified are indeed all of the irreducible representations of R. Finally, we classify the prime ideals of R.

5.1 Reduction to Three Variable Case

Throughout, let $n \geq 3$ and let x, y, and z stand for their images in various quotient algebras. We now show that to complete a classification of the prime and primitive spectra of algebras of the form

$$K\{x_1, \cdots, x_n\}/\langle x_i x_j - \beta_{ij}, i < j \rangle,$$

where K is an algebraically closed field, $n \geq 3$, and $\beta_{ij} \in K$, it suffices to classify the prime and primitive ideals of $R = K\{x, y, z\}/\langle xy, xz - 1, yz \rangle$. We consider the possible cases when each β_{ij} is taken to be zero or nonzero.

First, suppose that β_{ij} is nonzero for all i and j. We will assume that $\beta_{ij} = 1$ for all i and j and the results follow similarly for arbitrary nonzero values of the β_{ij}. Let

$$S_1 = K\{x_1, x_2, \cdots, x_n\}/\langle x_i x_j - 1, i < j \rangle.$$

In S_1, the equation $x_1 x_2 = 1$ holds. Multiplying both sides of this equation by x_k on the right yields $x_1 x_2 x_k = x_k$. In S_1, however, $x_2 x_k = 1$, for any $2 < k \leq n$. Therefore, $x_1 = x_k$ for all $2 < k \leq n$ and the equation $x_1 x_1 = 1$ also holds in S_1. Multiplying both sides of the equation $x_1 x_2 = 1$ by x_1 on the left yields $x_1 x_1 x_2 = x_1$ and, hence, $x_1 = x_2$. Thus $x_i = x_j$ for all $1 \leq i, j \leq n$ and $S_1 \cong K[x]/\langle x^2 - 1 \rangle$.

Next, suppose that $\beta_{ij} = 0$ for all i and j. That is, consider

$$S_2 = K\{x_1, x_2, \cdots, x_n\}/\langle x_i x_j, i < j \rangle.$$

Note that $x_j x_i$ is contained in the nilradical of S_2 for $1 \leq i < j \leq n$. Thus, modulo its nilradical, S_2 is commutative and hence the classification of the prime and primitive ideals is clear.

Let

$$S_3 = K\{x_1, x_2, \cdots, x_n\}/\langle x_i x_j - \beta_{ij}, i < j\rangle,$$

where at least two $\beta_{ij} \neq 1$ and at least one $\beta_{ij} = 0$. We will assume that nonzero $\beta_{ij} = 1$ for all i and j and the results follow similarly for arbitrary nonzero values of the β_{ij}.

Proposition 5.1.1. *If $n = 3$, then S_3 (as above) is trivial.*

Proof. We proceed in cases.

Case 1: Suppose $x_1 x_2 = 1$.

Subcase 1: Suppose $x_2 x_3 = 1$. Multiplying both sides of this equation by x_1 on the left yields $x_1 x_2 x_3 = x_1$. This implies that $x_3 = x_1$. The equation $x_1 x_3 = 0$ also holds in S_3. Hence $x_1^2 = 0$. Multiplying both sides of $x_1 x_2 = 1$ by x_1 on the left yields that $x_1 = 0$, contradicting that x_1 must be nonzero for the equation $x_1 x_2 = 1$ to hold. Hence S_3 is trivial.

Subcase 2: Suppose $x_2 x_3 = 0$. Then the equation $x_1 x_3 = 1$ also holds in S_3. Multiplying both sides of the equation $x_1 x_2 = 1$ by x_3 on the right yields that $x_1 x_2 x_3 = x_3$ and hence $x_3 = 0$, contradicting that x_3 must be nonzero for the equation $x_1 x_3 = 1$ to hold. Hence S_3 is trivial.

Case 2: Suppose $x_1 x_2 = 0$. Then the equations $x_1 x_3 = 1$ and $x_2 x_3 = 1$ hold in S_3. Multiplying both sides of the equation $x_2 x_3 = 1$ by x_1 on the left yields $x_1 x_2 x_3 = x_1$, implying that $x_1 = 0$. This equality contradicts that x_1 must be nonzero for the equation $x_1 x_3 = 1$ to hold in S_3 and, hence, S_3 is trivial. □

Corollary 5.1.2. *If n is any integer with $n \geq 3$, then S_3 is trivial.*

Proof. We proceed by induction on n. If $n = 3$, the corollary holds by Proposition 5.1.1. Let k be a positive integer greater than 3 and suppose the result is true for $n < k$. Then, if $n = k$, the algebra S_3 is an Ore extension of $S_3' = K\{x_1, x_2, \cdots, x_{k-1}\}/\langle x_i x_j - \beta_{ij}, i < j\rangle$, where $\beta_{ij} \in K$.

Case 1: If at least two of $\beta_{ij} = 1$ and at least one of the $\beta_{ij} = 0$ for $1 \leq i, j \leq k - 1$, then S_3' is trivial by the induction hypothesis and hence S_3 is trivial.

Case 2: If $\beta_{ij} = 1$ for all $1 \le i, j \le k-1$, then S_3' is isomorphic to $K[x]/\langle x^2 - 1\rangle$ by an above argument. But this implies that S_3 is an Ore extension of $K[x]/\langle x^2-1\rangle$ and hence cannot have at least two $\beta_{ij} = 1$ and at least one $\beta_{ij} = 0$, contradicting the assumptions on the β_{ij}.

Case 3: If there exists only one $\beta_{ij} = 1$ with $1 \le i, j \le k-1$, then there exists an $r < k$ such that $\beta_{rk} = 1$. Hence $x_k \neq 0$. Suppose that $x_\ell x_m = 1$ for $1 \le \ell < m \le k-1$. Then in S_3 the equation $x_\ell x_m x_k = x_k$ holds. If $x_m x_k = 0$, then $x_k = 0$ contradicting that x_k must be nonzero. If $x_m x_k = 1$, then $x_\ell = x_k$ and hence $S_3 \cong S_3'$. But this also contradicts the assumptions on the β_{ij}. Hence S_3 is trivial.

Case 4: Suppose that $\beta_{ij} = 0$ for all $1 \le i, j \le k-1$. Then there exists an r and an s with $1 \le r < s \le k-1$ such that the following equations hold in S_3.

1. $x_r x_k = 1$,
2. $x_s x_k = 1$, and
3. $x_r x_s = 0$.

Then $x_r x_s x_k = x_r$ holds in S_3. But this implies that $x_r = 0$ which contradicts that x_r must be nonzero for the equation $x_r x_k = 1$ to hold. Hence S_3 is trivial. □

Finally, let

$$S_4 = K\{x_1, x_2, \cdots, x_n\}/\langle x_i x_j - \beta_{ij}, i < j\rangle,$$

and suppose that there exists a k and an ℓ such that $\beta_{k\ell} \neq 0$ and $\beta_{ij} = 0$ whenever $(i,j) \neq (k,\ell)$. Note that by a change of variable we may assume that $\beta_{k\ell} = 1$.

Proposition 5.1.3. *The algebra S_4 (as above) is isomorphic to*

$$K\{x_k, x_{k+1}, \cdots, x_\ell\}/\langle x_i x_j - \gamma_{ij}, i < j\rangle,$$

where $\gamma_{k\ell} = 1$ and $\gamma_{ij} = 0$ whenever $(i,j) \neq (k,\ell)$.

Proof. Suppose $\beta_{k\ell} = 1$ and $\beta_{ij} = 0$ whenever $(i, j) \neq (k, \ell)$. If there exists an m such that $1 \leq m < k$, then $x_m x_k = 0$. Multiplying both sides of the equation $x_k x_\ell = 1$ by x_m on the left yields $x_m(x_k x_\ell) = x_m$. This implies that $(x_m x_k)x_\ell = x_m$ and thus that $x_m = 0$. By similar reasoning, $x_r = 0$ for all $\ell < r \leq n$. □

Thus for the remainder of this chapter, let

$$S_4 = K\{x_1, x_2, \cdots, x_n\}/\langle x_i x_j - \beta_{ij}, i < j\rangle,$$

where $\beta_{1n} = 1$ and $\beta_{ij} = 0$ for $(i, j) \neq (1, n)$.

Proposition 5.1.4. *Let P be a prime ideal of S_4. Then at most one of the elements $x_2, \cdots, x_{n-1}$ is not contained in P.*

Proof. Let P be any prime ideal of S_4. Suppose there exists a $2 \leq k \leq n - 1$ such that $x_k \notin P$. Elements of $\langle x_\ell\rangle\langle x_k\rangle$ are sums of elements of the form $r_1 x_\ell r_2 x_k r_3$, where $r_1, r_2, r_3 \in S_4$. If $k \neq 2$, then, for any $2 \leq \ell < k$, the product $x_\ell r_2 x_k = 0$. Hence $\langle x_\ell\rangle\langle x_k\rangle = 0$. Furthermore, since P is prime and $x_k \notin P$, the element $x_\ell \in P$. Similarly, if $k \neq n-1$, then for any $k < m \leq n-1$, the element $x_m \in P$. Hence, if $x_k \notin P$ for $2 \leq k \leq n - 1$, then $x_\ell \in P$ for all ℓ such that $2 \leq \ell \leq n - 1$ and $\ell \neq k$. Therefore any prime ideal P of S_4 will contain $x_2, \cdots, x_{n-1}$ or all but one of $x_2, \cdots, x_{n-1}$. □

Hence, using Theorem 2.2.14, a classification of the prime and primitive ideals of R, will completely determine the prime and primitive spectra of S_4. Furthermore, using Gerritzen's classification of the irreducible representations of $K\{x, y, \}/\langle xy - 1\rangle$ (see [11, Section 3]), a classification of the irreducible representations of R will complete such a classification for S_4.

5.2 Prime and Primitive Ideals

5.2.1 Notation and Preliminary Results

Proposition 5.2.1. *The set* $\{z^i y^j x^k : i,\ j,\ k$ *are nonnegative integers*$\}$ *is a* K*-linear basis for* R.

Proof. Let $A = K[x][y;\sigma_1]$ be a right Ore extension of $K[x]$ with $\sigma_1(x) = 0$. Then, by Remark 2.2.1, the set $\{y^j x^k : j$ and k are nonnegative integers$\}$ is a K-linear basis for A. Then $R = A[z;\sigma_2,\delta]$ is a right Ore extension of A, where $\sigma_2(y) = 0$, $\delta(y) = 0$, $\sigma_2(x) = 0$, and $\delta(x) = 1$. Thus, by Remark 2.2.1, $\{z^i y^j x^k : i,\ j,\ k$ are nonnegative integers$\}$ is a K-linear basis for R. □

Hence, whenever we refer to a nonzero element $\sum_{r=1}^{m} \lambda_r z^{i_r} y^{j_r} x^{k_r}$ in R, we will assume that the (i_r, j_r, k_r) are distinct for distinct r and that the λ_r are nonzero.

Proposition 5.2.2. *Any nonzero ideal of* R *contains a nonzero element of* $K[y]$.

Proof. Let I be a nonzero ideal of R and let $f = \sum_{r=0}^{n} \lambda_r z^{i_r} y^{j_r} x^{k_r}$ be a nonzero element of I.

Case 1: Suppose there exists a nonzero summand s_1 of f with x not a factor of s_1. Then fy will be nonzero and x will not be a factor of any summand of fy. If there is at least one nonzero summand, s_2, of fy with z not a factor of s_2, then yfy will be a nonzero element of I without x or z as a factor of any of its summands.

Suppose then that fy has z as a factor of every summand. Let α be the y-degree of fy. Note that $\alpha \geq 1$ (since we have multiplied f by y). Let β be the highest z-degree of the summands of fy with y-degree α. Note that there can be only one summand of fy with y-degree α and z-degree β. Also, note that $x^\beta fy$ contains a nonzero summand of the form λy^α, for some $\lambda \in K$, and this summand cannot possibly cancel with any other summands (using the fact that there is only one summand of fy with y-degree α and z-degree β). Therefore, $x^\beta fy$ is nonzero.

Also, $x^\beta fy$ is an element of I without x as a factor of any of its summands and there exists at least one nonzero summand of $x^\beta fy$ without z as a factor. Hence, as previously, $yx^\beta fy$ will be a nonzero element of I without x or z as a factor of any of its summands. Therefore, the desired result is true in this case.

Case 2: Suppose there exists a nonzero summand of f without z as a factor. Then proceed similarly to the above case.

Case 3: Suppose every summand of f has x and z as a factor. Let α be the minimum exponent of x appearing in any nonzero summand of f. Note that $\alpha \geq 1$. Then fz^α will be a nonzero element of I where not every summand has x as a factor. Hence proceed as in Case 1.

Therefore, every nonzero ideal of R contains a nonzero element of $K[y]$. □

Corollary 5.2.3. *R is prime.*

Proof. Suppose I and J are nonzero ideals of R with $IJ = 0$. By Proposition 5.2.2, I contains a nonzero polynomial in y, say f, and J contains a nonzero polynomial in y, say g. Then $fg \in IJ$ and $fg \neq 0$ (the product of two nonzero polynomials in y cannot be zero), contradicting the assumption that $IJ = 0$. □

Lemma 5.2.4. *In R, the ideal $\langle y^i \rangle = \langle y \rangle^i$, where i is any integer greater than or equal to zero.*

Proof. The proof is shown for $i = 2$ and the general case follows analogously. It is obvious that $\langle y^2 \rangle \subseteq \langle y \rangle^2$. Suppose $a \in \langle y \rangle^2$. Then a is a finite sum of elements of the form $r_1 y z^i y^j x^k y r_2$ where r_1 and r_2, are elements of R and i, j, and k are nonnegative integers. If either i or k is nonzero, then

$$r_1 y z^i y^j x^k y r_2 = 0 \in \langle y^2 \rangle.$$

If $i = 0$ and $k = 0$, then $r_1 y z^i y^j x^k y r_2 = r_1 y^{j+2} r_2 \in \langle y^2 \rangle$. □

Lemma 5.2.5. *Let P be any nonzero proper prime ideal of R not containing y. Then there exists a nonzero $c \in K$ such that $\langle y^2 - cy \rangle \subseteq P$.*

Proof. If $\langle y^2 - cy \rangle \subseteq P$ for some $c \in K$, then, by Lemma 5.2.4, c is nonzero. By Proposition 5.2.2, P contains a nonzero polynomial in y, say g. If g has a nonzero constant term, then xgz is equal to a nonzero scalar and $xgz \in P$, contradicting that P is a proper ideal of R. Hence, g must have a zero constant term, i.e., $g = yf$ for some polynomial $f \in K[y]$.

Suppose then that $f = (y - \alpha_1)(y - \alpha_2) \cdots (y - \alpha_t)$ is a factorization of f into irreducible polynomials over K. Since $\langle g(y) \rangle \subseteq P$ and P is prime, we need only prove

$$\langle y(y-\alpha_1)\rangle\langle y(y-\alpha_2)\rangle ... \langle y(y-\alpha_t)\rangle \subseteq \langle g(y)\rangle$$

to prove the lemma. We proceed by induction on t.

If $t = 2$, let $a \in \langle y(y-\alpha_1)\rangle\langle y(y-\alpha_2)\rangle$. Then a is equal to a finite sum of elements of the form

$$r_1 y(y-\alpha_1) z^i y^j x^k y(y-\alpha_2) r_2,$$

for some elements r_1 and $r_2 \in R$ and nonnegative integers i, j, and k. If i or k is nonzero, then

$$r_1 y(y-\alpha_1) z^i y^j x^k y(y-\alpha_2) r_2 = 0 \in \langle y(y-\alpha_1)(y-\alpha_2)\rangle.$$

If i and k are zero, then

$$r_1 y(y-\alpha_1) z^i y^j x^k y(y-\alpha_2) r_2 = r_1 y^{j+2}(y-\alpha_1)(y-\alpha_2) \in \langle y(y-\alpha_1)(y-\alpha_2)\rangle.$$

Hence, $a \in \langle y(y-\alpha_1)(y-\alpha_2)\rangle$ and the desired result is true for $t = 2$.

Let s be an integer greater than two. Suppose the claim is true for $t < s$. Using the induction hypothesis,

$$\langle y(y-\alpha_1)\rangle\langle y(y-\alpha_2)\rangle \ldots \langle y(y-\alpha_{s-1})\rangle \subseteq \langle y(y-\alpha_1)\cdots(y-\alpha_{s-1})\rangle.$$

Hence,

$$\langle y(y-\alpha_1)\rangle\langle y(y-\alpha_2)\rangle \cdots \langle y(y-\alpha_s)\rangle \subseteq \langle y(y-\alpha_1)\ldots(y-\alpha_{s-1})\rangle\langle y(y-\alpha_s)\rangle.$$

Suppose

$$a \in \langle y(y-\alpha_1)\dots(y-\alpha_{s-1})\rangle\langle y(y-\alpha_s)\rangle.$$

Then a is a finite sum of elements of the form

$$r_1(y-\alpha_1)\cdots(y-\alpha_{s-1})y(z^iy^jx^k)y(y-\alpha_s)r_2,$$

for elements r_1 and $r_2 \in R$ and nonnegative integers i, j, and k. If i or k is nonzero,

$$r_1(y-\alpha_1)\cdots(y-\alpha_{s-1})y(z^iy^jx^k))y(y-\alpha_s)r_2 = 0.$$

If i and k are zero, then

$$\begin{aligned} &r_1(y-\alpha_1)\cdots(y-\alpha_{s-1})y(z^iy^jx^k))y(y-\alpha_s)r_2 \\ &= r_1y^{j+2}(y-\alpha_1)\cdots(y-\alpha_{s-1})(y-\alpha_s) \\ &\in \langle y(y-\alpha_1)\cdots(y-\alpha_s)\rangle \subseteq P. \end{aligned}$$

Hence, $a \in P$. This implies that

$$\langle y(y-\alpha_1)\rangle\langle y(y-\alpha_2)\rangle\cdots\langle y(y-\alpha_s)\rangle \subseteq P.$$

Since P is prime, there exists an $1 \leq i \leq s$ such that $\langle y(y-\alpha_i)\rangle \subseteq P$, as desired. □

Remark 5.2.6. Note that $\langle y\rangle$ is a prime and primitive ideal of R (see Theorem 2.2.14).

Remark 5.2.7. Also note the following corollary to Proposition 2.2.13.

Corollary 5.2.8. *All cofinite dimensional primitive ideals of R are of the form $\langle y, x-\lambda, z-\lambda^{-1}\rangle$, where λ is a nonzero element of K.*

5.2.2 Infinite Dimensional Irreducible Representations

To show the existence of infinite dimensional irreducible representations of R, we begin by explicitly constructing such a representation, using a method of Irving (see [18, Section 7]). Let λ be a nonzero element of K and let M_λ be the infinite dimensional K-vector space with basis v_0, v_1, v_2,.... By the following action, M_λ is a $K\{x, y, z\}$-module.

1. $zv_n = v_{n+1}$ for all $n \in \mathbb{N}$ with $n \geq 0$,

2. $xv_0 = 0$

3. $xv_n = v_{n-1}$ for all $n \in \mathbb{N}$ with $n \geq 1$,

4. $yv_0 = \lambda v_0$, and

5. $yv_n = 0$ for all $n \in \mathbb{N}$ with $n \geq 1$.

Let $v \in M_\lambda$. Since $v_0, v_1, v_2, \ldots$ is a K-linear basis for M_λ, we can write v in the form $v = \sum_{i=0}^{\ell} c_i v_i$, for some $c_i \in K$ and some nonnegative integer ℓ. Whenever we write an element of M_λ in this form, we are assuming that $c_\ell \neq 0$ and that the v_i are the K-basis vectors of M_λ. Therefore,

$$xy \cdot v = xy \cdot \sum_{i=0}^{\ell} c_i v_i = x \cdot c_0 \lambda v_0 = 0.$$

Hence, $xy \in ann_{K\{x,y,z\}}(M_\lambda)$. Similarly, yz and $xz - 1$ are elements of $ann_{K\{x,y,z\}}(M_\lambda)$. Hence, M_λ is an R-module.

Let $v = \sum_{i=0}^{\ell} c_i v_i$, (where $c_i \in K$ and v_i are K-basis vectors of M_λ), be an arbitrary element of M_λ with $c_\ell \neq 0$. Then $z^m x^\ell v = c_\ell v_m$, which implies that $Rv = M_\lambda$ or, equivalently since v is arbitrary, that M_λ is simple.

Proposition 5.2.9. *The ideal* $ann_R(M_\lambda) = \langle \lambda zx + y - \lambda \rangle$.

Proof. The proof is shown for $\lambda = 1$ and follows analogously for other values of λ. An easy check shows that $\langle zx+y-1 \rangle \subseteq ann_R(M_\lambda)$. Let $R' = R/\langle zx+y-1 \rangle$ and let x, y, and z also denote their images in R'. The set $\{z^i y^j x^k : i, j, k$ are integers$\}$ spans R' over K. However, $y^2 - y \in \langle zx+y-1 \rangle$ and hence $y^2 - y = 0$ in R'. Therefore,

$$\{z^i y^j x^k : i, j \text{ and } k \text{ are nonnegative integers with } j = 0 \text{ or } j = 1\}$$

spans R' over K. Also, $zx = 1 - y$ in R'. Thus, $\{z^i y x^k\} \bigcup \{z^i\} \bigcup \{x^k\}$, where i and j are nonnegative integers, spans R' over K. Hence, if r is any nonzero element of R', we can write r in the form

$$r = \sum_a \lambda_a z^{i_a} y x^{k_a} + \sum_b \gamma_b z^b + \sum_c \mu_c x^c,$$

for nonzero λ_a, γ_b, and $\mu_c \in K$ and for (i_a, k_a) distinct for distinct a.

Suppose

$$0 \neq p = \sum_a \lambda_a z^{i_a} y x^{k_a} + \sum_b \gamma_b z^b + \sum_c \mu_c x^c \in ann_{R'}(M_\lambda),$$

where the λ_a, γ_b, and μ_c are nonzero elements of K. Let q be greater than k_a for all a and greater than c for all c. Then

$$pv_q = \sum \gamma_b v_{q+b} + \sum \mu_c v_{q-c},$$

which must be zero since $p \in ann_{R'}(M_\lambda)$.

This implies that for each b, there exists a c such that $\gamma_b v_{q+b} = -\mu_c v_{q-c}$. This, in turn, implies that $q + b = q - c$ which can only happen if $b = c = 0$ since $b, c \geq 0$. Similarly for each c, there exists a b such that $\gamma_b v_{q+b} = -\mu_c v_{q-c}$. Therefore, $c = b = 0$. Hence, $p = \sum_a \lambda_a z^{i_a} y x^{k_a}$.

Let r be the highest exponent of x appearing in p. Then

$$pv_r = \sum_a \lambda_a z^{i_a} y x^{k_a} v_r = \sum_{(a:k_a=r)} \lambda_a z^{i_a} y v_0 = \sum_{(a:k_a=r)} \lambda_a z^{i_a} v_0 = \sum_{(a:k_a=r)} \lambda_a v_{i_a}.$$

However, for each a such that $k_a = r$, the i_a must be distinct (otherwise there would be two summands with the same x, y, and z-degrees). This implies that $\sum_{(a:k_a=r)} \lambda_a v_{i_a}$ cannot possibly be zero, which contradicts that $p \in ann_R(M_\lambda)$. Hence, $0 = ann_{R'}(M_\lambda)$, which implies that $\langle zx + y - 1\rangle \supseteq ann_R(M_\lambda)$ and that $ann_R(M_\lambda) = \langle zx + y - 1\rangle$, as desired. □

Thus, R has an infinite family of infinite dimensional irreducible representations and we wish to explore the existence of others. Since a classification of the irreducible infinite dimensional representations V of R with $y \in ann_R V$ can be found in [11, Section 3], we now concentrate on irreducible infinite dimensional representations of R without y in their annihilator.

Proposition 5.2.10. *Let M be a nonzero infinite dimensional simple R-module with $ann_R(M) \neq 0$ and $y \notin ann_R(M)$. Then M is isomorphic to M_λ (as defined above) for some $\lambda \neq 0 \in K$.*

Proof. Let M be any nonzero infinite dimensional simple R-module. Suppose $ann_R(M) \neq 0$ and $y \notin ann_R(M)$. Let $A = K\langle x, y\rangle$ and $M' = (AyxA)M$.

Case 1: Suppose $M' = 0$. This implies that yx is in the annihilator in R of M. Hence $yxz = y \in ann_R(M)$, contradicting the assumption that $y \notin ann_R(M)$.

Case 2: Suppose $M' \neq 0$. Then $x \in ann_A(M')$. Therefore, M' is a $K[y]$-module. By Proposition 5.2.2, there exists a polynomial in y, say $f(y)$, with $f(y) \in ann_R(M)$. Hence, $f(y)M' = 0$. Therefore, for $0 \neq m \in M'$, the A-module Am is finite dimensional over K. Thus Am contains a simple A-module, say Av_0, for some $v_0 \in M'$. Then $xv_0 = 0$ and $yv_0 = \lambda v_0$ for some $\lambda \neq 0 \in K$.

Choose a basis for M over K that includes v_0, say $v_0, v_1, v_2, \ldots$. Since $xzv_i = v_i$ for all i, the element $v_0 \notin zM$. Let n be an arbitrary positive integer. Since M is a simple R-module, there exists an element, $\sum_{\ell=0}^{m} \alpha_\ell z^{i_\ell} y^{j_\ell} x^{k_\ell}$, of R such that

$$\left(\sum_{\ell=0}^{m} \alpha_\ell z^{i_\ell} y^{j_\ell} x^{k_\ell}\right) v_0 = v_n.$$

Since $xv_0 = 0$, the exponent $k_\ell = 0$ for all ℓ. Using the fact that $yv_0 = \lambda v_0$, the product $\sum_{\ell=0}^{m} \alpha_\ell \lambda^{j_\ell} z^{i_\ell} \cdot v_0 = v_n$. Thus, there exists a polynomial in z, say $g(z) = \sum_{\ell=0}^{r} \alpha_\ell z^\ell$, such that $g(z) \cdot v_0 = v_n$. Then

$$x^r \left(\sum_{\ell=0}^{r} \alpha_r z^\ell\right) \cdot v_0 = x^r v_n$$

and thus $\alpha_r v_0 = x^r v_n$.

However, we also have from the relations in R that $x^r z^r \alpha_r v_0 = \alpha_r v_0$. Thus, $x^r z^r \alpha_r v_0 = x^r v_n$. This implies that

$$x^r \in ann_R(z^r \alpha_r v_0 - v_n),$$

which implies that

$$1 \in ann_R(z^r \alpha_r v_0 - v_n).$$

Hence, $z^r \alpha_r v_0 = v_n$. Let $w_0 = v_0$ and $w_i = z^i v_0$ for all $i > 0$. Note that the set $\{w_0, w_1, \ldots\}$ will contain all basis vectors of M over K and hence the basis $w_0, w_1, \ldots$ is just a renumbering of the original basis, $v_0, v_1, \ldots$. Let $u_0, u_1, \ldots$ be the basis for M_λ, with R-action:

1. $zu_n = u_{n+1}$ for all $n \in \mathbb{N}$ with $n \geq 0$,
2. $xu_0 = 0$,
3. $xu_n = u_{n-1}$ for all $n \in \mathbb{N}$ with $n \geq 1$,
4. $yu_0 = u_0$, and
5. $yu_n = 0$ for all $n \in \mathbb{N}$ with $n \geq 1$.

Define $\phi : M \to M_\lambda$ by $\phi(w_i) = u_i$. This is an isomorphism of R-modules. □

Thus, all infinite dimensional irreducible representations of R that are not faithful have been classified. To complete a classification of the primitive ideals of R, we need to ascertain whether or not $\langle 0 \rangle$ is a primitive ideal.

Proposition 5.2.11. *R is not primitive.*

Proof. Suppose that M is a faithful, simple nonzero R-module. Let $A = K\langle x, y\rangle$ and $M' = (AyxA)M$, as above. Note that $M' \neq 0$ since M is faithful. Thus, for all $v \in M'$ and for all nonzero $f(y) \in K[y]$, the product $f(y) \cdot v \neq 0$, or else $M = M_\lambda$ for some $\lambda \in K$, as in the proof of Proposition 5.2.10.

Choose $v \neq 0 \in M'$. Then, since M is simple, $Ryv = M$. This implies that there exists an $a = \sum_{\ell=0}^{m} \alpha_\ell z^{i_\ell} y^{j_\ell} x^{k_\ell} \in R$ such that $ayv = v$.

Then

$$v = \left(\sum_{\ell=0}^{m} \alpha_\ell z^{i_\ell} y^{j_\ell} x^{k_\ell} \right) y \cdot v = \left(\sum_{\ell : k_\ell = 0}^{m} \alpha_\ell z^{i_\ell} y^{j_\ell + 1} \right) \cdot v.$$

Let r be the minimum exponent of z appearing in any nonzero summand of

$$\sum_{\ell:k_\ell=0}^{m} \alpha_\ell z^{i_\ell} y^{j_\ell+1}.$$

Then

$$x^r \cdot v = x^r \left(\sum_{\ell:k_\ell=0}^{m} \alpha_\ell z^{i_\ell} y^{j_\ell+1} \right) \cdot v.$$

Since $x \in ann_R(M')$, the product $x^r \cdot v = 0$ and hence

$$x^r \left(\sum_{\ell:k_\ell=0}^{m} \alpha_\ell z^{i_\ell} y^{j_\ell+1} \right) \cdot v = 0.$$

This implies that

$$\left(\sum_{\ell:k_\ell=0, i_\ell>r} \alpha_\ell z^{i_\ell - r} y^{j_\ell+1} + \sum_{\ell:k_\ell=0, i_\ell=r} \alpha_\ell y^{j_\ell+1} \right) \cdot v = 0.$$

Note that $\sum_{\ell:k_\ell=0, i_\ell=r} \alpha_\ell y^{j_\ell+1} \neq 0$. Multiplying both sides of the equation

$$\left(\sum_{\ell:k_\ell=0, i_\ell>r} \alpha_\ell z^{i_\ell - r} y^{j_\ell+1} + \sum_{\ell:k_\ell=0, i_\ell=r} \alpha_\ell y^{j_\ell+1} \right) \cdot v = 0$$

by y on the left yields

$$\left(\sum_{\ell:k_\ell=0, i_\ell=r} \alpha_\ell y^{j_\ell+2} \right) \cdot v = 0,$$

contradicting that $f(y) \cdot v \neq 0$, for all nonzero $f(y) \in K[y]$. □

5.2.3 Classification of Prime Ideals

Proposition 5.2.12. *Any nonzero prime ideal of R not containing y contains $\langle \lambda zx - \lambda + y \rangle$ for some nonzero $\lambda \in K$.*

Proof. Let P be a prime ideal of R not containing y. By Lemma 5.2.5, there exists a $\lambda \in K$ such that $y^2 - \lambda y \in P$. If $\langle \lambda zx - \lambda + y \rangle \langle y \rangle \subseteq P$, then, because P is prime and P does not contain y, the ideal $\langle \lambda zx - \lambda + y \rangle \subseteq P$. Let $a \in \langle \lambda zx - \lambda + y \rangle \langle y \rangle$. Then a is equal to a finite sum of elements of the form

$$r_1(\lambda zx - \lambda + y)(z^i y^j x^k)(y) r_2,$$

for some elements r_1 and $r_2 \in R$ and some nonnegative integers i, j, and k. If $k > 0$, then

$$r_1(\lambda zx - \lambda + y)(z^i y^j x^k)(y) r_2 = 0 \in P.$$

If $k = 0$ and $i > 0$, then

$$r_1(\lambda zx - \lambda + y)(z^i y^j x^k) y r_2 = r_1(\lambda zx - \lambda + y)(z^i y^{j+1}) r_2$$

$$= r_1(\lambda z^i y^{j+1} - \lambda z^i y^{j+1} + y z^i y^{j+1}) r_2 = 0 \in P.$$

If $k = 0$ and $i = 0$, then

$$r_1(\lambda zx - \lambda + y)(z^i y^j x^k) y r_2 = r_1(\lambda zx - \lambda + y)(y^{j+1}) r_2$$

$$= r_1(\lambda zxy^{j+1} - \lambda y^{j+1} + y^{j+2}) r_2 = r_1(-\lambda y^{j+1} + y^{j+2}) r_2 \in \langle y^2 - \lambda y \rangle \subseteq P.$$

Hence, $a \subseteq P$. □

Thus a classification of the prime ideals of $R/\langle \lambda zx - \lambda - y \rangle$ for all $\lambda \in K$ will complete the classification of the prime ideals of R.

Proposition 5.2.13. *All nonzero ideals of R properly containing $\langle \lambda zx - \lambda + y \rangle$ contain y.*

Proof. Let P be a nonzero ideal of $R/\langle \lambda zx - \lambda + y \rangle$. Suppose $a \neq 0 \in P$. Then, as in the proof of Proposition 5.2.9, a can be written in the form

$$a = \sum_{\ell_1=0}^{m_1} \alpha_{\ell_1} z^{i_{\ell_1}} y x^{k_{\ell_1}} + \sum_{\ell_2=0}^{m_2} \alpha_{\ell_2} z^{i_{\ell_2}} y + \sum_{\ell_3=0}^{m_3} \alpha_{\ell_3} y x^{k_{\ell_3}}$$

$$+ \sum_{\ell_4=0}^{m_4} \alpha_{\ell_4} z^{i_{\ell_4}} + \sum_{\ell_5=0}^{m_5} \alpha_{\ell_5} x^{k_{\ell_5}} + \lambda_1 y + \lambda_2,$$

with $\ell_n, k_n > 0$ for all $1 \leq n \leq 5$, $\alpha_n \in K$ for all $1 \leq n \leq 5$, and $\lambda_1, \lambda_2 \in K$.

Case 1: Suppose $\lambda_1 \neq 0$ and $\lambda_2 = 0$. Then $yay = \lambda_1 y \in P$, using the fact that $\lambda_1 y^2 = \lambda_1 y$ in $R/\langle \lambda zx - \lambda - y \rangle$.

Case 2: Suppose $\lambda_1 = 0$ and $\lambda_2 \neq 0$. Then $yay = \lambda_2 y \in P$.

Case 3: Suppose $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Let γ be strictly greater than the highest exponent of z appearing in a. Then

$$x^\gamma a = \sum_{\ell_4=0}^{m_4} \alpha_{\ell_4} x^{\gamma - i_{\ell_4}} + \sum_{\ell_5=0}^{m_5} x^{\gamma + k_{\ell_5}} + \lambda_2 x^\gamma.$$

Therefore,

$$x^\gamma a z^\gamma = \sum_{\ell_4=0}^{m_4} \alpha_{\ell_4} z^{i_{\ell_4}} + \sum_{\ell_5=0}^{m_5} x^{k_{\ell_5}} + \lambda_2.$$

Thus, $y x^\gamma a z^\gamma y = \lambda_2 \in P$.

Case 4: Suppose $\lambda_1 = 0$ and $\lambda_2 = 0$.

Subcase a: Suppose $\sum_{\ell_5=0}^{m_5} \alpha_{\ell_5} x^{k_{\ell_5}} \neq 0$. Let γ be as in Case 2. Then

$$x^\gamma a = \sum_{\ell_4=0}^{m_4} \alpha_{\ell_4} x^{\gamma - i_{\ell_4}} + \sum_{\ell_5=0}^{m_5} \alpha_{\ell_5} x^{\gamma + k_{\ell_5}}.$$

Let δ be the minimum exponent of x appearing in any nonzero summand of $x^\gamma a$. Then

$$z^\delta x^\gamma a = \sum_{\ell_4 : \gamma - i_{\ell_4} > \delta}^{m_4} \alpha_{\ell_4} x^{\gamma - i_{\ell_4} - \delta} + \sum_{\ell_5=0}^{m_5} \alpha_{\ell_5} x^{\gamma + k_{\ell_5} - \delta} + \epsilon,$$

for some $\epsilon \in K$. Finally, $z^\delta x^\gamma a y = \epsilon y \in P$.

Subcase b: Suppose $\sum_{\ell_4=0}^{m_4} \alpha_{\ell_4} z^{i_{\ell_4}} \neq 0$. Then we can proceed similarly to Subcase a to get $y \in P$.

Subcase c: Suppose

$$\sum_{\ell_5=0}^{m_5} \alpha_{\ell_5} x^{k_{\ell_5}} = 0 = \sum_{\ell_4=0}^{m_4} \alpha_{\ell_4} z^{i_{\ell_4}}.$$

Subsubcase i: Suppose $\sum_{\ell_2=0}^{m_2} \alpha_{\ell_2} z^{i_{\ell_2}} y \neq 0$. Then

$$ay = \sum_{\ell_2=0}^{m_2} \alpha_{\ell_2} z^{i_{\ell_2}} y.$$

Let δ be the minimum exponent of z appearing in any nonzero summand of ay. Then

$$x^\delta a y = \sum_{\ell_2 : \ell_2 > \delta}^{m_2} \alpha_{\ell_2} z^{i_{\ell_2}} y + \epsilon y,$$

for some $\epsilon \in K$. Hence, $yx^\delta ay = \epsilon y \in P$.

Subsubcase ii: Suppose $\sum_{\ell_3=0}^{m_3} \alpha_{\ell_3} yx^{k_{\ell_3}} \neq 0$. Proceed similarly to Subsubcase i.

Subsubcase iii: Suppose

$$\sum_{\ell_2=0}^{m_2} \alpha_{\ell_2} z^{i_{\ell_2}} y = 0 = \sum_{\ell_3=0}^{m_3} \alpha_{\ell_3} yx^{k_{\ell_3}}.$$

Let δ be the minimum exponent of z appearing in any nonzero summand of a. Then

$$x^\delta a = \sum_{\ell_1 : i_{\ell_1} > \delta}^{m_1} \alpha_{\ell_1} z^{i_{\ell_1} - \delta} yx^{k_{\ell_1}} + \sum_{\ell_1 : i_{\ell_1} = \delta}^{m_1} \alpha_{\ell_1} yx^{k_{\ell_1}}.$$

Hence,

$$yx^\delta a = \sum_{\ell_1 : i_{\ell_1} = \delta}^{m_1} \alpha_{\ell_1} yx^{k_{\ell_1}}$$

and we can then proceed as in Subsubcase ii to get $y \in P$.

Therefore, any ideal of $R/\langle \lambda zx - \lambda - y \rangle$ contains y. □

Thus, the only nonzero prime ideals of R are those containing y and those of the form $\langle \lambda zx - \lambda + y \rangle$ for $\lambda \neq 0 \in K$. Therefore, we may consider the classification of the prime ideals, primitive ideals, and irreducible representations of $K\{x_1, \ldots, x_n\}/\langle x_i x_j - \beta_{ij}, i < j \rangle$, where the $\beta_{ij} \in K$, complete.

REFERENCES CITED

[1] Alev, J. and Cauchon, G., editors. *Algèbre non commutative, groupes quantiques et invariants*, volume 2 of *Séminaires et Congrès [Seminars and Congresses]*, Paris, 1997. Société Mathématique de France.

[2] Berliner, A. Private communication.

[3] Brown, L. and Rechenberg, H. Paul Dirac and Werner Heisenberg—a partnership in science. In *Reminiscences about a great physicist: Paul Adrien Maurice Dirac*, pages 117–162. Cambridge Univ. Press, Cambridge, 1987.

[4] Cauchon, G. Centraliseurs dans les anneaux de polynômes différentiels formels et leurs corps de fractions. In *Ring theory (Granada, 1986)*, volume 1328 of *Lecture Notes in Math.*, pages 60–68. Springer, Berlin, 1988.

[5] Cauchon, G. Idéaux bilatères et centre des anneaux de polynômes de Ore sur les anneaux quasi-simples. In *Séminaire d'Algèbre Paul Dubreil 31ème année (Paris, 1977–1978)*, volume 740 of *Lecture Notes in Math.*, pages 397–407. Springer, Berlin, 1979.

[6] Cauchon, G. Idéaux et modules sur certaines algèbres filtrées. In *Mathematics*, CTHS: Bull. Sec. Sci., III, pages 105–123. Bib. Nat., Paris, 1981.

[7] Cauchon, G. Effacement des dérivations et spectres premiers des algèbres quantiques. *J. Algebra*, 260(2):476–518, 2003.

[8] Dirac, P. *The collected works of P. A. M. Dirac: 1924–1948.* Cambridge University Press, Cambridge, 1995. Edited and with a preface by R. H. Dalitz.

[9] Dixmier, J. Sur les algèbres de Weyl. *Bull. Soc. Math. France*, 96:209–242, 1968.

[10] Dixmier, J. *Enveloping algebras*, volume 11 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 1996. Revised reprint of the 1977 translation.

[11] Gerritzen, L. Modules over the algebra of the noncommutative equation $yx = 1$. *Arch. Math. (Basel)*, 75(2):98–112, 2000.

[12] Goodearl, K. Prime ideals in skew polynomial rings and quantized Weyl algebras. *J. Algebra*, 150(2):324–377, 1992.

[13] Goodearl, K. and Letzter, E. Prime ideals in skew and q-skew polynomial rings. *Mem. Amer. Math. Soc.*, 109(521):vi+106, 1994.

[14] Goodearl, K. and Letzter, E. Prime and primitive spectra of multiparameter quantum affine spaces. In *Trends in ring theory (Miskolc, 1996)*, volume 22 of *CMS Conf. Proc.*, pages 39–58. Amer. Math. Soc., Providence, RI, 1998.

[15] Goodearl, K. and Letzter, E. The Dixmier-Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras. *Trans. Amer. Math. Soc.*, 352(3):1381–1403, 2000.

[16] Goodearl, K. and Warfield, R. *An introduction to noncommutative Noetherian rings*, volume 16 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 1989.

[17] Irving, R. Prime ideals of Ore extensions over commutative rings. *J. Algebra*, 56(2):315–342, 1979.

[18] Irving, R. Prime ideals of Ore extensions over commutative rings. II. *J. Algebra*, 58(2):399–423, 1979.

[19] Jacobson, N. A note on non-commutative polynomials. *Ann. of Math. (2)*, 35(2):209–210, 1934.

[20] Jordan, D. Simple skew Laurent polynomial rings. *Comm. Algebra*, 12(1-2):135–137, 1984.

[21] Joseph, A. *Quantum groups and their primitive ideals*, volume 29 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1995.

[22] Krause, G. and Lenagan, H. *Growth of algebras and Gelfand-Kirillov dimension*, volume 22 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, revised edition, 2000.

[23] Leroy, A. and Matczuk, J. Prime ideals of Ore extensions. *Comm. Algebra*, 19(7):1893–1907, 1991.

[24] McConnell, J. and Robson, J. *Noncommutative Noetherian rings*. Pure and Applied Mathematics. John Wiley & Sons Ltd., Chichester, 1987. With the cooperation of L. W. Small, A Wiley-Interscience Publication.

[25] Noether, E. and Schmeidler, A. Moduln in nichtkommutativen Bereichen, insbesondere aus Differential - und Differenzenausdrucken. *Math. Zeitschrift*, 8:1–35, 1920.

[26] Ore, O. Theory of non-commutative polynomials. *Ann. of Math. (2)*, 34(3):480–508, 1933.

[27] Stafford, J. Noncommutative projective geometry. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 93–103, Beijing, 2002. Higher Ed. Press.

[28] Weyl, H. Der Zusammenhang zwischen der symmetrischen und der linearen Gruppe. *Ann. of Math. (2)*, 30(1-4):499–516, 1928/29.

APPENDIX A

64 CASES

This appendix contains a complete list of the 64 cases that occur when each α_i and β_j in

$$K\{x, y, z\}/\langle xy - \alpha_1 yx - \beta_1, xz - \alpha_2 xz - \beta_2, yz - \alpha_3 zy - \beta_3\rangle$$

is either a zero or nonzero element of the algebraically closed field K. References for previously studied and noetherian cases are given. For the remainder of this appendix, variables stand for their images in the quotient algebras and the α_i and β_j are assumed to be nonzero elements of K. The cases are divided as follows.

- **Cases 1 - 11:** These cases are discussed in Chapters 4 and 5.
- **Cases 12 - 17:** These cases are isomorphic to K.
- **Cases 18 - 26:** These cases are isomorphic to $K[x, y]/\langle yx - 1\rangle$.
- **Cases 27 - 32:** These cases are isomorphic to $K\{x, y\}/\langle yx - 1\rangle$.
- **Cases 33 - 41:** These cases are isomorphic to quantized Weyl algebras.
- **Cases 42 - 49:** These cases are noetherian Ore extensions of quantized Weyl algebras.

- **Cases 50 - 55:** The study of the prime ideal theory of these algebras reduces to known cases because every prime ideal of these algebras contains x or y.
- **Cases 56 - 64:** These cases are trivial.

Cases:

1. $K\{x, y, z\}/\langle xy - \beta_1, xz - \alpha_2 zx, yz - \alpha_3 zy\rangle$

   This algebra is discussed in Chapter 4.

2. $K\{x, y, z\}/\langle xy - \alpha_1 yx, xz - \alpha_2 zx, yz - \beta_3\rangle$

   This algebra is similar to the above case (discussed in Chapter 4).

3. $K\{x, y, z\}/\langle xy - \alpha_1 yx, xz - \beta_2, yz - \alpha_3 zy\rangle$

   This algebra is similar to the above two cases (discussed in Chapter 4).

4. $K\{x, y, z\}/\langle xy, xz, yz\rangle$

   This algebra is discussed in Chapter 5.

5. $K\{x, y, z\}/\langle xy - \beta_1, xz, yz\rangle$

   This algebra is discussed in Chapter 5.

6. $K\{x, y, z\}/\langle xy, xz - \beta_2, yz\rangle$

   This algebra is discussed in Chapter 5.

7. $K\{x, y, z\}/\langle xy, xz, yz - \beta_3\rangle$

   This algebra is discussed in Chapter 5.

8. $K\{x, y, z\}/\langle xy - \beta_1, xz - \beta_2, yz\rangle$

   This algebra is discussed in Chapter 5.

9. $K\{x, y, z\}/\langle xy - \beta_1, xz, yz - \beta_3\rangle$

   This algebra is discussed in Chapter 5.

10. $K\{x, y, z\}/\langle xy, xz - \beta_2, yz - \beta_3\rangle$

    This algebra is discussed in Chapter 5.

11. $K\{x, y, z\}/\langle xy - \beta_1, xz - \beta_2, yz - \beta_3\rangle$

    This algebra is discussed in Chapter 5.

12. $K\{x, y, z\}/\langle xy - \beta_1, xz - \beta_2, yz - \alpha_3 zy - \beta_3\rangle$

    This algebra is isomorphic to K as follows. The equation $yz = \alpha_3 zy + \beta_3$ holds in this algebra. Multiplying through by x on the left yields $xyz = \alpha_3 xzy + \beta_3 x$. This equation implies that $\beta_1 z = \alpha_3\beta_2 y + \beta_3 x$. Multiplying through by x on the left again yields $\beta_1\beta_2 = \alpha_3\beta_2\beta_1 + \beta_3 x^2$. Hence, $x^2 = (\beta_1\beta_2 - \alpha_3\beta_2\beta_1)/\beta_3$. Note that $x = 0$ implies that $\beta_2 = 0$, contradicting the assumption that $\beta_2 \neq 0$. Thus, $x = ((\beta_1\beta_2 - \alpha_3\beta_2\beta_1)/\beta_3)^{1/2}$ is nonzero and $y = z = ((\beta_1\beta_2 - \alpha_3\beta_2\beta_1)/\beta_3)^{-1/2}$. Hence, this algebra is isomorphic to K.

13. $K\{x, y, z\}/\langle xy - \alpha_1 yx - \beta_1, xz - \beta_2, yz - \beta_3\rangle$

    This algebra is isomorphic to K similarly to the previous case.

14. $K\{x, y, z\}/\langle xy - \beta_1, xz - \alpha_2 zx - \beta_2, yz - \alpha_3 zy - \beta_3\rangle$

    This algebra is isomorphic to K and discussed further in Chapter 3.

15. $K\{x, y, z\}/\langle xy - \alpha_1 yx - \beta_1, xz - \beta_2, yz - \alpha_3 zy - \beta_3\rangle$

    This algebra is isomorphic to K similarly to the above case (discussed in Chapter 3).

16. $K\{x, y, z\}/\langle xy - \alpha_1 yx - \beta_1, xz - \alpha_2 zx - \beta_2, yz - \beta_3\rangle$

    This algebra is isomorphic to K similarly to the above two cases (discussed in Chapter 3).

17. $K\{x, y, z\}/\langle xy - \beta_1, xz - \alpha_2 zx - \beta_2, yz - \beta_3\rangle$

    This algebra is isomorphic to K as follows. In this algebra, $yz = \beta_3$. Multiplying through by x on the left yields the equation $xyz = \beta_3 x$ which implies that $z = \beta_3/\beta_1 x$. Substituting $\beta_3/\beta_1 x$ for z in $xz = \alpha_2 zx + \beta_2$ yields $\beta_3/\beta_1 x^2 = \alpha_2\beta_3/\beta_1 x^2 + \beta_2$. This equation implies that $\beta_3/\beta_1(1-\alpha_2)x^2 = \beta_2$. If $\alpha_2 = 1$ then $\beta_2 = 0$, contradicting that β_2 is assumed to be nonzero. Hence $\alpha_2 \neq 1$ and $x = [\beta_2\beta_1/(\beta_3(1-\alpha_2)]^{1/2}$ is an element of K. Thus, $y = \beta_1[\beta_2\beta_1/(\beta_3(1-\alpha_2)]^{-1/2}$ and $z = \beta_3/\beta_1[\beta_2\beta_1/(\beta_3(1-\alpha_2)]^{1/2}$. These equations imply that this algebra is isomorphic to K.

18. $K\{x,y,z\}/\langle xy - \beta_1, xz - \alpha_2 zx, yz - \beta_3\rangle$

    This algebra is isomorphic to $K[x,y]/\langle yx - 1\rangle$ as follows. In this algebra, $yz = \beta_3$. Multiplying through by x on the left yields the equation $xyz = \beta_3 x$ which implies that $z = \beta_3/\beta_1 x$. Substituting $\beta_3/\beta_1 x$ for z in $yz = \beta_3$ yields $yx = \beta_1$. Hence this algebra is isomorphic to $K[x,y]/\langle yx - 1\rangle$.

19. $K\{x,y,z\}/\langle xy - \beta_1, xz - \beta_2, yz - \alpha_3 zy\rangle$

    This algebra is isomorphic to $K[x,y]/\langle yx - 1\rangle$ similarly to the above case.

20. $K\{x,y,z\}/\langle xy - \alpha_1 yx, xz - \beta_2, yz - \beta_3\rangle$

    This algebra is isomorphic to $K[x,y]/\langle yx - 1\rangle$ similarly to the above two cases.

21. $K\{x,y,z\}/\langle xy - \beta_1, xz - \alpha_2 zx, yz - \alpha_3 zy - \beta_3\rangle$

    This algebra is isomorphic to $K[x,y]/\langle yx - 1\rangle$ as follows. In this algebra, $yz = \alpha_3 zy + \beta_3$. Multiplying through by x on the left yields $xyz = \alpha_3 xzy + \beta_3 x$ implying that $\beta_1 z = \alpha_3\alpha_2\beta_1 z + \beta_3 x$. Hence $\beta_1(1-\alpha_3\alpha_2)z = \beta_3 x$. If $\alpha_3 = \alpha_2^{-1}$, then $x = 0$ implying that $\beta_1 = 0$, which contradicts the assumption that $\beta_1 \neq 0$. Hence $\alpha_3 \neq \alpha_2^{-1}$. Thus $z^2 = \alpha_2 z^2$ implying that $z = 0$ or $\alpha_2 = 1$. If $z = 0$ then $\beta_3 = 0$, which contradicts our assumption that β_3 is nonzero. Hence $\alpha_2 = 1$ and $z = (\beta_3/(\beta_1(1-\alpha_3)))x$. Substituting $(\beta_3/(\beta_1(1-\alpha_3)))x$ for z in $yz = \alpha_3 zy + \beta_3$ yields that $yx = \beta_1$. This equation implies that this algebra is isomorphic to $K[x,y]/\langle yx - 1\rangle$.

22. $K\{x, y, z\}/\langle xy - \beta_1, xz - \alpha_2 zx - \beta_2, yz - \alpha_3 zy\rangle$

This algebra is isomorphic to $K[x, y]/\langle yx - 1\rangle$ similarly to the above case.

23. $K\{x, y, z\}/\langle xy - \alpha_1 yx, xz - \beta_2, yz - \alpha_3 zy - \beta_3\rangle$

This algebra is isomorphic to $K[x, y]/\langle yx - 1\rangle$ similarly to the above two cases.

24. $K\{x, y, z\}/\langle xy - \alpha_1 yx, xz - \alpha_2 zx - \beta_2, yz - \beta_3\rangle$

This algebra is isomorphic to $K[x, y]/\langle yx - 1\rangle$ similarly to the above three cases.

25. $K\{x, y, z\}/\langle xy - \alpha_1 yx - \beta_1, xz - \beta_2, yz - \alpha_3 zy\rangle$

This algebra is isomorphic to $K[x, y]/\langle yx - 1\rangle$ similarly to the above four cases.

26. $K\{x, y, z\}/\langle xy - \alpha_1 yx - \beta_1, xz - \alpha_2 zx, yz - \beta_3\rangle$

This algebra is isomorphic to $K[x, y]/\langle yx - 1\rangle$ similarly to the above five cases.

27. $K\{x, y, z\}/\langle xy - \alpha_1 yx, xz - \beta_2, yz\rangle$

This algebra is isomorphic to $K\{x, y\}/\langle yx - 1\rangle$ as follows. In this algebra, $xy = \alpha_1 yx$. Multiplying through by z on the right yields $xyz = \alpha_1 yxz$ which implies that $0 = \alpha_1 \beta_2 y$. Hence this algebra is isomorphic to $K\{x, y\}/\langle yx - 1\rangle$ (see Theorem 2.2.14 and [11, section 3]).

28. $K\{x, y, z\}/\langle xy - \alpha_1 yx, xz, yz - \beta_3\rangle$

This algebra is isomorphic to $K\{x, y\}/\langle yx - 1\rangle$ similarly to the above case.

29. $K\{x, y, z\}/\langle xy - \beta_1, xz, yz - \alpha_3 zy\rangle$

This algebra is isomorphic to $K\{x, y\}/\langle yx - 1\rangle$ similarly to the above two cases.

30. $K\{x,y,z\}/\langle xy, xz-\beta_2, yz-\alpha_3 zy\rangle$

    This algebra is isomorphic to $K\{x,y\}/\langle yx-1\rangle$ similarly to the above three cases.

31. $K\{x,y,z\}/\langle xy-\beta_1, xz-\alpha_2 zx, yz\rangle$

    This algebra is isomorphic to $K\{x,y\}/\langle yx-1\rangle$ as follows. In this algebra $xy=\beta_1$. Multiplying by z on the right yields $xyz=\beta_1 z$. Since $yz=0$ in this algebra, the equation $\beta_1 z=0$ also holds. Hence $z=0$ and this algebra is isomorphic to $K\{x,y\}/\langle yx-1\rangle$.

32. $K\{x,y,z\}/\langle xy, xz-\alpha_2 zx, yz-\beta_3\rangle$

    This algebra is isomorphic to $K\{x,y\}/\langle yx-1\rangle$ similarly to the above case.

33. $K\{x,y,z\}/\langle xy-\alpha_1 yx-\beta_1, xz, yz\rangle$

    This algebra is isomorphic to a quantized Weyl algebra as follows. In this algebra, $xy=\alpha_1 yx+\beta_1$. Multiplying through by z on the right yields the equation $xyz=\alpha_1 yxz+\beta_1 z$ which implies that $\beta_1 z=0$. Hence, this algebra is isomorphic to a quantized Weyl algebra, which is discussed in Chapter 2.

34. $K\{x,y,z\}/\langle xy, xz-\alpha_2 zx-\beta_2, yz\rangle$

    This algebra is isomorphic to a quantized Weyl algebra, similarly to the above case (see Chapter 2).

35. $K\{x,y,z\}/\langle xy, xz, yz-\alpha_3 zy-\beta_3\rangle$

    This algebra is isomorphic to a quantized Weyl algebra, similarly to the above two cases (see Chapter 2).

36. $K\{x,y,z\}/\langle xy-\alpha_1 yx-\beta_1, xz-\alpha_2 zx, yz\rangle$

    This algebra is isomorphic to a quantized Weyl algebra, similarly to the above three cases.

37. $K\{x,y,z\}/\langle xy-\alpha_1 yx-\beta_1, xz, yz-\alpha_3 zy\rangle$

    This algebra is isomorphic to a quantized Weyl algebra, similarly to the above four cases.

38. $K\{x,y,z\}/\langle xy-\alpha_1 yx, xz-\alpha_2 zx-\beta_2, yz\rangle$

    This algebra is isomorphic to a quantized Weyl algebra, similarly to the above five cases.

39. $K\{x,y,z\}/\langle xy-\alpha_1 yx, xz, yz-\alpha_3 zy-\beta_3\rangle$

    This algebra is isomorphic to a quantized Weyl algebra, similarly to the above six cases.

40. $K\{x,y,z\}/\langle xy, xz-\alpha_2 zx-\beta_2, yz-\alpha_3 zy\rangle$

    This algebra is isomorphic to a quantized Weyl algebra as follows. In this algebra, $yz = \alpha_3 zy$. Multiplying through by x on the left yields $xyz = \alpha_3 xzy$ implying that $0 = \alpha_3(\alpha_2 zx + \beta_2)y = \alpha_3\alpha_2 zxy + \alpha_3\beta_2 y = \alpha_3\beta_2 y$. Hence this algebra is isomorphic to a quantized Weyl algebra, which is discussed in Chapter 2.

41. $K\{x,y,z\}/\langle xy, xz-\alpha_2 zx, yz-\alpha_3 zy-\beta_3\rangle$

    This algebra is isomorphic to a quantized Weyl algebra, similarly to the above case.

42. $K\{x,y,z\}/\langle xy-\alpha_1 yx-\beta_1, xz-\alpha_2 xz-\beta_2, yz-\alpha_3 zy-\beta_3\rangle$

    This algebra is a noetherian Ore extension of a quantized Weyl algebra (see [13]).

43. $K\{x,y,z\}/\langle xy-\alpha_1 yx, xz-\alpha_2 zx-\beta_2, yz-\alpha_3 zy-\beta_3\rangle$

    This algebra is a noetherian Ore extension of a quantized Weyl algebra (see [13]).

44. $K\{x,y,z\}/\langle xy-\alpha_1 yx-\beta_1, xz-\alpha_2 zx, yz-\alpha_3 zy-\beta_3\rangle$

This algebra is a noetherian Ore extension of a quantized Weyl algebra (see [13]).

45. $K\{x,y,z\}/\langle xy-\alpha_1yz-\beta_1, xz-\alpha_2zx-\beta_2, yz-\alpha_3zy\rangle$

    This algebra is a noetherian Ore extension of a quantized Weyl algebra (see [13]).

46. $K\{x,y,z\}/\langle xy-\alpha_1yz-\beta_1, xz-\alpha_2zx, yz-\alpha_3zy\rangle$

    This algebra is a noetherian Ore extension of a quantized Weyl algebra, and is discussed more in Chapter 3.

47. $K\{x,y,z\}/\langle xy-\alpha_1yx, xz-\alpha_2zx, yz-\alpha_3zy-\beta_3\rangle$

    This algebra is a noetherian Ore extension of a quantized Weyl algebra similar to the above case (see Chapter 3).

48. $K\{x,y,z\}/\langle xy-\alpha_1yx, xz-\alpha_2zx-\beta_2, yz-\alpha_3zy\rangle$

    This algebra is a noetherian Ore extension of a quantized Weyl algebra similar to the above two cases (see Chapter 3).

49. $K\{x,y,z\}/\langle xy-\alpha_1yx, xz-\alpha_2zx, yz-\alpha_3zy\rangle$

    This algebra is multiparameter quantum three space (see [14]).

50. $K\{x,y,z\}/\langle xy, xz, yz-\alpha_3zy\rangle$

    The study of the prime ideal theory of this algebra reduces to known cases and is discussed in Chapter 3.

51. $K\{x,y,z\}/\langle xy, xz-\alpha_2zx, yz\rangle$

    The study of the prime ideal theory of this algebra reduces to known cases similarly to the above case (discussed in Chapter 3).

52. $K\{x,y,z\}/\langle xy-\alpha_1yx, xz, yz\rangle$

    The study of the prime ideal theory of this algebra reduces to known cases similarly to the above two cases (discussed in Chapter 3).

53. $K\{x, y, z\}/\langle xy - \alpha_1 yx, xz - \alpha_2 zx, yz\rangle$

The study of the prime ideal theory of this algebra reduces to known cases similarly to the above three cases (discussed in Chapter 3).

54. $K\{x, y, z\}/\langle xy - \alpha_1 yx, xz, yz - \alpha_3 zy\rangle$

The study of the prime ideal theory of this algebra reduces to known cases similarly to the above four cases (discussed in Chapter 3).

55. $K\{x, y, z\}/\langle xy, xz - \alpha_2 zx, yz - \alpha_3 zy\rangle$

The study of the prime ideal theory of this algebra reduces to known cases similarly to the above five cases (discussed in Chapter 3).

56. $K\{x, y, z\}/\langle xy - \beta_1, xz, yz - \alpha_3 zy - \beta_3\rangle$

This algebra is trivial as follows. In this algebra, $yz = \alpha_3 zy + \beta_3$. Multiplying through by x on the left yields $xyz = \alpha_3 xzy + \beta_3 x$ implying that $\beta_1 z = \beta_3 x$. This equation implies that $\beta_1/\beta_3 z^2 = 0$ and hence that $z = 0$. If $z = 0$ then $\beta_3 = 0$, contradicting the assumption that $\beta_3 \neq 0$.

57. $K\{x, y, z\}/\langle xy - \alpha_1 yx - \beta_1, xz, yz - \beta_3\rangle$

This algebra is trivial similarly to the above case.

58. $K\{x, y, z\}/\langle xy, xz - \beta_2, yz - \alpha_3 zy - \beta_3\rangle$

This algebra is trivial similarly to the above two cases.

59. $K\{x, y, z\}/\langle xy - \alpha_1 yx - \beta_1, xz - \beta_2, yz\rangle$

This algebra is trivial similarly to the above three cases.

60. $K\{x, y, z\}/\langle xy, xz - \alpha_2 zx - \beta_2, yz - \alpha_3 zy - \beta_3\rangle$

This algebra is trivial similarly to the above four cases.

61. $K\{x, y, z\}/\langle xy - \alpha_1 yx - \beta_1, xz, yz - \alpha_3 zy - \beta_3\rangle$

This algebra is trivial similarly to the above five cases.

62. $K\{x,y,z\}/\langle xy-\alpha_1 yx-\beta_1, xz-\alpha_2 zx-\beta_2, yz\rangle$

    This algebra is trivial similarly to the above six cases.

63. $K\{x,y,z\}/\langle xy-\beta_1, xz-\alpha_2 zx-\beta_2, yz\rangle$

    This algebra is trivial as follows. The equation $xy=\beta_1$ holds in this algebra. Multiplying through by z on the right yields $xyz=\beta_1 z$. Since $yz=0$ holds in this algebra, $z=0$, implying that $\beta_2=0$ contradicting the assumption that $\beta_2\neq 0$.

64. $K\{x,y,z\}/\langle xy, xz-\alpha_2 zx-\beta_2, yz-\beta_3\rangle$

    This algebra is trivial similarly to the above case.