Temple University
Doctoral Dissertation
Submitted to the Graduate Board

Title of Dissertation: (Please type) Polynomial Detection of Matrix Subalgebras

Author: (Please type) Alberto Daniel Birmajer

Date of Defense: (Please type) May 2, 2003

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Date Submitted to Graduate Board:

Accepted by the Graduate Board of Temple University in partial fulfillment of the requirements for the degree of **Doctor of Philosophy.**

Date

(Dean of the Graduate School)

Polynomial detection of matrix subalgebras

A Dissertation
Submitted to
the Temple University Graduate Board

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in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

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by
Alberto Daniel Birmajer
May, 2003

UMI Number: 3097674

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ABSTRACT

Polynomial detection of matrix subalgebras

Alberto Daniel Birmajer

DOCTOR OF PHILOSOPHY

Temple University, May, 2003

Professor Edward Letzter, Chair

In this dissertation we present some results on polynomial identities, along with their applications to algorithmic representation theory.

The Amitsur-Levitski theorem asserts that $M_n(F)$ satisfies a polynomial identity of degree $2n$. (Here, F is a field and $M_n(F)$ is the algebra of $n \times n$ matrices over F). It is easy to give examples of subalgebras of $M_n(F)$ that do satisfy an identity of lower degree and subalgebras of $M_n(F)$ that satisfy no polynomial identity of degree $\leq 2n-1$. In this dissertation we give a full classification of the subalgebras of $n \times n$ matrices that satisfy no nonzero polynomial of degree less that $2n$.

Second, the double Capelli polynomial of total degree $2t$ is

$$\sum \left\{ (\operatorname{sg} \sigma\tau) X_{\sigma(1)} Y_{\tau(1)} X_{\sigma(2)} Y_{\tau(2)} \cdots X_{\sigma(t)} Y_{\tau(t)} \middle| \, \sigma, \tau \in S_t \right\}.$$

Formanek pointed out that the double Capelli polynomial of total degree $4n-2$

is not a polynomial identity for $M_n(F)$. Later, Giambruno-Sehgal and Chang proved that the double Capelli polynomial of total degree $4n$ is a polynomial identity for $M_n(F)$. We show that the double Capelli polynomial of total degree $4n-2$ is a polynomial identity for any proper F-subalgebra of $M_n(F)$. Subsequently, we construct polynomial tests for nonsplit non-self extensions of full matrix algebras. Then we use these results to construct effective algorithmic procedures in representation theory of finitely presented algebras, expanding on ideas found in the work by Letzter [**Le01**] and [**Le02**].

Finally, the algorithmic complexity of the proposed procedures leads to the so-called Paz conjecture. In the last chapter we study a specific example along these lines.

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my advisor, Professor Edward Letzter, for his constant support and guidance throughout the preparation of this thesis. I would also like to thank him for recommending this problem to me, and for being so generous with his time, insight, experience, and valuable suggestions.

Professor Boris Datskovsky has helped me in many ways. I have benefitted greatly from his expertise and encouragement during my studies at Temple University.

I am particularly indebted to Professor Martin Lorenz for all the algebra that I have learned from his insightful and profound lectures. He has been a source of good advice and mathematics.

I want to thank Professor Ching-Li Chai, from the University of Pennsylvania, for his time and courtesy.

At this time I would also like to acknowledge all of the other members of the Temple faculty who have made my time here productive and enjoyable. I am particularly grateful to Professor Weih Shih Yang, for helping me develop my interest in probability theory. I would also like to thank Professors Eric Grinberg and Jack Schiller who had a great impact on my early studies and first years of graduate work.

In memory of my father, Boris Birmajer.

He gave me courage and conviction.

To my mother, Ana Perla Trau de Birmajer.

She gave life and gives me passion.

To my wife, Susana Hild de Birmajer.

She gives me love and fulfillment.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

This dissertation is concerned with the study of polynomial identity algebras (PI-algebras) and algorithmic methods applicable to representation theory. The theory of multilinear polynomial identities has played a prominent role throughout modern noncommutative algebra, beginning with a 1948 article of Kaplansky [**Ka48**] and the Amitsur-Levitski theorem in 1950 (see [**AL50**]).

Let F be the underlying field of an algebra A and $X_1, \ldots, X_t$ a set of noncommuting indeterminates. Consider a polynomial $f(X_1, \ldots X_t)$ over F, that is, an element of the free algebra $F\{X_1, \ldots X_t\}$ generated by the indeterminates X_i over the field F. If this polynomial is not identically zero and if the equation

$$f(r_1, \ldots, r_t) = 0 \tag{1.1}$$

is satisfied by all choices of elements $r_1, \ldots r_t$ in A, then we say that the polynomial identity (1.1) holds in A. Such identities are satisfied, for example, by every commutative algebra, every algebraic algebra of bounded degree, and every finite dimensional algebra. The reader is referred to [**Fo91**] and [**Ro80**] for a comprehensive treatment of the subject.

Kaplansky (see [**Ka48**]) showed that if A satisfies a polynomial identity of degree d, then it satisfies a multilinear polynomial of degree d. This is true in any characteristic and reduces the study of polynomial identities to multilinear ones. Two of the most famous multilinear polynomials are the *standard polynomials* and the *Capelli polynomials.*

The standard polynomial of degree t is

$$s_t(X_1, \ldots, X_t) = \sum_{\sigma \in S_t} (\mathrm{sg}\sigma) X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(t)},$$

where S_t is the symmetric group on $\{1, \ldots, t\}$ and $(\mathrm{sg}\sigma)$ is the sign of the permutation $\sigma \in S_t$. The standard polynomial s_t is homogeneous of degree t, multilinear and alternating.

Let $M_n(F)$ denote the algebra of $n \times n$ matrices over a field F. The Amitsur-Levitski theorem asserts that $M_n(F)$ satisfies any standard polynomial of degree $2n$ or higher. A short and elegant proof of this theorem was obtained by Rosset, using an exterior algebra over F in an ingenious way [**Ro76**]. The standard polynomial s_{2n} is a minimal identity for the $n \times n$ matrices, in the sense that $M_n(F)$ satisfies no polynomial identity of degree less than $2n$. More generally, if A is a subalgebra of $M_n(F)$ isomorphic to a full block upper triangular matrix algebra,

$$\begin{pmatrix} \boxed{*} & & & * \\ & \boxed{*} & & \\ & & \ddots & \\ & 0 & & \boxed{*} \end{pmatrix},$$

then A satisfies no polynomial identity of degree less than $2n$. To prove this assertion, note that every full block upper triangular matrix algebra contains the "staircase sequence" of $2n-1$ matrix units

$$e_{11}, e_{12}, e_{22}, e_{23}, \ldots, e_{(n-1)(n-1)}, e_{(n-1)n}, e_{nn},$$

which gives a nonzero product, e_{1n}, only when multiplied in the given order. It follows that any full block upper triangular matrix algebra of $M_n(F)$ satisfies no multilinear polynomial of degree $\leq 2n-1$.

When studying the class of finite dimensional algebras over a field, one encounters the following question: Suppose A is a subalgebra of $M_n(F)$. Obviously A satisfies the standard identity of degree $2n$, however, s_{2n} does not need to be a minimal identity for A. For instance, the algebra of diagonal matrices is commutative, thus, satisfies the identity $X_1X_2 - X_2X_1$. Therefore, it is natural to ask whether one can give a full characterization of the subalgebras of $n \times n$ matrices not satisfying an identity of degree $2n-1$. In Chapter

2 we give a complete solution to this question by proving a theorem that can be viewed as a converse of the Amitsur-Levitski identity:

Theorem 1.0.1 *If a matrix subalgebra of $M_n(F)$ does not satisfy the standard polynomial s_{2n-2}, then it is isomorphic, as F-algebra, to a full block upper triangular matrix algebra.*

In Chapter 3 we discuss *polynomial tests* (p-tests) for algebras. This definition is our own, but the idea of a "polynomial test" has already been noted, for example, in Rowen [**Ro80**]. A polynomial $f(X_1, \ldots, X_t) \in F\{X\}$ is a *polynomial test* for an F-algebra R if it is not an identity for R, but is an identity for every proper F-subalgebra of R.

A consequence of Theorem 1.0.1 is that the standard polynomial s_{2n-2} is a polynomial test for the algebra $U_n(F)$ of upper triangular matrices, for each n, for all fields F.

The *Capelli polynomials* are defined by

$$c_{2t-1}(X_1, \ldots, X_t, Y_1, \ldots, Y_{t-1}) = \sum_{\sigma \in S_t} (\mathrm{sg}\sigma) X_{\sigma(1)} Y_1 X_{\sigma(2)} Y_2 \cdots X_{\sigma(t-1)} Y_{t-1} X_{\sigma(t)}$$

and

$$c_{2t}(X_1, \ldots, X_t, Y_1, \ldots, Y_t) = c_{2t-1}(X_1, \ldots, X_t, Y_1, \ldots, Y_{t-1}) Y_t.$$

The Capelli polynomials were introduced by Razmylov in [**Ra74**] and have important applications in PI-theory, in particular, the development of central polynomials for $M_n(F)$ by Razmylov [**Ra74**]. A polynomial of positive degree $f \in F\{X\}$, is *central* for a ring R if it is not an identity for R, and if the values of $f(R)$ lie in the center of R. The original central polynomials for $M_n(F)$ were discovered by Formanek in 1972 [**Fo72**]. Central polynomials provide a link between PI-theory and commutative ring theory, and led to a revolution in the subject through the application of classical methods of commutative ring theory.

It is well known that c_{2n^2} is a polynomial test for $M_n(F)$ (cf. [**Fo91**], Proposition 29). Furthermore, central polynomials for $M_n(F)$ are also polynomial tests for $M_n(F)$ (cf. §§3.2.1). Polynomial tests may play a role in the algorithmic representation theory of finitely presented algebras over a computable field. Studies, from an algorithmic perspective, on matrix representations of finitely presented algebras appear in [**Le01**] and [**Le02**]. In this setting, the question of efficiency and algorithmic complexity is crucial, and the question naturally arises of looking for a polynomial test of minimal degree for $M_n(F)$. To lower the degree, we need to go from the Capelli polynomial to the double Capelli polynomial.

The *double Capelli polynomials* are defined by

$$h_{2t-1}(X_1,\ldots,X_t,Y_1,\ldots,Y_{t-1}) = \sum_{\sigma\in S_t,\tau\in S_{t-1}} (\mathrm{sg}\sigma\tau) X_{\sigma(1)}Y_{\tau(1)}X_{\sigma(2)}Y_{\tau(2)}\cdots X_{\sigma(t-1)}Y_{\tau(t-1)}X_{\sigma(t)},$$

and

$$h_{2t}(X_1, \ldots, X_t, Y_1, \ldots, Y_t)$$
$$= \sum_{\sigma,\tau \in S_t} (\mathrm{sg}\sigma\tau) X_{\sigma(1)} Y_{\tau(1)} X_{\sigma(2)} Y_{\tau(2)} \cdots X_{\sigma(t-1)} Y_{\tau(t-1)} X_{\sigma(t)} Y_{\tau(t)}.$$

Formanek pointed out that h_{4n-2} is not a polynomial identity for $M_n(F)$ and asked for the least integer m such that h_m is a polynomial identity for $M_n(F)$. Chang [**CH88**] proved that both double Capelli polynomials h_{2t-1} and h_{2t} are consequences of the standard polynomial s_t, implying that h_{4n-1} and h_{4n} are polynomial identities for $M_n(F)$. That h_{4n} is a polynomial identity for $M_n(F)$ was also proved by Giambruno-Sehgal [**GS89**] using a variation of Rosset's method.

The second basic result presented in this dissertation is the following.

Theorem 1.0.2 *h_{4n-2} is an identity for any proper subalgebra of $M_n(F)$.*

It follows from this theorem that the double Capelli polynomial of total degree $4n-2$ is a polynomial test for $M_n(F)$. Following this Theorem, P-tests for nonsplit non-self extensions of full matrix algebras are explicitely constructed in § 3.3.

Chapter 4 contains a detailed discussion of the application of the results obtained in Chapter 2 and Chapter 3 to algorithmic representation theory. More precisely, algorithmic procedures are presented to determine the existence or not of certain types of finite dimensional representations of a finitely presented (but not necessarily finitely dimensional) associative algebra over a computable field. If $R = F\{X_1, \ldots, X_s\}/\langle f_1, \ldots, f_t\rangle$ is a finitely presented algebra over a filed F, then it is easy to see that the n-dimensional representations of R amount to solutions to a system of tn^2 commutative polynomial equations in sn^2 variables. Moreover, n-dimensional irreducible representations and full block upper triangular representations of R can also be explicitely parameterized by finite systems of commutative polynomial equations using P-tests. Consequently, the techniques of computational algebraic geometry (and in particular, Groebner basis methods) can be used to study the n-dimensional representation theory of R. When the desired n-dimensional representation exists, it is possible (in principle) to produce explicit constructions. Examples of these algorithmic procedures are implemented in §§4.3.5, using the computer algebra package Macaulay2.

Considerations of the complexity of these algorithms leads to another topic, presented in Chapter 5. Let F be a field, and let A be a finite-dimensional F-algebra. Set $d = dim_F A$. Since A is finite-dimensional over F, it is obviously finitely generated. Let S be a finite generating set for A as an F-algebra. We shall write $A = F\{S\}$ to denote this. Writing $S = \{a_1, \ldots, a_t\}$ we shall adopt the convention that 1 is a word in S of length zero, and write S^i for the set of all words in S of length $\leq i$. We have the obvious containment $S^i \subseteq S^j$ for $i \leq j$, also $S^i S^j = S^{i+j}$. Writing FS^i for the F-linear span of S^i, we have the following chain of containments (noting that $S^0 = 1$, so $FS^0 = F$):

$$F = FS^0 \subseteq FS^1 \subseteq \cdots \subseteq FS^i \subseteq FS^{i+1} \subseteq \cdots \subseteq F\{S\} = A. \tag{1.2}$$

Since A is assumed finite-dimensional over F, there is an integer k such that

$$FS^k = FS^{k+1} = FS^{k+2} = \cdots = F\{S\} = A. \tag{1.3}$$

We define the *length* of the generating set, written $\ell(S)$, to be the smallest k for which $FS^k = A$, and define $\ell = \max_S \ell(S)$, where the maximum is taken over all finite generating sets, to be the length of A. For the algebra of $n \times n$ matrices over F, Pappacena ([**Pa97**]) has proved that ℓ is bounded above by a function in $O(n^{3/2})$ and Paz ([**Paz84**]) has conjectured that $\ell \leq 2n - 2$.

In our last result in this dissertation, it is demonstrated that the length of the set

$$S = \{\mathrm{SupDiag}_n, \mathrm{SubDiag}_n\}$$

is n (it is easy to verify that S generate $M_n(F)$ as a F-algebra). I hope that the ideas presented in the proof of this result serve to continue to study other examples along these lines, gathering data on this difficult problem.

For the convenience of the reader, each chapter is self contained and starts with background material and preliminary results needed for the statements and proofs developed therein.

CHAPTER 2

On subalgebras of $n \times n$ matrices not satisfying identities of degree $2n - 2$.

The Amitsur-Levitski theorem asserts that $M_n(F)$ satisfies a polynomial identity of degree $2n$. (Here, F is a field and $M_n(F)$ is the algebra of $n \times n$ matrices over F). It is easy to give examples of subalgebras of $M_n(F)$ that do satisfy an identity of lower degree and subalgebras of $M_n(F)$ that satisfy no polynomial identity of degree $\leq 2n - 1$. In this chapter we give a full classification of the subalgebras of $n \times n$ matrices that satisfy no nonzero polynomial of degree less that $2n$.

2.1 Introduction

Let F be a field, $M_n(F)$ the algebra of $n \times n$ matrices over F, and $F\{X\} = F\{X_1, X_2, \dots\}$ the free associative algebra over F in countably many variables.

A nonzero polynomial $f(X_1, \dots X_m) \in F\{X\}$ is a *polynomial identity* for an F-algebra R (or, R *satisfies* f) if $f(r_1, \dots, r_m) = 0$ for all $r_1, \dots, r_m \in R$.

It is well known that if R satisfies a polynomial of degree d, then it satisfies a multilinear polynomial of degree d. The study of identities for R therefore reduces to the multilinear case.

The *standard polynomial* of degree t is

$$s_t(X_1, \dots, X_t) = \sum_{\sigma \in S_t} (\mathrm{sg}\sigma) X_{\sigma(1)} X_{\sigma(2)} \dots X_{\sigma(t)},$$

where S_t is the symmetric group on $\{1, \dots, t\}$ and $(\mathrm{sg}\sigma)$ is the sign of the permutation $\sigma \in S_t$. The standard polynomial s_t is homogeneous of degree t, multilinear and alternating. If t is odd then $s_t(1, X_2, \dots, X_t) = s_{t-1}(X_2, \dots, X_t)$. Thus s_{2t} is an identity of R if and only if s_{2t+1} is an identity of R.

The Amitsur-Levitski theorem asserts that $M_n(F)$ satisfies any standard polynomial of degree $2n$ or higher. Moreover, if $M_n(F)$ satisfies a polynomial of degree $2n$, then it is a scalar multiple of s_{2n} (cf. [**AL50**]).

The standard polynomial s_{2n} is a minimal identity in the sense that $M_n(F)$ satisfies no polynomial identity of degree less than $2n$. More generally, if A

is a subalgebra of $M_n(F)$ isomorphic to a full block upper triangular matrix algebra,

$$\begin{pmatrix} \boxed{*} & & * & \\ & \boxed{*} & & \\ & & \ddots & \\ & 0 & & \boxed{*} \end{pmatrix},$$

then A satisfies no polynomial identity of degree less than $2n$. To prove this assertion, note that every full block upper triangular matrix algebra contains the "staircase sequence" $e_{11}, e_{12}, e_{22}, e_{23}, \ldots, e_{(n-1)(n-1)}, e_{(n-1)n}, e_{nn}$, and

$$s_{2n-1}\left(e_{11}, e_{12}, e_{22}, e_{23}, \ldots, e_{(n-1)(n-1)}, e_{(n-1)n}, e_{nn}\right) = e_{1n}, \tag{2.1}$$

where the e_{ij} are the standard matrix units.

The aim of this chapter is to present and prove a "converse" of the Amitsur-Levitski theorem:

Theorem: *If a matrix subalgebra of $M_n(F)$ does not satisfy a multilinear polynomial of degree $2n-2$, then it is isomorphic, as F-algebra, to a full block upper triangular matrix algebra.*

In §2.2 we provide the building blocks for the main theorem of this chapter and its proof. This proof and some of its consequences are presented in §2.3.

2.2 Building Blocks

Lemma 2.2.1 *Let A be a simple F-subalgebra of $M_n(F)$. Then either $A = M_n(F)$ or A satisfies the identity $s_{2n-2}(A) = 0$.*

Proof. By assumption, A is a a finite dimensional central simple algebra over its center k. Let K denote the algebraic closure of k; then $A \otimes_k K$ is a simple K-algebra in a natural way (cf. **[Ro80]**, §1.8), with $\dim_K (A \otimes_k K) = \dim_k(A)$. Also, $A \otimes_k K \cong M_t(K)$ for some $t \leq n$. Suppose that A is a proper subalgebra of $M_n(F)$. It follows that $t < n$. Hence, by the Amitsur-Levitski theorem, $A \otimes_k K$ satisfies s_{2n-2}, and the result follows since A is embedded as a k algebra in $A \otimes_k K$. □

Let ℓ, m be positive integers such that $\ell + m = n$ and set

$$E_{(\ell,m)}(F) = \begin{bmatrix} M_\ell(F) & M_{\ell \times m}(F) \\ 0 & M_m(F) \end{bmatrix},$$

an F-subalgebra of $M_n(F)$.

(i) Associated to $E_{(\ell,m)}(F)$ are canonical F-algebra homomorphisms

$$\pi_\ell \colon E_{(\ell,m)}(F) \to M_\ell(F) \quad \text{and} \quad \pi_m \colon E_{(\ell,m)}(F) \to M_m(F).$$

Further identify $M_\ell(F)$ and $M_m(F)$ with

$$\begin{bmatrix} M_\ell(F) & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & M_m(F) \end{bmatrix},$$

respectively.

(ii) Associated to a subalgebra A of $E_{(\ell,m)}(F)$ are homomorphic image subalgebras A_ℓ and A_m in $M_\ell(F)$ and $M_m(F)$ respectively.

(iii) Set

$$T_{(\ell,m)}(F) = \begin{bmatrix} 0 & M_{\ell\times m} \\ 0 & 0 \end{bmatrix},$$

the Jacobson radical of $E_{(\ell,m)}(F)$.

Lemma 2.2.2 *Let A be a subalgebra of $E_{(\ell,m)}(F)$ such that A_ℓ satisfies s_q for some $q \leq 2\ell$ and A_m satisfies s_r for some $r \leq 2m$. Then A satisfies s_{q+r}.*

Proof. Let $t = q+r$. As an F-vector space, $E_{(\ell,m)}(F) = M_\ell(F) \oplus T_{(\ell,m)}(F) \oplus M_m(F)$. Thus each matrix x in A can be written as $x = a+b+c$, with $a \in A_\ell$, $b \in T_{(\ell,m)}$ and $c \in A_m$. Using linearity, we expand completely $s_t(x_1, \ldots, x_t)$ and further use the following rules to simplify some of the terms:

1. $T_{(\ell,m)}(F)$ is a nilpotent ideal of $E_{(\ell,m)}(F)$, with $T^2_{(\ell,m)}(F) = 0$, and so each term in the expansion containing more than one entry in $T_{(\ell,m)}(F)$ equals 0.

2. $M_\ell(F)M_m(F) = M_m(F)M_\ell(F) = 0$.

3. $M_m(F)T_{(\ell,m)}(F) = T_{(\ell,m)}(F)M_\ell(F) = 0$.

We obtain

$$s_t(x_1, \ldots, x_n) = \sum_{i=0}^{t+1} \sum_{\sigma \in S_t} (\mathrm{sg}\sigma) a_{\sigma(1)} \cdots a_{\sigma(i-1)} b_{\sigma(i)} c_{\sigma(i+1)} \cdots c_{\sigma(t)}. \tag{2.2}$$

Fixing $i > q$, and given $\tau, \sigma \in S_t$, we say that τ is *i-equivalent* to σ, if τ restricted to the final interval $[i, t]$ equals the restriction of σ to the same domain. In symbols,

$$\tau \sim_i \sigma \iff \tau|_{[i,t]} = \sigma|_{[i,t]}.$$

For each $i > q$, the relation $\sim_i$ yields a partition of S_t into disjoint subsets P_i^k, $k = 1, \ldots, \frac{t!}{(i-1)!}$. Then, we have

$$\sum_{\sigma \in S_t} (\mathrm{sg}\sigma) a_{\sigma(1)} \ldots a_{\sigma(i-1)} b_{\sigma(i)} c_{\sigma(i+1)} \ldots c_{\sigma(t)} =$$

$$= \sum_k \sum_{\sigma \in P_i^k} (\mathrm{sg}\sigma) a_{\sigma(1)} \ldots a_{\sigma(i-1)} b_{\sigma(i)} c_{\sigma(i+1)} \ldots c_{\sigma(t)}$$

$$= \sum_k (\mathrm{sg}\sigma_k) s_{i-1}(a_{\sigma_k(1)}, \ldots, a_{\sigma_k(i-1)}) b_{\sigma_k(i)} c_{\sigma_k(i+1)} \ldots c_{\sigma_k(t)},$$

where σ_k is a representative of the class P_i^k. The last equality follows from the fact that for any $\sigma \in P_i^k$, $\sigma = \tau \circ \sigma_k$ for some $\tau \in S_{i-1} \subseteq S_t$, and $(\mathrm{sg}\sigma) = (\mathrm{sg}\tau)(\mathrm{sg}\sigma_k)$. By assumption, A_ℓ satisfies s_q, and since $i - 1 \geq q$ we obtain

$$\sum_{\sigma \in S_t} (\mathrm{sg}\sigma) a_{\sigma(1)} \ldots a_{\sigma(i-1)} b_{\sigma(i)} c_{\sigma(i+1)} \ldots c_{\sigma(t)} = 0.$$

This shows that

$$\sum_{i=q+1}^{t+1} \sum_{\sigma \in S_t} \mathrm{sg}(\sigma)\, a_{\sigma(1)} \ldots a_{\sigma(i-1)} b_{\sigma(i)} c_{\sigma(i+1)} \ldots c_{\sigma(t)} = 0. \qquad (2.3)$$

For $i \leq q$ we have that $t - i \geq r$. Applying a similar argument to the above, and using the fact that A_m satisfies s_r, we see that also

$$\sum_{i=0}^{q}\sum_{\sigma\in S_t}(\mathrm{sg}\sigma)a_{\sigma(1)}\cdots a_{\sigma(i-1)}b_{\sigma(i)}c_{\sigma(i+1)}\cdots c_{\sigma(t)}=0. \tag{2.4}$$

Together, Equations (2.3) and (2.4) ensure that $s_t(x_1,\ldots,x_n)=0$, given Equation (2.2). □

2.2.1

We now consider the case when A contains a "repetition". We will need some more notation.

(i) Let $M_1,\ldots M_t$ be matrices in A, with

$$M_k=\begin{bmatrix} a_k & b_k & c_k \\ 0 & e_k & d_k \\ 0 & 0 & a_k \end{bmatrix},\ a_k\in M_\ell(F), e_k\in M_m(F), b_k\in M_{\ell\times m}(F), d_k\in M_{m\times\ell}(F).$$

Given $1\le i<j\le t$ and $\sigma\in S_t$, set

$$m_t^\sigma[i,j]=(\mathrm{sg}\sigma)\,a_{\sigma(1)}\cdots a_{\sigma(i-1)}b_{\sigma(i)}e_{\sigma(i+1)}\cdots e_{\sigma(j-1)}d_{\sigma(j)}a_{\sigma(j+1)}\cdots a_{\sigma(t)},$$

and denote by W the set of all matrix products

$$\{m_t^\sigma[i,j]:\sigma\in S_t\,\text{and}\,1\le i<j\le t\}.$$

(ii) The projection ur returns the $\ell\times\ell$ upper right block of a matrix in A:

$$ur\begin{bmatrix} a & b & c \\ 0 & e & d \\ 0 & 0 & a \end{bmatrix}=c$$

(iii) Given $n \times n$ matrices $M_1, \ldots, M_t$, we say that a matrix product $M_1 \cdots M_t$ *formally contains* the factor $A_1 \cdots A_s$ if $A_1 = M_\ell, A_2 = M_{\ell+1}, \ldots, A_s = M_{\ell+s-1}$, for some $1 \le \ell \le t$. This notation is to distinguish from the case when $CA_1 \cdots A_s D = M_1 \cdots M_t$ as $n \times n$ matrices, for some matrices C and D. Further, if $\ell = 1$, we say that $M_1 \cdots M_t$ formally contains $A_1 \cdots A_s$ as a *left factor.*

This is a good place to record a Lemma extracted from [**AL50**], which will be used later.

Lemma 2.2.3 [AL50, Lemma 1, 450-451] *If for an odd positive integer r we put $Y = X_{i+1} \cdots X_{i+r}$, and if s' denotes the sum of all terms of $s_m(X)$ containing the common factor Y, then*

$$s' = s_{m-r+1}(X_1, \ldots, X_i, Y, X_{i+r+1}, \ldots, X_m).$$

Lemma 2.2.4 *Set $t = 2(\ell + m)$, and let $M_1, \ldots, M_t$ be matrices in A such that for all $1 \le k \le t$,*

$$M_k = \begin{bmatrix} a_k & b_k & 0 \\ 0 & e_k & d_k \\ 0 & 0 & a_k \end{bmatrix}, \quad \text{for } a_k \in M_\ell(F), e_k \in M_m(F), b_k \in M_{\ell\times m}(F), d_k \in M_{m\times\ell}(F).$$

Then $ur\left[s_t(M_1, \ldots, M_t)\right] = 0$.

Proof. First we observe that

$$ur[M_1 \cdots M_t] = \sum_{1\le i<j\le t} a_{\sigma(1)} \ldots a_{\sigma(i-1)} b_{\sigma(i)} e_{\sigma(i+1)} \ldots e_{\sigma(j-1)} d_{\sigma(j)} a_{\sigma(j+1)} \ldots a_{\sigma(t)},$$

which implies that

$$ur\,[s_t(M_1,\ldots,M_t)] = \sum_{\sigma\in S_t}\ \sum_{1\le i<j\le t} m_t^\sigma[i,j]. \tag{2.5}$$

To prove that $ur\,[s_t(M_1,\ldots,M_t)] = 0$, we split the right hand side into two summands:

$$ur\,[s_t(M_1,\ldots,M_t)] =$$

$$\sum_{\sigma\in S_t}\ \sum_{\substack{1\le i<j\le t\\ j-i-1\ge 2m}} m_t^\sigma[i,j] + \sum_{\sigma\in S_t}\ \sum_{\substack{1\le i<j\le t\\ j-i\le 2m}} m_t^\sigma[i,j] \tag{2.6}$$

Our goal is to show that each summand in (2.6) is zero. To handle the first summand we introduce the following new equivalence relation on S_t. Given fixed $1\le i<j\le t$, such that $j-i-1\ge 2m$, and given $\tau,\sigma\in S_t$, say that τ is $[i,j]$-*equivalent* to σ if τ restricted to the initial and final intervals $[1,i]$ and $[j,t]$ equals the restriction of σ to the same domain. In symbols,

$$\tau \sim_{[i,j]} \sigma \iff \tau|_{[1,i]} = \sigma|_{[1,i]} \text{ and } \tau|_{[j,t]} = \sigma|_{[j,t]}$$

For each pair i,j, such that $1\le i<j\le t$ and $j-i-1\ge 2m$, the relation $\sim_{[i,j]}$ yields a partition of S_t into disjoint subsets $P^k_{[i,j]}$, $k=1,\ldots,\frac{t!}{(j-i-1)!}$. Then, we have

$$B_\delta = \{D^{n-\delta}, UD^{n-\delta+1}, D^{n-\delta+1}U, U^2D^{n-\delta+2}, D^{n-\delta+1}U^2,\ldots$$
$$\ldots, U^{\frac{\delta-1}{2}}D^{n-\frac{\delta+1}{2}}, D^{n-\frac{\delta+1}{2}}U^{\frac{\delta-1}{2}}\}$$

$$\sum_{\sigma\in S_t}\sum_{\substack{1\le i<j\le t\\ j-i-1\ge 2m}} m_t^\sigma[i,j] = \sum_{\substack{1\le i<j\le t\\ j-i-1\ge 2m}}\sum_k\sum_{\sigma\in P^k_{[i,j]}} m_t^\sigma[i,j] =$$

$$\sum_{\substack{1\le i<j\le t\\ j-i-1\ge 2m}}\sum_k\sum_{\sigma\in P^k_{[i,j]}} (\mathrm{sg}\sigma)\, a_{\sigma(1)}\cdots a_{\sigma(i-1)}b_{\sigma(i)}e_{\sigma(i+1)}\cdots e_{\sigma(j-1)}d_{\sigma(j)}a_{\sigma(j+1)}\cdots a_{\sigma(t)}$$

$$=$$

$$\sum_{\substack{1\le i<j\le t\\ j-i-1\ge 2m}}\sum_k (\mathrm{sg}\sigma_k)\, a_{\sigma_k(1)}\cdots a_{\sigma_k(i-1)}b_{\sigma_k(i)}\, s\, d_{\sigma_k(j)}a_{\sigma_k(j+1)}\cdots a_{\sigma_k(t)},$$

where $s = s_{i-j+1}(e_{\sigma_k(i+1)},\ldots,e_{\sigma_k(j-1)})$ and σ_k is a representative of the class $P^k_{[i,j]}$. Since $j-i-1\ge 2m$,

$$s_{i-j+1}(e_{\sigma_k(i+1)},\ldots,e_{\sigma_k(j-1)}) = 0 \quad \text{for all } k,$$

hence

$$\sum_{\sigma\in S_t}\sum_{\substack{1\le i<j\le t\\ j-i-1\ge 2m}} m_t^\sigma[i,j] = 0.$$

This takes care of the first term in (2.6). We now turn to the second summand. For a given q, with $2\le q\le t$, denote by R_q the set of all q-tuples $r = (r_1,\ldots,r_q)$ of different elements from $\{1,\ldots,t\}$ and by $T_{(r_1,\ldots,r_q)}$ the set of matrix products w formally containing the common factor $b_{r_1}e_{r_2}\cdots e_{r_{q-1}}d_{r_q}$. Considering all possible q and q-tuples, the sets $T_{(r_1,\ldots,r_q)}$ form a partition of

W. We are interested in the case when $q \leq 2m+1$. Observe that

$$\sum_{\sigma \in S_t} \sum_{\substack{1 \leq i < j \leq t \\ j-i \leq 2m}} m_t^\sigma[i,j] = \sum_{q=2}^{2m+1} \sum_{r \in R_q} \sum_{w \in T_{(r_1,\ldots,r_q)}} w.$$

Fix q odd, a q-tuple $(r_1, \ldots, r_q)$, and the corresponding set of matrix products $T_{(r_1,\ldots,r_q)}$. Then, $\sum_{w \in T_{(r_1,\ldots,r_q)}} w$ is the sum of all matrix products formally containing the common factor $y = b_{r_1} e_{r_2} \cdots e_{r_{q-1}} d_{r_q}$. Each matrix product $w \in T_{(r_1,\ldots,r_q)}$ corresponds uniquely to a permutation $\sigma \in S_t$ and a pair (i,j), such that the q-tuple $(r_1, \ldots, r_q)$ is the image under σ of $(i, \ldots, j)$. Explicitly, the correspondence is $w = m_t^\sigma[i,j]$. We can now apply Lemma 2.2.3 and the alternating property of the standard polynomials. If $\sigma_0 \in S_t$ is a fixed permutation such that $\sigma_0 : i \to r_i$, for $1 \leq i \leq q$, we have

$$\sum_{w \in T_{(r_1,\ldots,r_q)}} w = (\operatorname{sg} \sigma_0)\, s_{t-q+1}\left(y, a_{\sigma_0(q+1)}, \ldots, a_{\sigma_0(t)}\right),$$

where $y = b_{r_1} e_{r_2} \cdots e_{r_{q-1}} d_{r_q}$. Since $t - q + 1 \geq 2\ell$, and since all the arguments of s_{t-q+1} in the last equation are $\ell \times \ell$ matrices, it follows that

$$\sum_{w \in T_{(r_1,\ldots,r_q)}} w = 0, \text{ when } q \text{ is odd and } (r_1, \ldots, r_q) \text{ is a fixed } q\text{-tuple.} \qquad (2.7)$$

Therefore

$$\sum_{\substack{q=2 \\ q \text{ odd}}}^{2m+1} \sum_{r \in R_q} \sum_{w \in T_{(r_1,\ldots,r_q)}} w = 0.$$

Suppose now that q is even, so $q \leq 2m$, and fix an arbitrary q-tuple $r = (r_1, \ldots, r_q)$. We will split further the sets T_r. First consider all $w \in T_r$ formally containing in common the left factor $y = b_{r_1}e_{r_2}\cdots e_{r_{q-1}}d_{r_q}$, and call this subset L_r. Then, for each $r_0 \notin \{r_1, \ldots, r_q\}$ consider the $(q+1)$-tuple (r_0, r) and the subset $G_{(r_0,r)}$ of $w \in T_r$ formally containing in common the factor $y = a_{r_0}b_{r_1}e_{r_2}\cdots e_{r_{q-1}}d_{r_q}$. The sum of all matrix products in the set T_r can be split as

$$\sum_{w\in T_r} w = \sum_{w\in L_r} w + \sum_{r_0:r_0\neq r_1,\ldots,r_q} \sum_{w\in G_{(r_0,r)}} w.$$

For the terms in L_r we have

$$\sum_{w\in L_{(r_1,\ldots,r_q)}} w = (\operatorname{sg}\sigma_0)\, y\, s_{t-q}\left(a_{\sigma_0(q+1)}, \ldots, a_{\sigma_0(t)}\right), \tag{2.8}$$

where $y = b_{r_1}e_{r_2}\cdots e_{r_{q-1}}d_{r_q}$, and where $\sigma_0 \in S_t$ is a fixed permutation such that $\sigma_0 : i \to r_i$, for $1 \le i \le q$.

Since $t - q \ge 2\ell$, we obtain

$$\sum_{w\in L_r} w = 0. \tag{2.9}$$

Finally, for a suitable fixed r_0, the sequence (r_0, r) has odd length, so we can argue as in (2.7) to obtain

$$\sum_{w\in G_{(r_0,r)}} w = (\operatorname{sg}\sigma_0)\, s_{t-q+1}\left(y, a_{\sigma_0(q+2)}, \ldots, a_{\sigma_0(t)}\right) = 0,$$

where $y = a_{r_0}b_{r_1}e_{r_2}\cdots e_{r_{q-1}}d_{r_q}$, and where $\sigma_0 \in S_t$ is a fixed permutation such that

$$\sigma_0 = \begin{cases} 1 \to r_0, \\ i \to r_{i-1}, \quad \text{for} \quad 2 \le i \le q+1. \end{cases}$$

This finishes the proof of Lemma 2.2.4. □

Proposition 2.2.5 *Let*

$$A = \left\{ \begin{bmatrix} a & b & c \\ 0 & e & d \\ 0 & 0 & a \end{bmatrix} : a, c \in M_\ell(F), e \in M_m(F), b \in M_{\ell\times m}(F), d \in M_{m\times\ell}(F) \right\}.$$

Then, A satisfies $s_{2(\ell+m)}$.

Proof. For any t and matrices $M_k \in A$, $k = 1 \ldots t$, set

$$M_k = \begin{bmatrix} a_k & b_k & c_k \\ 0 & e_k & d_k \\ 0 & 0 & a_k \end{bmatrix}, \; a_k \in M_\ell(F), e_k \in M_m(F), b_k \in M_{\ell\times m}(F), d_k \in M_{m\times\ell}(F).$$

By direct calculations, we obtain

$$ur[s_t(M_1, \ldots, M_t)] =$$

$$= \sum_{i=1}^{t} s_t(a_1, \ldots, a_{i-1}, c_i, a_{i+1}, \ldots, a_t) + \sum_{\sigma\in S_t} \sum_{1\le i<j\le t} m_t^\sigma[i,j].$$

Now set $t = 2(\ell+m)$. It follows from (2.5) that

$$\sum_{\sigma\in S_t} \sum_{1\le i<j\le t} m_t^\sigma[i,j] = ur\left[s_t(M_1', \ldots, M_t')\right] = 0,$$

where M_k' is the matrix in A obtained by replacing the upper right corner c_k of M_k by $0 \in M_\ell(F)$. Suitable applications of the Amitsur-Levitski identity give us

$$ur\left[s_t(M_1, \ldots, M_t)\right] = 0,$$

$$s_t\left(\begin{bmatrix} a_1 & b_1 \\ 0 & e_1 \end{bmatrix}, \ldots, \begin{bmatrix} a_t & b_t \\ 0 & e_t \end{bmatrix}\right) = 0,$$

and

$$s_t\left(\begin{bmatrix} e_1 & d_1 \\ 0 & a_1 \end{bmatrix}, \ldots, \begin{bmatrix} e_t & d_t \\ 0 & a_t \end{bmatrix}\right) = 0.$$

Combining the three equations, it follows that $s_t(M_1, \ldots, M_t) = 0$. □

2.3 Main Theorem

In this section we prove that if a matrix subalgebra of $M_n(F)$ does not satisfy the standard polynomial s_{2n-2}, then it is isomorphic as F-algebra to a full block upper triangular matrix algebra.

2.3.1

We first introduce our notation and review some necessary background (cf. [**Le02**]).

(i) Let t be a positive integer, let $\ell_1, \ell_2, \cdots, \ell_t$ be positive integers summing up to n, and set

$$E_{(\ell_1,\ell_2,\ldots,\ell_t)}(F) = \begin{bmatrix} M_{\ell_1}(F) & M_{\ell_1\times\ell_2}(F) & \cdots & M_{\ell_1\times\ell_{t-1}}(F) & M_{\ell_1\times\ell_t}(F) \\ 0 & M_{\ell_2}(F) & \cdots & M_{\ell_2\times\ell_{t-1}}(F) & M_{\ell_2\times\ell_t}(F) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & M_{\ell_{t-1}}(F) & M_{\ell_{t-1}\times\ell_t}(F) \\ 0 & 0 & \cdots & 0 & M_{\ell_t}(F) \end{bmatrix},$$

a full block upper triangular matrix subalgebra of $M_n(F)$.

(ii) Recall that every F-algebra automorphism τ of $M_n(F)$ is inner (i.e., there exists an invertible Q in $M_n(F)$ such that $\tau(a) = QaQ^{-1}$ for all $a \in M_n(F)$). We will say that two F-subalgebras A, A' of $M_n(F)$ are *equivalent* provided there exists an automorphism τ of $M_n(F)$ such that $\tau(A) = A'$.

(iii) Associated to $E_{(\ell_1,\ell_2,\ldots,\ell_t)}(F)$ are canonical F-algebra homomorphisms

$$\pi_{ij} : E_{(\ell_1,\ell_2,\ldots,\ell_t)}(F) \to E_{(\ell_i,\ell_{i+1},\ldots,\ell_j)}(F), \quad \text{for } 1 \leq i \leq j \leq t.$$

When $i = j$ we write π_i for π_{ii}. For a subalgebra Λ of $E_{(\ell_1,\ell_2,\ldots,\ell_t)}(F)$, we have the homomorphic images:

$$\Lambda_{ij} := \pi_{ij}(\Lambda),$$

embedded in $E_{(\ell_i,\ell_{i+1},\ldots,\ell_j)}$.

(iv) We will say that a subalgebra Λ of $E_{(\ell_1,\ell_2,\ldots,\ell_t)}(F)$ is an $(\ell_1, \ell_2, \ldots, \ell_t)$-*extension of simple blocks* if the restrictions $\pi_i : \Lambda \to M_{\ell_i}(F)$, for $1 \leq i \leq t$, are all irreducible representations (when F is algebraically closed, of course, the representation π_i is irreducible if and only if $\pi_i(\Lambda) = M_{\ell_i}$). Note that every F-subalgebra A of $M_n(F)$ is equivalent to an $(\ell_1, \ell_2, \ldots, \ell_t)$-extension of simple blocks Λ for some suitable $(\ell_1, \ell_2, \ldots, \ell_t)$.

(v) Further, we will say that Λ *contains a repetition* when $\pi_i : \Lambda \to M_{\ell_i}$ and $\pi_j : \Lambda \to M_{\ell_j}$ are equivalent representations, for some $1 \leq i < j \leq t$ (and so $\ell_i = \ell_j$). Also, Λ is *uniserial* when $\Lambda_{i(i+1)}$ is not semisimple, for all $1 \leq i \leq (t-1)$.

Lemma 2.3.1 *If an extension of simple blocks* Λ *contains a repetition, then the standard identity* $s_{2n-2} = 0$ *holds for* Λ.

Proof. Assume $\pi_i : \Lambda \to M_{\ell_i}$ and $\pi_j : \Lambda \to M_{\ell_j}$ are equivalent representations for some $1 \le i < j \le t$. Then we can choose an F-algebra automorphism τ of $M_n(F)$ such that $\pi_{ij}(\tau(\Lambda))$ is a subalgebra of

$$\left\{ \begin{bmatrix} a & b & c \\ 0 & e & d \\ 0 & 0 & a \end{bmatrix} : a, c \in M_{\ell_i}(F), e \in M_{\ell_{i+1}}(F), b \in M_{\ell_i \times \ell_{i+1}}(F), d \in M_{\ell_{i+1} \times \ell_i}(F) \right\}.$$

The result now follows from Lemma 2.2.2 and Proposition 2.2.5. □

Lemma 2.3.2 *If an extension of simple blocks* Λ *is not uniserial, then the standard identity* $s_{2n-2} = 0$ *holds for* Λ.

Proof. Follows immediately from Lemma 2.2.2. □

The main theorem in this chapter is:

Theorem 2.3.3 *Let* F *be a field and let* A *be an* F*-subalgebra of* $M_n(F)$. *If* A *does not satisfy the standard polynomial* s_{2n-2}, *then* A *is equivalent to a full block upper triangular matrix algebra.*

Proof. It suffices to show that the only $(\ell_1, \ell_2, \ldots, \ell_t)$-extension of simple blocks Λ for which the standard polynomial s_{2n-2} is not an identity is the full block upper triangular matrix algebra $E_{(\ell_1,\ell_2,\ldots,\ell_t)}(F)$. By Lemma 2.2.1, $\Lambda_i = M_{\ell_i}(F)$ for $1 \le i \le t$. By Lemma 2.3.2 and Lemma 2.3.1, $\Lambda_{i(i+1)}(F)$ is not semisimple and does not contain a repetition, for each $1 \leq i \leq t-1$. We conclude (cf. **[Le02]**, Lemma 3.6) that

$$\Lambda_{i(i+1)}(F) = M_{\ell_i \times \ell_{i+1}}(F), \quad \text{for each} \quad 1 \leq i \leq t-1.$$

Therefore, Λ contains the staircase unit matrices (c.f. (2.1)), and every unit matrix e_{ij}, for $j > i$ can be expressed as a product of those. The Theorem now follows. □

Corollary 2.3.4 *The standard polynomial s_{2n-2} is an identity for any proper subalgebra of $U_n(F)$, the algebra of upper triangular matrices over the field F.*

Proof. Immediate from Theorem 2.3.3. □

Remark. The standard polynomial of degree $2n-2$ is not necessarily an identity for any proper subalgebra of $U_n(C)$ when C is a commutative ring: Let I be a nonzero ideal of C, and consider the C-subalgebra B of $U_n(C)$ defined by the property that the $(1,2)$-entry of matrices in B lie in I. A staircase argument shows that $s_{2n-2}(B) \neq 0$.

CHAPTER 3

Polynomial detection of matrix subalgebras.

In this chapter we develop the concept of polynomial test, along with some applications to algorithmic representation theory. The definition of "polynomial test" is our own, but this notion has previously appeared, e.g. in Rowen [**Ro80**]. First, the double Capelli polynomial of total degree $2t$ is

$$\sum \left\{ (\operatorname{sg} \sigma\tau) X_{\sigma(1)} Y_{\tau(1)} X_{\sigma(2)} Y_{\tau(2)} \cdots X_{\sigma(t)} Y_{\tau(t)} \mid \sigma,\ \tau \in S_t \right\}.$$

It was proved by Giambruno-Sehgal and Chang that the double Capelli polynomial of total degree $4n$ is a polynomial identity for $M_n(F)$. (Here, F is a field and $M_n(F)$ is the algebra of $n \times n$ matrices over F). In this chapter we show that the double Capelli polynomial of total degree $4n-2$ is a polynomial identity for any proper F-subalgebra of $M_n(F)$. Subsequently, we present polynomial tests for nonsplit non-self extensions of full matrix algebras.

3.1 Introduction

Let F be a field and $M_n(F)$ the algebra of $n \times n$ matrices over F. $F\{X\} = F\{X_1, X_2, \dots\}$ denotes the free associative algebra over F in countably many variables $X_1, X_2, \dots$. Sometimes we will use other variables X, Y, Z, X_i, Y_i for notational simplicity. A nonzero polynomial $f(X_1, \dots X_m) \in F\{X\}$ is a *polynomial identity* for an F-algebra R if $f(r_1, \dots, r_m) = 0$ for all $r_1, \dots, r_m \in R$. A *T-ideal* is an ideal of $F\{X\}$ which is closed under endomorphisms of $F\{X\}$. If $f_1, \dots, f_t$ are polynomial identities for R, so is every polynomial f in the T-ideal generated by $f_1, \dots, f_t$. In this case we say that the identity $f = 0$ in R is a *consequence* of the identities $f_i = 0$, for $1 \le i \le t$.

Two of the most important multilinear polynomials in the theory of associative algebras with polynomial identities are the *standard polynomials* and the *Capelli polynomials.* The standard polynomial of degree t has the form

$$s_t(X_1, \dots, X_t) = \sum_{\sigma \in S_t} (\text{sg}\sigma) X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(t)},$$

where S_t is the symmetric group on $\{1, \dots, t\}$ and $(\text{sg}\sigma)$ is the sign of the permutation $\sigma \in S_t$. The standard polynomial s_t is homogeneous of degree t, multilinear and alternating.

Recall that the Amitsur-Levitski identity asserts that $M_n(F)$ satisfies any standard polynomial of degree $2n$ or higher. A short and elegant proof of this theorem was obtained by Rosset [**Ro76**]. Rosset's proof uses an exterior algebra over F in an ingenious way. The standard polynomial s_{2n} is a minimal identity in the sense that $M_n(F)$ satisfies no polynomial identity of degree less than $2n$.

The *Capelli polynomials* are

$$c_{2t-1}(X_1,\ldots,X_t,Y_1,\ldots,Y_{t-1}) = \sum_{\sigma\in S_t}(\mathrm{sg}\sigma)X_{\sigma(1)}Y_1X_{\sigma(2)}Y_2\cdots X_{\sigma(t-1)}Y_{t-1}X_{\sigma(t)},$$

and

$$c_{2t}(X_1,\ldots,X_t,Y_1,\ldots,Y_t) = c_{2t-1}(X_1,\ldots,X_t,Y_1,\ldots,Y_{t-1})\,Y_t\,;$$

c_{2t-1} and c_{2t} are multilinear and alternating as a function of $X_1,\ldots,X_t$.

We will say that a multilinear polynomial $f(X_1,\ldots,X_t) \in F\{X\}$ is a *polynomial test* for an F-algebra R if it is not a polynomial identity for R but it is an identity for every proper F-subalgebra of R.

A polynomial $f(X_1,\ldots,X_t) \in F\{X\}$ is a *central polynomial* for an F-algebra R if (1) for any $r_1,\ldots,r_t \in R$, $f(r_1,\ldots,r_t)$ lies in the center of R, (2) f is not a polynomial identity for R, and (3) the constant term of f is zero. Central polynomials for $M_n(F)$ are also polynomial tests for $M_n(F)$, as is discussed in Section § 3.2.

The Capelli polynomial was introduced by Razmylov in [**Ra74**] and has important applications in PI-theory, in particular, the development of central polynomials for $M_n(F)$ by Razmylov [**Ra73**] and Amitsur [**Am78**]. The original central polynomials for $M_n(F)$ were discovered by Formanek in [**Fo72**]. The following proposition (extracted from [**Fo91**]) shows that the Capelli polynomial c_{2n^2} is a polynomial test for $M_n(F)$.

Proposition 3.1.1 *(a) The Capelli polynomial c_{2n^2+1} is a PI for $M_n(F)$.*

(b) The Capelli polynomial c_{2n^2} is a PI for any proper F-subalgebra of $M_n(F)$.

(c) The Capelli polynomial c_{2n^2} is not a PI for $M_n(F)$.

Proof. (a) and (b) hold because $M_n(F)$ has dimension n^2 over F.

(c) Evaluate $c_{2n^2}(x_1, \dots, x_{n^2}, y_1, \dots y_{n^2})$ with

$$(x_1, x_2, \dots, x_n, x_{n+1}, \dots x_{n^2-1}, x_{n^2}) = \left(e_{11}, e_{12}, \dots, e_{1n}, e_{21}, \dots e_{n(n-1)}, e_{nn}\right),$$

$$(y_1, \dots, y_n, \dots y_{n^2-1}, y_{n^2}) = \left(e_{11}, \dots, e_{n2}, \dots e_{(n-1)n}, e_{n1}\right).$$

Here $y_1 = e_{11}$, $y_{n^2} = e_{n1}$, and $y_2, \dots y_{n^2-1}$ are the unique choices of matrix units such that the monomial with $\sigma = 1$ is nonzero, so c_{2n^2} takes on the value $e_{11} \neq 0$. □

In Corollary 2.3.4, it is proved that the standard polynomial of degree $2n - 2$ is a polynomial test for the subalgebra of upper triangular matrices of $M_n(F)$.

Define the *double Capelli polynomials* by

$$h_{2t-1}(X_1,\ldots,X_t,Y_1,\ldots,Y_{t-1})$$
$$= \sum_{\sigma\in S_t,\tau\in S_{t-1}} (\mathrm{sg}\sigma\tau)X_{\sigma(1)}Y_{\tau(1)}X_{\sigma(2)}Y_{\tau(2)}\cdots X_{\sigma(t-1)}Y_{\tau(t-1)}X_{\sigma(t)}$$

and

$$h_{2t}(X_1,\ldots,X_t,Y_1,\ldots,Y_t)$$
$$= \sum_{\sigma,\tau\in S_t} (\mathrm{sg}\sigma\tau)X_{\sigma(1)}Y_{\tau(1)}X_{\sigma(2)}Y_{\tau(2)}\cdots X_{\sigma(t-1)}Y_{\tau(t-1)}X_{\sigma(t)}Y_{\tau(t)}.$$

Note that h_{2t-1} and h_{2t} are multilinear and alternating in the X_i and also in the Y_j.

Formanek pointed out that h_{4n-2} is not a polynomial identity for $M_n(F)$ and asked for the least integer m such that h_m is a polynomial identity for $M_n(F)$. Chang [CH88] proved that both double Capelli polynomials h_{2t-1} and h_{2t} are consequences of the standard polynomial s_t, implying that h_{4n-1} and h_{4n} are polynomial identities for $M_n(F)$. A different proof that h_{4n} is a polynomial identity for $M_n(F)$, that uses a variation of Rosset's method, was given by Giambruno-Sehgal [GS89]. To see that h_{4n-2} is not a polynomial identity for $M_n(F)$, consider the substitution (double staircase)

$$x_1 = e_{11}, y_1 = e_{12}, x_2 = e_{22}, y_2 = e_{23}, \ldots, x_n = e_{nn}$$

$$y_n = e_{nn}, x_{n+1} = e_{n(n-1)}, y_{n+1} = e_{(n-1)(n-1)}, \ldots, x_{2n-1} = e_{21}, y_{2n-1} = e_{11}$$

where the e_{ij} are the standard matrix units. The only nonzero monomials in $h_{4n-2}(x_i, y_i)$ are the $2n-1$ even cyclic permutations of $x_1y_1 \dots x_{2n-1}y_{2n-1}$, and they all have positive sign. Thus

$$h_{4n-2}(x_1, \dots, x_{2n-1}, y_1, \dots, y_{2n-1}) = 2I - e_{11}.$$

We finish this section with some useful properties of the double Capelli polynomials:

Proposition 3.1.2 *Let t be a positive integer.*

(a) h_{2t} lies in the T-ideal of h_{2t-1}.

(b) h_{2t+1} lies in the T-ideal of h_{2t}.

(c) The identity h_q is a consequence of the identity h_r for any $q \geq r$.

Proof. For (a) and (b) we can give an explicit relation

$$h_{2t}(X_1, \dots, X_t, Y_1, \dots, Y_{t-1}, Y_t) = \sum_{i=1}^{t} (-1)^{t-i} h_{2t-1}(X_1, \dots, X_t, Y_1, \dots, \hat{Y}_i, \dots, Y_{t-1}) Y_i,$$

where $\hat{Y}_i$ means that Y_i does not participate in the expression, and

$$h_{2t+1}(X_1, \dots, X_t, Y_1, \dots, Y_t, X_{t+1}) = \sum_{i=1}^{t} (-1)^{t+1-i} h_{2t}(X_1, \dots, \hat{X}_i, \dots, X_t, Y_1, \dots, Y_t) X_i.$$

(c) is immediate from (a) and (b). □

3.2 A polynomial test for the full matrix algebra

The main goal of this section is to prove that h_{4n-2} is a polynomial test for $M_n(F)$. Before proceeding to the proof of this theorem we need some preliminaries. First, we fix our notation.

3.2.1

Let ℓ, m be positive integers such that $\ell + m = n$ and set

$$E_{(\ell,m)}(F) = \begin{bmatrix} M_\ell(F) & M_{\ell\times m}(F) \\ 0 & M_m(F) \end{bmatrix},$$

an F-subalgebra of $M_n(F)$.

(i) Associated to $E_{(\ell,m)}(F)$ are canonical F-algebra homomorphisms

$$\pi_\ell : E_{(\ell,m)}(F) \to M_\ell(F) \quad \text{and} \quad \pi_m : E_{(\ell,m)}(F) \to M_m(F).$$

Further identify $M_\ell(F)$ and $M_m(F)$ with

$$\begin{bmatrix} M_\ell(F) & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & M_m(F) \end{bmatrix},$$

respectively.

(ii) Associated to a subalgebra A of $E_{(\ell,m)}(F)$ are homomorphic image subalgebras A_ℓ and A_m of $M_\ell(F)$ and $M_m(F)$ respectively.

(iii) Set

$$T_{(\ell,m)}(F) = \begin{bmatrix} 0 & M_{\ell\times m}(F) \\ 0 & 0 \end{bmatrix},$$

the Jacobson radical of $E_{(\ell,m)}(F)$.

(iv) Recall that every F-algebra automorphism τ of $M_n(F)$ is *inner* (i.e., there exists an invertible Q in $M_n(F)$ such that $\tau(a) = QaQ^{-1}$ for all $a \in M_n(F)$). Two F-subalgebras A, A' of $M_n(F)$ are *equivalent* (or isomorphic) provided there exists an automorphism τ of $M_n(F)$ such that $\tau(A) = A'$.

(v) $E_{(\ell,m)}(F)$ has no central polynomials: Let $c(r_1, \ldots, r_t) = \alpha I_n$, where $\alpha \in F$, $r_1, \ldots, r_t \in E_{(\ell,m)}(F)$. Notice that $\alpha e_{11}, \ldots, \alpha e_{\ell\ell}$ depend on the first ℓ rows and columns of $r_1, \ldots, r_t$ only, and do not depend on the $m \times m$ lower-right block. If

$$r_i = \begin{bmatrix} a_i & b_i \\ 0 & c_i \end{bmatrix},$$

let r_i' the matrix obtained from r_i by replacing the block c_i by the 0 block. From the evaluation of the polynomial c in $r_1', \ldots, r_t'$ we conclude that $\alpha = 0$.

Lemma 3.2.1 *Let A be a subalgebra of $E_{(\ell,m)}(F)$ such that A_ℓ satisfies h_{2q-1} for some $1 \leq q \leq 2\ell$ and A_m satisfies h_{2r-1} for some $1 \leq r \leq 2m$. Then A satisfies $h_{2(q+r)-2}$.*

Proof. Let $t = q + r$. Then $M_n(F) = M_\ell(F) \oplus T_{(\ell,m)}(F) \oplus M_m(F)$ as F-vector space, and so each matrix x (resp. y) in A can be written as $x = a+b+c$ with $a \in A_\ell$, $b \in T_{(\ell,m)}$ and $c \in A_m$ (resp. $y = \tilde{a} + \tilde{b} + \tilde{c}$). Using linearity, we expand completely $h_{2t-1}(x_1, \ldots, x_t, y_1, \ldots, y_{t-1})$ and further use the following rules to simplify some of the terms:

1. $T_{(\ell,m)}(F)$ is a nilpotent ideal of $E_{(\ell,m)}(F)$, with $T^2_{(\ell,m)}(F) = (0)$, and we see that each term in the expansion containing more than one entry in $T_{(\ell,m)}(F)$ equals 0.

2. $M_\ell(F)M_m(F) = M_m(F)M_\ell(F) = 0$.

3. $M_m(F)T_{(\ell,m)}(F) = T_{(\ell,m)}(F)M_\ell(F) = 0$.

We obtain

$$h_{2t-2}(x_1, \ldots, x_{t-1}, y_1, \ldots, y_{t-1}) = \tag{3.1}$$

$$\sum_{i=0}^{t-1} \sum_{\sigma,\tau \in S_{t-1}} (\mathrm{sg}\sigma\tau)\, a_{\sigma(1)}\tilde{a}_{\tau(1)} \ldots a_{\sigma(i-1)}\tilde{a}_{\tau(i-1)} b_{\sigma(i)} \tilde{c}_{\tau(i)} c_{\sigma(i+1)} \cdots c_{\sigma(t-1)}\tilde{c}_{\tau(t-1)}$$

$$+ \sum_{i=1}^{t} \sum_{\sigma,\tau \in S_{t-1}} (\mathrm{sg}\sigma\tau)\, a_{\sigma(1)}\tilde{a}_{\tau(1)} \cdots \tilde{a}_{\tau(i-1)} a_{\sigma(i)} \tilde{b}_{\tau(i)} c_{\sigma(i+1)} \cdots c_{\sigma(t-1)}\tilde{c}_{\tau(t-1)}$$

We want to show that $h_{2t-2}(x_1, \ldots, x_{t-1}, y_1, \ldots, y_{t-1}) = 0$. To do so, we will examine the above summands for each fixed value of i.

Case 1: $i > q$. In this case we partition the pairs of permutations $(\sigma, \tau) \in S_{t-1} \times S_{t-1}$ by the equivalence relation:

$$(\sigma_1, \tau_1) \sim_i (\sigma_2, \tau_2) \quad \text{iff} \quad \sigma_1|_{[i,(t-1)]} = \sigma_2|_{[i,(t-1)]} \text{ and } \tau_1|_{[i,(t-1)]} = \tau_2|_{[i,(t-1)]}.$$

For each $i > q$, the relation $\sim_i$ yields a partition of $S_{t-1} \times S_{t-1}$, into disjoint subsets P_i^k. Then, we have

$$\sum_{\sigma,\tau\in S_{t-1}} (\mathrm{sg}\sigma\tau)\, a_{\sigma(1)} \dots \tilde{a}_{\tau(i-1)} b_{\sigma(i)} \tilde{c}_{\tau(i)} c_{\sigma(i+1)} \dots \tilde{c}_{\tau(t-1)}$$

$$+ \sum_{\sigma\in S_{t-1},\tau\in S_{t-1}} (\mathrm{sg}\sigma\tau)\, a_{\sigma(1)} \tilde{a}_{\tau(1)} \dots a_{\sigma(i)} \tilde{b}_{\tau(i)} c_{\sigma(i+1)} \dots \tilde{c}_{\tau(t-1)}$$

$$= \sum_k \sum_{(\sigma,\tau)\in P_i^k} (\mathrm{sg}\sigma\tau)\, a_{\sigma(1)} \tilde{a}_{\tau(1)} \dots \tilde{a}_{\tau(i-1)} b_{\sigma(i)} \tilde{c}_{\tau(i)} c_{\sigma(i+1)} \dots \tilde{c}_{\tau(t-1)}$$

$$+ \sum_k \sum_{(\sigma,\tau)\in P_i^k} (\mathrm{sg}\sigma\tau)\, a_{\sigma(1)} \tilde{a}_{\tau(1)} \dots \tilde{a}_{\tau(i-1)} a_{\sigma(i)} \tilde{b}_{\tau(i)} c_{\sigma(i+1)} \dots \tilde{c}_{\tau(t-1)}$$

$$= \sum_k (\mathrm{sg}\sigma_k\tau_k)\, h_{2i-2}\left(a_{\sigma_k(1)}, \dots, \tilde{a}_{\tau_k(i-1)}\right) b_{\sigma_k(i)} \tilde{c}_{\tau_k(i)} c_{\sigma_k(i+1)} \dots \tilde{c}_{\tau_k(t-1)}$$

$$+ \sum_k (\mathrm{sg}\sigma_k\tau_k)\, h_{2i-2}\left(a_{\sigma_k(1)}, \dots, \tilde{a}_{\tau_k(i-1)}\right) a_{\sigma_k(i)} \tilde{b}_{\tau_k(i)} c_{\sigma_k(i+1)} \dots \tilde{c}_{\tau_k(t-1)},$$

where (σ_k, τ_k) is a representative of the class P_i^k.

By assumption, A_ℓ satisfies h_{2q-1}. It follows that h_{2i-2} is an identity for A_ℓ for all $q < i \leq t$, and hence each sum in Equation (3.1), corresponding to the case $i > q$, equals 0.

Case 2: $i < q$.

In this case $(t-1) - i \geq r$, so we partition the pairs of permutations $(\sigma, \tau) \in S_{t-1} \times S_{t-1}$ by the equivalence relation:

$$(\sigma_1, \tau_1) \sim_i (\sigma_2, \tau_2) \quad \text{iff} \quad \sigma_1|_{[1,i]} = \sigma_2|_{[1,i]} \text{ and } \tau_1|_{[1,i]} = \tau_2|_{[1,i]}.$$

For each $i < q$, the relation $\sim_i$ yields a partition of $S_{t-1} \times S_{t-1}$ into disjoint subsets P_i^k. Then, we have

$$\sum_{\sigma,\tau \in S_{t-1}} (\mathrm{sg}\sigma\tau)\, a_{\sigma(1)} \cdots \tilde{a}_{\tau(i-1)} b_{\sigma(i)} \tilde{c}_{\tau(i)} c_{\sigma(i+1)} \cdots \tilde{c}_{\tau(t-1)}$$

$$+ \sum_{\sigma,\tau \in S_{t-1}} (\mathrm{sg}\sigma\tau)\, a_{\sigma(1)} \cdots \tilde{a}_{\tau(i-1)} a_{\sigma(i)} \tilde{b}_{\tau(i)} c_{\sigma(i+1)} \cdots \tilde{c}_{\tau(t-1)}$$

$$= \sum_k \sum_{(\sigma,\tau) \in P_i^k} (\mathrm{sg}\sigma\tau)\, a_{\sigma(1)} \cdots \tilde{a}_{\tau(i-1)} b_{\sigma(i)} \tilde{c}_{\tau(i)} c_{\sigma(i+1)} \cdots \tilde{c}_{\tau(t-1)}$$

$$+ \sum_{k,j} \sum_{\sigma \in P_i^k, \tau \in Q_i^j} (\mathrm{sg}\sigma\tau)\, a_{\sigma(1)} \cdots \tilde{a}_{\tau(i-1)} a_{\sigma(i)} \tilde{b}_{\tau(i)} c_{\sigma(i+1)} \cdots \tilde{c}_{\tau(t-1)}$$

$$= \sum_k (\mathrm{sg}\sigma_k\tau_k)\; a_{\sigma_k(1)} \cdots \tilde{a}_{\tau_k(i-1)} b_{\sigma_k(i)} \tilde{c}_{\tau_k(i)}\; h_{2(t-1-i)} \left(c_{\sigma_k(i+1)}, \ldots, \tilde{c}_{\tau_k(t-1)}\right)$$

$$+ \sum_k (\mathrm{sg}\sigma_k\tau_k)\; a_{\sigma_k(1)} \cdots \tilde{a}_{\tau(i-1)} a_{\sigma_k(i)} \tilde{b}_{\tau_k(i)} h_{2(t-1-i)} \left(c_{\sigma_k(i+1)}, \ldots, \tilde{c}_{\tau_k(t-1)}\right),$$

where (σ_k, τ_k) is a representative of the class P_i^k.

By assumption, A_m satisfies h_{2r-1}. It follows that $h_{2(t-i-1)}$ is an identity for A_m for all $0 \leq i < q$, and hence each summand in (3.1), corresponding to the case $i < q$, equals 0.

Case 3: $i = q$.

In this case we need to examine two summands:

$$\sum_{\sigma \in S_{t-1}, \tau \in S_{t-1}} (\mathrm{sg}\sigma\tau)\, a_{\sigma(1)} \cdots \tilde{a}_{\tau(q-1)} b_{\sigma(q)} \tilde{c}_{\tau(q)} c_{\sigma(q+1)} \cdots \tilde{c}_{\tau(t-1)} +$$

$$\sum_{\sigma \in S_{t-1}, \tau \in S_{t-1}} (\mathrm{sg}\sigma\tau)\, a_{\sigma(1)} \cdots \tilde{a}_{\tau(q-1)} a_{\sigma(q)} \tilde{b}_{\tau(q)} c_{\sigma(q+1)} \cdots \tilde{c}_{\tau(t-1)}.$$

For the first summand we consider the equivalence relation:

$$(\sigma_1, \tau_1) \sim_q (\sigma_2, \tau_2) \quad \text{iff} \quad \sigma_1|_{[1,q]} = \sigma_2|_{[1,q]} \text{ and } \tau_1\text{-}[1, q-1] = \tau_2|_{[1,q-1]}.$$

The relation $\sim_q$ yields a partition of $S_{t-1} \times S_{t-1}$, into disjoint subsets P_q^k. Then, we have

$$\sum_{\sigma \in S_{t-1}, \tau \in S_{t-1}} (\mathrm{sg}\sigma\tau)\, a_{\sigma(1)} \cdots \tilde{a}_{\tau(q-1)} b_{\sigma(q)} \tilde{c}_{\tau(q)} c_{\sigma(q+1)} \cdots \tilde{c}_{\tau(t-1)}$$

$$= \sum_k \sum_{(\sigma,\tau) \in P_q^k} (\mathrm{sg}\sigma\tau)\, a_{\sigma(1)} \cdots \tilde{a}_{\tau(q-1)} b_{\sigma(q)} \tilde{c}_{\tau(q)} c_{\sigma(q+1)} \cdots \tilde{c}_{\tau(t-1)}$$

$$= \sum_k (\mathrm{sg}\sigma_k\tau_k)\, a_{\sigma_k(1)} \tilde{a}_{\tau_k(1)} \cdots b_{\sigma_k(q)} h_{2r-1}\left(\tilde{c}_{\tau_k(q)}, c_{\sigma_k(q+1)}, \cdots, \tilde{c}_{\tau_k(t-1)}\right),$$

where (σ_k, τ_k) is a representative of the class P_q^k.

Since h_{2r-1} is an identity in A_m the first summand equals 0. For the second summand we consider the equivalence relation:

$$(\sigma_1, \tau_1) \sim_q (\sigma_2, \tau_2) \quad \text{iff} \quad \sigma_1|_{[q+1,t]} = \sigma_2|_{[q+1,t]} \text{ and } \tau_1|_{[q,t]} = \tau_2|_{[q,t]}.$$

The relation $\sim_q$ yields a partition of $S_{t-1} \times S_{t-1}$, into disjoint subsets P_q^k. Then, we have

$$\sum_{\sigma \in S_{t-1}, \tau \in S_{t-1}} (\mathrm{sg}\sigma\tau)\, a_{\sigma(1)} \cdots \tilde{a}_{\tau(q-1)} a_{\sigma(q)} \tilde{b}_{\tau(q)} c_{\sigma(q+1)} \cdots \tilde{c}_{\tau(t-1)}$$

$$= \sum_k \sum_{\sigma \in P_q^k} (\mathrm{sg}\sigma\tau)\, a_{\sigma(1)} \cdots \tilde{a}_{\tau(q-1)} a_{\sigma(q)} \tilde{b}_{\tau(q)} c_{\sigma(q+1)} \cdots \tilde{c}_{\tau(t-1)}$$

$$= \sum_k (\mathrm{sg}\sigma_k\tau_k)\, h_{2q-1}\left(a_{\sigma_k(1)}, \ldots, \tilde{a}_{\tau(q-1)}, a_{\sigma(q)}\right) \tilde{b}_{\tau(q)} c_{\sigma(q+1)} \cdots \tilde{c}_{\tau(t-1)},$$

where (σ_k, τ_k) is a representative of the class P_q^k.

Since h_{2q-1} is an identity in A_ℓ, we conclude that the second summand equals 0. This finishes the proof of the lemma. □

Theorem 3.2.2 *h_{4n-2} is an identity for any proper subalgebra of $M_n(F)$.*

Proof. Let A be a proper subalgebra of $M_n(F)$. If A is simple, then A satisfies the standard polynomial s_{2n-2} (cf. Lemma 2.2.1), hence A satisfies h_{4n-5}. Otherwise, A can be embedded as F-algebra in $E_{(\ell,m)}(F)$ for some suitable positive integers ℓ and m. Since $h_{4\ell-1}$ and h_{4m-1} are identities for $M_\ell(F)$ and $M_m(F)$ respectively, we apply Lemma 3.2.1 to obtain that h_{4n-2} is an identity for A. □

3.3 A Polynomial test for $E_{(\ell,m)}$

In this section we show that the double Capelli polynomial h_{4n-3} is a polynomial test for the subalgebra $E_{(\ell,m)}$ of $M_n(F)$ for any positive integers ℓ, m such that $\ell + m = n$.

Lemma 3.3.1 *Let A be a subalgebra of $E_{(\ell,m)}(F)$ such that A_ℓ satisfies h_q for some $1 \leq q \leq 4\ell$, and A_m satisfies h_r for some $1 \leq r \leq 4m$. then A satisfies $h_{(q+r)}$.*

Proof. It follows by a combinatorial argument similar to that in the proof of Lemma 3.2.1. □

Proposition 3.3.2 *h_{4n-3} is an identity for every proper subalgebra of $E_{(\ell,m)}$.*

Proof. We consider all possible proper subalgebras of $E_{(\ell,m)}(F)$.

Let us first consider a subalgebra A of $E_{(\ell,m)}$ such that A_ℓ is a proper subalgebra of $M_\ell(F)$. Then $h_{4\ell-2}$ is an identity for A_ℓ as established in Theorem 3.2.2, and h_{4m-1} is an identity for $M_m(F)$. Hence h_{4n-3} is an identity for

$$\begin{bmatrix} A_\ell & M_{\ell\times m}(F) \\ 0 & M_m(F) \end{bmatrix},$$

hence an identity for A.

Similarly, h_{4n-3} is an identity for every subalgebra of $E_{(\ell,m)}$ such that A_m is a proper subalgebra of $M_m(F)$. Clearly, h_{4n-4} is an identity for the semisimple case

$$\begin{bmatrix} M_\ell(F) & 0 \\ 0 & M_m(F) \end{bmatrix}.$$

In Proposition 2.2.5, it is proved that the standard polynomial $s_{2\ell}$ is an identity for the self-extension of irreducible representations:

$$A = \left\{ \begin{bmatrix} a & c \\ 0 & a \end{bmatrix} : a, c \in M_\ell(F) \right\},$$

hence, h_{4n-4} is an identity for A. □

Remark In general, h_{4n-3} is not an identity for $E_{(\ell,m)}$. For instance, if $n = 3$ and $A = E_{(1,2)}$, we have

$$h_9\left(e_{11}, e_{11}, e_{12}, e_{22}, e_{22}, e_{23}, e_{33}, e_{33}, e_{32}\right) = 2e_{12}.$$

CHAPTER 4

Effective detection of n-dimensional representations.

In this chapter, we describe several algorithmic procedures, using polynomial tests (and elementary computational commutative algebra), for determing the existence of certain types of n-dimensional representations of finitely presented algebras. This approach is largely influenced by Letzter's papers [**Le01**] and [**Le02**], and extends the results therein. The basic strategy is to reduce each of the considered representation theoretic decision problems to the problem of deciding whether a particular finite set of commutative polynomials has a common zero. Standard methods of computational algebraic geometry can then be applied (in principle).

4.1 Introduction

Let $R = F\{X_1, \ldots, X_s\}/\langle f_1, \ldots, f_t\rangle$ be a finitely presented algebra over the field F. Assume that F is computable and K is the algebraic closure of F. It is easy to see that the n-dimensional representations of R amount to solutions to a system of tn^2 commutative polynomial equations in sn^2 variables. Moreover, n-dimensional irreducible representations and full block upper triangular representations of R can also be explicitely parameterized by finite systems of commutative polynomial equations using P-tests. Consequently, the techniques of computational algebraic geometry (and in particular, Groebner basis methods) can be used to study the n-dimensional representation theory of R. When the desired n-dimensional representation exists, it is possible (in principle) to produce explicit constructions. An example of these algorithmic procedures is implemented in §§4.3.5, using the computer algebra package Macaulay 2.

4.2 Preliminaries

In this section we develop our notation (which will remain fixed for the remainder) and quickly review some necessary background.

4.2.1 Notation

(i) We will use the expression (n-dimensional) representation of R only to refer to F-algebra homomorphisms $\rho : R \to M_n(K)$; the representation is irreducible when $K_\rho(R) = M_n(K)$. This approach allows us to consider the K-representation theory of R while restricting our calculations to F; in our algorithmic procedures below we will assume that F is computable and that K is the algebraic closure of F

(ii) Let $\mathcal{P}(n)$ denote the minimum positive integer with the following property: For all positive integers q, and for all $a_1, \ldots, a_q \in M_n(K)$, the K-algebra $K\{a_1, \ldots, a_q\}$ is K-linearly spanned by products of the $a_1, \ldots, a_q$ having length no greater that $\mathcal{P}(n)$ (the identity matrix is the product of length zero). It is easy to check that $\mathcal{P}(n) \leq n^2 - 1$, and in **[Pa97]** it is proved that $\mathcal{P}(n)$ is bounded above by the function $f(n) = n\sqrt{2n^2/(n-1) + \frac{1}{4}} + n/2 - 2$.

(iii) Let $\rho : R \to M_n(K)$ be a representation, and set $\Lambda = K_\rho(R)$. It follows from (i) that Λ is K-linearly spanned by the images of the monomials (in the X_i) having length no greater that $\mathcal{P}(n)$. Also, the Cayley-Hamilton Theorem tells us that the nth power of an $n \times n$ matrix is a linear combination of its lower powers. Therefore, Λ is K-linearly spanned by the image under ρ of

$$\{X_{j_1}^{i_1}\cdots X_{j_p}^{i_p} : X_{j_1},\ldots,X_{j_p} \in \{X_1,\ldots,X_s\}; i_1+\cdots+i_p \leq \mathcal{P}(n);$$
$$0 \leq i_1,\ldots,i_p < n\}$$

(iv) For $1 \leq \mu \leq s$, let x_μ denote the generic $n \times n$ matrix $(x_{ij}(\mu))$ (i.e, the $n \times n$ matrix whose ijth entry is the indeterminate $x_{ij}(\mu)$), and set $x = (x_1,\ldots,x_s)$. Note that R has an n-dimensional representation if and only if the entries of $f_1(x),\ldots,f_t(x)$ have a common zero.

4.3 Effective detection

4.3.1 Effective detection of full block upper triangular representations

Ingredient: A subalgebra of $M_n(K)$ does not satisfy the standard identity s_{2n-2} if and only if it is equivalent to a full block upper triangular matrix algebra. (See Theorem 2.3.3).

Application: Decide whether or not a finitely presented algebra has a full block upper triangular representation of size n.

(i) Let $W =$

$$\{x_{j_1}^{i_1} \cdots x_{j_p}^{i_p} : x_{j_1}, \ldots, x_{j_p} \in \{x_1, \ldots, x_s\}; i_1 + \cdots + i_p \leq \mathcal{P}(n);$$
$$0 \leq i_1, \ldots, i_p < n\}$$

(ii) Let w be an indeterminate. For each choice of $w_1, \ldots, w_{2n-2} \in W$ we can construct a subtest that returns "true" if the entries of

$$f_1(x_1, \ldots, x_s), \ldots, f_t(x_1, \ldots, x_s)$$
$$w\,[s_{2n-2}(w_1, \ldots, w_{2n-2})]_{1n} - 1$$

have a common zero. The subtest returns "false" if no common zero exist. It is easy to check that the following are equivalent: (1) at least one of the possible choices of $w_1, \ldots, w_{2n-2}$ produces a "true" in the subtest, (2) there exists a representation for $R \to M_n(K)$ for which the polynomial s_{2n-2} is not satisfied, (3) there exists a full block upper triangular representation $R \to M_n(K)$.

4.3.2 Effective detection of irreducible representations

This algorithm provides an alternate approach to that found in [**Le01**].

Ingredient: h_{4n-2} is a polynomial test for $M_n(K)$. (See Theorem 3.2.2).

Application: Decide whether or not a finitely presented algebra has an irreducible n-dimensional representation.

(i) Let $W =$

$$\{x_{j_1}^{i_1} \cdots x_{j_p}^{i_p} : x_{j_1}, \ldots, x_{j_p} \in \{x_1, \ldots, x_s\};\ i_1 + \cdots + i_p \leq \mathcal{P}(n);$$
$$0 \leq i_1, \ldots, i_p < n\}$$

(ii) Let u, v be indeterminates. For each choice of $w_1, \ldots, w_{2n-2} \in W$ and $v_1, \ldots, v_{2n-2} \in W$, we can construct a subtest that returns "true" if the entries of

$$f_1(x_1, \ldots, x_s), \ldots, f_t(x_1, \ldots, x_s)$$
$$u\,[h_{4n-2}(w_1, \ldots, w_{2n-1}, v_1, \ldots, v_{2n-1})]_{11} - 1$$

have a common zero. The subtest returns "false" if no common zero exists. It is easy to check that the following are equivalent: (1) at least one of the possible choices of $w_1, v_1 \ldots, w_{2n-1}, v_{2n-1}$ produces a "true" in the subtest, (2) there exists an irreducible n-dimensional representation of R.

4.3.3 Effective detection of full upper triangular representations

Ingredient: s_{2n-2} is a polynomial test for the algebra of upper triangular matrices. (See Corollary 2.3.4).

Application: Devise an algorithmic test for deciding whether R has a full upper triangular n-dimensional representation.

(i) For $1 \leq \mu \leq s$, let x_μ denote the $n \times n$ matrix whose

$$ij\text{th entry} = \begin{cases} \text{the indeterminate } x_{ij}(\mu) \text{ if } 1 \leq i \leq j \leq n, \\ 0 \text{ otherwise.} \end{cases}$$

(ii) Let $U(n) =$

$$\{x_{j_1}^{i_1} \cdots x_{j_p}^{i_p} : x_{j_1}, \ldots, x_{j_p} \in \{x_1, \ldots, x_s\};\ i_1 + \cdots + i_p \leq \mathcal{P}(n);$$
$$0 \leq i_1, \ldots, i_p < n\}$$

(iii) Choose $w_1, \ldots, w_{2n-2} \in U(n)$ Let u be an indeterminate, we can construct a test that returns "true" if the entries of

$$f_1(x_1, \ldots, x_s), \ldots, f_t(x_1, \ldots, x_s)$$
$$u[s_{2n-2}(w_1, \ldots, w_{2n-2})]_{1n} - 1$$

have a common zero. The subtest returns "false" if no common zero exists. It is easy to check that the following are equivalent: (1) at least one of the possible choices of $w_1, \ldots w_{2n-2}$ produces a "true" in the subtest, (2) there exists a full upper triangular n-dimensional representation of R.

4.3.4 Nonsplit (ℓ, m)-extension of inequivalent irreducible representations test

This algorithm provides an alternate approach to that found in [**Le02**].

Ingredient: h_{4n-3} is a polynomial test for the nonsplit (ℓ, m)-extension of inequivalent irreducible representations. (See Lemma 3.3.2)

Application: Device an algorithmic test for deciding whether R has a nonsplit, n-dimensional, (ℓ, m)-extension of inequivalent irreducible representations, for fixed ℓ and m, with $\ell + m = n$.

(i) For $1 \leq \mu \leq s$, let x_μ denote the $n \times n$ matrix whose

$$ij\text{th entry} = \begin{cases} \text{the indeterminate } x_{ij}(\mu) \text{ if } i \leq \ell \text{ or } j \geq m, \\ 0 \text{ otherwise.} \end{cases}$$

(ii) Let $U(\ell + m) =$

$$\{x_{j_1}^{i_1} \cdots x_{j_p}^{i_p} : x_{j_1}, \ldots, x_{j_p} \in \{x_1, \ldots, x_s\};\ i_1 + \cdots + i_p \leq \mathcal{P}(n);$$
$$0 \leq i_1, \ldots, i_p < n\}$$

(iii) Let u be an indeterminate. For each choice of $w_1, \ldots, w_{2n-1}, v_1, \ldots, v_{2n-2} \in U(\ell + m)$ we can construct a subtest that returns "true" if the entries of

$$f_1(x_1, \ldots, x_s), \ldots, f_t(x_1, \ldots, x_s)$$
$$u\left[h_{4n-3}(w_1, \ldots, w_{2n-1}, v_1, \ldots, v_{2n-2})\right]_{ij} - 1$$

have a common zero. The subtest returns "false" if no common zero exists. It is easy to check that the following are equivalent: (1) at least one of the possible choices of (i, j) and $w_1, v_1 \ldots, w_{2n-1}, v_{2n-1}$ produces a "true" in the subtest, (2) there exists a nonsplit (ℓ, m)-extension of distinct irreducible representations $R \to E_{(\ell,m)}(K)$

4.3.5 An example

Determine algorithmically whether a given finitely presented algebra has a representation of a particular type.

Ingredients: P-test and elementary computational commutative algebra.

Example: A three-dimensional representation.

Set

$$R = \mathbf{Q}\{X, Y\}/\langle X^3, Y^2 \rangle,$$

Can we find a full block upper triangular 3×3 representation for R?

A Macaulay2 Session

```
F=QQ
R=F[a..z]

M = R^3

i4 : X = matrix{a*M_0, b*M_0+c*M_1, d*M_0+e*M_1+f*M_2}

o4 = | a b d |
     | 0 c e |
     | 0 0 f |

               3       3
o4 : Matrix R   <--- R

i5 : Y = matrix{g*M_0+h*M_1+i*M_2, j*M_1+k*M_2, l*M_2}

o5 = | g 0 0 |
     | h j 0 |
```

```
     | i k l |

               3       3
o5 : Matrix R  <--- R

length2=(X*X, X*Y,Y*X)
length3=(X^2*Y, X*Y*X, Y*X^2, Y*X*Y)

s1=(x)-> x
s2=(x,y)->x*s1(y)-y*s1(x)
s3=(x,y,z)->x*s2(y,z)-y*s2(x,z)+z*s2(x,y)
s4=(x,y,z,w)->x*s3(y,z,w)-y*s3(x,z,w)+z*s3(x,y,w)-w*s3(x,y,z)

f1 = X^3
f2 = Y^2
MatrixRel = f1|f2


w1=X
w2=Y
w3=length2#0
w4=length2#1
W=s4(w1,w2,w3,w4)
r =u*( last flatten entries W^{0})-1
Rel = append(flatten entries MatrixRel,r)
RelIdeal = ideal(Rel)

i23 : 1 % RelIdeal

o23 = 1

o23 : R

--There exists a full block upper triangular representation

S =transpose gens gb RelIdeal

i30 : -- we found the solution

      X =substitute(X,{a=>0,b=>1,c=>0,d=>0,e=>1,f=>0})

o30 = | 0 1 0 |
```

```
      | 0 0 1 |
      | 0 0 0 |

              3       3
o30 : Matrix R  <--- R

i31 : Y =substitute(Y,{g=>0,h=>0,i=>0,j=>0,k=>1,l=>0})

o31 = | 0 0 0 |
      | 0 0 0 |
      | 0 1 0 |

              3       3
o31 : Matrix R  <--- R

i32 :
      ---------------------------END----------------
```

It is easy to verify that X and Y generates a full $(1,2)$-block upper triangular matrix algebra.

CHAPTER 5

The length of the Super-Diagonal and Sub-Diagonal matrices.

The length of an algebra generating set was studied by Paz [**Paz84**] and by Pappacena [**Pa97**]. In §§4.2.1 we saw an application of this notion. In this chapter we study the length of the sub-diagonal and super-diagonal matrices.

Let F be a field, and let A be a finite-dimensional F-algebra. Set $d = dim_F A$. Since A is finite-dimensional over F, it is obviously finitely generated. Let S be a finite generating set for A as an F-algebra. We shall write $A = F\{S\}$ to denote this. Writing $S = \{a_1, \ldots, a_t\}$ we shall adopt the convention that 1 is a word in S of length zero, and write S^i for the set of all words in

S of length $\leq i$. We have the obvious containment $S^i \subseteq S^j$ for $i \leq j$, also $S^iS^j = S^{i+j}$. Writing FS^i for the F-linear span of S^i, we have the following chain of containments (noting that $S^0 = 1$, so $FS^0 = F$):

$$F = FS^0 \subseteq FS^1 \subseteq \cdots \subseteq FS^i \subseteq FS^{i+1} \subseteq \cdots \subseteq F\{S\} = A. \tag{5.1}$$

Since A is assumed finite-dimensional over F, there is an integer k such that

$$FS^k = FS^{k+1} = FS^{k+2} = \cdots = F\{S\} = A. \tag{5.2}$$

We define the *length* of the generating set, written $\ell(S)$, to be the smallest k for which $FS^k = A$, and define $\ell = \max_S \ell(S)$, where the maximum is taken over all finite generating sets, to be the length of A. For the algebra of $n \times n$ matrices over F, Pappacena ([**Pa97**]) has proved that ℓ is bounded above by a function in $O(n^{3/2})$ and Paz ([**Paz84**]) has conjectured that $\ell \leq 2n - 2$.

5.1 Preliminaries

Notation: Henceforth we consider $M_n(F)$, the algebra of $n \times n$ matrices over F, and S the subset of $M_n(F)$ consisting of the following two particular matrices: The super-diagonal matrix U is defined by the law:

$$u_{ij} = \begin{cases} 1 & \text{if} \quad i = j - 1, \\ 0 & \text{otherwise;} \end{cases}$$

and the lower-diagonal matrix $D = U^t$ is defined by the law:

$$d_{ij} = \begin{cases} 1 & \text{if} \quad i = j + 1, \\ 0 & \text{otherwise;} \end{cases}$$

for $1 \leq i, j \leq n$.

Given a matrix $A \in M_n(F)$, we shall call the *δ-diagonal* of A to the set of entries

$$\{a_{ij},\ i = j + n - \delta,\ j = 1 \dots \delta\}, \quad \text{for} \quad \delta = 1 \dots n,$$

$$\{a_{ij},\ j = i + \delta - n,\ i = 1 \dots 2n - \delta\}, \quad \text{for} \quad \delta = n \dots 2n - 1.$$

If all but the δ-diagonal entries of A are equal to 0, we call it a δ-diagonal matrix. If δ is out of the range $1 \dots 2n - 1$, we adopt the convention that a δ diagonal matrix is the zero matrix. The F-subspace of $M_n(F)$ consisting of all δ-diagonal matrix is denoted Δ_δ. Clearly,

$$\dim_F \Delta_\delta = \begin{cases} \delta & \text{if } \delta = 1 \dots n, \\ 2n - \delta & \text{if } \delta = n \dots 2n - 1, \\ 0 & \text{otherwise.} \end{cases}$$

The δ-diagonal matrix with 1's in all of the diagonal entries is called the *δ-identity*. The δ-diagonal matrix with 1's in the first i entries (starting from the left) of the δ-diagonal and 0's in the final $\delta - i$ entries of the δ-diagonal is called the *i-initial segment* of the δ-diagonal, for $1 \leq i \leq \delta$. The δ-diagonal matrix with 0 in the initial $(\delta - i)$ entries (starting from the left) of the δ-diagonal and 1's in the final i entries of the δ-diagonal is called the *i-terminal segment* of the δ-diagonal, for $1 \leq i \leq \delta$. Notice that the δ-identity is the δ-initial and δ-terminal segment of the δ-diagonal.

In order to understand S^i, the sets of words in S of length up to i, it is useful to have in mind what exactly is the action of multiplying U and D on the left and on the right of any matrix A.

(a) UA is the matrix whose i^{th} row equals the $(i+1)^{\text{th}}$ row of A, and its last row is zero (left multiplication by U "pushes up" one row)

(b) AU is the matrix whose $(j+1)^{\text{th}}$ row equals the j^{th} column of A, and its first column is zero (right multiplication by U "pushes right" one column)

(c) DA is the matrix whose $(i+1)^{\text{th}}$ row equals the i^{th} row of A and its first row is zero (left multiplication by D "pushes down" one row)

(d) AD is the matrix whose j^{th} column equals the $(j+1)^{\text{th}}$ column of A and its last column is zero (right multiplication by D "pushes left" one column)

Notice that $U^{(n-1)}$ is the matrix whose $(1\,n)$ entry is 1 and all other entries are zero. With this elementary matrix in hand, and with the help of rules

(a) - (d), we can produce any other elementary matrix. This shows (the well known fact) that S is a generating set for $M_n(F)$.

Let

$$B_1 = \{D^{n-1}\},$$

$$B_2 = \{UD^{n-1}, D^{n-1}U\},$$

$$B_3 = \{D^{n-3},\ UD^{n-2}, D^{n-2}U\},$$

$$\vdots$$

In general, for $\delta = 1 \ldots n$, we define:

$$B_\delta = \{D^{n-\delta}, UD^{n-\delta+1}, D^{n-\delta+1}U, U^2D^{n-\delta+2}, D^{n-\delta+1}U^2, \ldots$$
$$\ldots, U^{\frac{\delta-1}{2}}D^{n-\frac{\delta+1}{2}}, D^{n-\frac{\delta+1}{2}}U^{\frac{\delta-1}{2}}\}$$

if δ is odd, and

$$B_\delta = \{UD^{n-\delta+1}, D^{n-\delta+1}U, U^2D^{n-\delta+2}, D^{n-\delta+1}U^2, \ldots, U^{\frac{\delta}{2}}D^{n-\frac{\delta}{2}}, D^{n-\frac{\delta}{2}}U^{\frac{\delta}{2}}\}$$

if δ is even.

Proposition 5.1.1 *The set B_δ is a basis of the subspace Δ_δ, for $\delta = 1 \ldots n$.*

Proof. Assume for the moment that δ is even. Notice that, for $i = 1 \ldots \frac{\delta}{2}$, the matrix $U^iD^{n-\delta+i}$ is the $(\delta - i)$-initial segment of the δ-diagonal. Similarly, for $i = 1 \ldots \frac{\delta}{2}$, the matrix $D^{n-\delta+i}U^i$ is the $(\delta - i)$-terminal segment of the δ-diagonal. These matrices form a basis for Δ_δ. The argument is similar when δ is an odd number. □

Proposition 5.1.2 *The length of S is n.*

Proof. Since $U = D^t$, we can apply Proposition 5.1.1 to show that for $1 \le i \le n$, the set B_i^t, consisting of the transpose of each matrix in B_i, is a basis for the F-subspace of $(2n - i)$-diagonal matrices. Since each matrix in $M_n(F)$ is a linear combination of δ-diagonal matrices, we conclude that n is an upper bound for $l(S)$. For $n \ge 2$, in order to generate the 2-diagonal, words of length at least n are required, namely UD^{n-1} and $D^{n-1}U$. This completes the result. □

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