January 2022 Applied Mathematics Qualifying Written Exam

PART I. Do three of the following four problems.

1. Compute a two term asymptotic approximation in ϵ for the solution of the equation

$$\frac{d^2h}{dt^2} = -\frac{1}{(1+\epsilon h)^2}, \quad h(0) = 0, \; \frac{dh}{dt}(0) = 1.$$

2. Given the system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

   (a) Compute the eigenvalues and eigenvectors of the system.

   (b) Sketch the phase portrait.

3. Nondimensionalize and solve the problem

$$k\nabla^2 u = -1$$

   on the domain $\{x^2 + y^2 < 1\}$ with boundary conditions $u = 0$ on $x^2 + y^2 = 1$.

4. Let

$$X = \{y \in C^2([0,1]) : y(0) = y(1) = 0\},$$

   and define a functional $J : X \to R$ by

$$J[y] = \int_0^1 xy\,dx.$$

   Find the minimizer of J on X subject to the constraint

$$\int_0^1 (y')^2 dx = 1$$

PART II. Do two of the following three problems.

1. Determine the leading order approximation and first correction for all roots of the equation
$$\epsilon x^5 - x^3 - 1 = 0.$$

2. Find the first variation of the functional
$$J(y) = (y(2))^2 + \int_1^2 (xy + (y')^2)dx$$
with admissible set $\{y \in C^2[1,2],\ y(1) = 3\}$. What is the boundary value problem for its extrema? (Do not solve.)

3. (a) Find the adjoint operator for the operator
$$Lu = u_{xx} + 4\pi^2 u, \quad u(0) = u(1) = 0.$$
(b) Find conditions on the parameters α and β for which a solution to the problem
$$u_{xx} + 4\pi^2 u = f(x), \quad u(0) = u(1) = 0$$
exists, where $f(x) = \alpha \sin(2\pi x) - \beta \sin^3(2\pi x)$.