Applied Mathematics Qualifying Written Exam (August 2023)

1 Part I: do 3 of 4

1. Consider Maxwell's equations in the medium

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \quad \nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t}, \quad \nabla \cdot \boldsymbol{D} = 0$$

together with the constitutive relations $\boldsymbol{D} = \varepsilon \boldsymbol{E}$, $\boldsymbol{B} = \mu \boldsymbol{H}$, where $\mu \geq 1$ and $\varepsilon > 1$ are the constant relative magnetic permeability and the electric permittivity of the medium, respectively.

   (a) Show that both the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ satisfy the 3 dimensional wave equation.

   (b) According to your wave equation, what is the speed of wave propagation?

2. Let the velocity vector field of a moving fluid be $\boldsymbol{v}(\boldsymbol{x}, t)$, where $\boldsymbol{x}$ is the Eulerian coordinate in $\mathbb{R}^3$. Let $\rho(\boldsymbol{x}, t)$ be its density. Derive the balance of mass equation in an imaginary volume $\Omega \subset \mathbb{R}^3$ through which the fluid moves. Explain each step of the derivation.

3. Consider the incompressible Navier-Stokes system

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\nabla \boldsymbol{v})\boldsymbol{v} = -\frac{\nabla p}{\rho_0} + \nu \Delta \boldsymbol{v}, \quad \nabla \cdot \boldsymbol{v} = 0$$

defined on a large spatial domain $\Omega_L = L\Omega = \{L\boldsymbol{x} : \boldsymbol{x} \in \Omega\}$. Rescale the above system, so that the rescaled system would be again a Navier-Stokes system, but with $\rho_0 = 1$, $\nu = 1$, and defined on the spatial domain Ω. Write the explicit relation between the solution of the rescaled system and the solution of the original one.

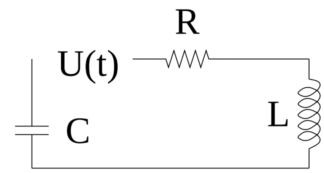
4. Consider the RCL circuit shown in the figure below. The voltage $U(t)$ and current $I(t)$ in the circuit are related by the differential equation

$$RI' + LI'' + \frac{1}{C}I = U'(t). \tag{1}$$

Consider the effect of switching on the source of constant voltage $U(t) = U_0 H(t) = U_0 \chi_{(0,+\infty)}(t)$. Assume that $I(t) = 0$ for all $t < 0$. Then, for all $t > 0$ the current will satisfy the ODE

$$RI' + LI'' + \frac{1}{C}I = 0.$$

Derive the initial conditions for this ODE that determine the current $I(t)$ uniquely, using the fact that equation (1) holds in the sense of distributions for all $t \in \mathbb{R}$. You don't need to solve the resulting initial value problem.



2 Part II: do 2 of 3

1. Consider the following optimal control problem. Let $u(t)$ be governed by the dynamics

$$u'(t) = \alpha(t), \quad u(0) = u_0,$$

where $\alpha(t)$ is the control, satisfying the constraint $|\alpha'(t)| \le m$. The goal is to maximize

$$P[\alpha] = \int_0^1 u(t)dt,$$

provided that $u(1) = u_1$. Assuming that $|u_0 - u_1| < m$ find the optimal control $\alpha^*(t)$ using the Pontryagin maximum principle. Compute the maximal possible payoff $P[\alpha^*]$.

2. Suppose that in an imaginary universe, the force of gravity decays according to the inverse cube law, i.e. the potential energy of a unit point mass at a distance r from the point mass $M \gg 1$ is given by

$$\Pi = -\frac{GM}{r^2}.$$

If $\boldsymbol{x}(t) \in \mathbb{R}^3$ describes the position of the unit point mass at time t then the the action functional will be

$$A[\boldsymbol{x}] = \int_{t_0}^{t_1} \left\{ \frac{|\dot{\boldsymbol{x}}|^2}{2} + \frac{GM}{|\boldsymbol{x}|^2} \right\} dt.$$

(a) Prove that the family of transformations

$$T(t, \boldsymbol{x}; \epsilon) = e^{2\epsilon} t, \qquad \boldsymbol{X}(t, \boldsymbol{x}; \epsilon) = e^{\epsilon} \boldsymbol{x}$$

is a variational symmetry of $A[\boldsymbol{x}]$.

(b) What is the conserved quantity corresponding to this variational symmetry?

3. Find the leading term asymptotics of the integral

$$I(x) = \int_0^1 e^{-(t^2-t)x} dt$$

as $x \to +\infty$.