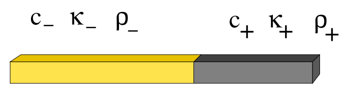
Applied Mathematics Qualifying Written Exam (August 2022)

1 Part I: do 3 of 4

1. Consider a possibly non-uniform cable with linear density $\rho(x)$ moving in the three-dimensional space. Let $\boldsymbol{r}(x,t)$ be the position vector of the material point $x \in [0,L]$ of the cable at time t.
   (a) Give a definition of the tension vector $\boldsymbol{T}(x,t)$;
   (b) Apply Newton's law of motion $\boldsymbol{F} = m\boldsymbol{a}$ to each segment $[x, x+\Delta x]$ of the cable to derive the equation of motion of the cable relating functions $\rho(x)$, $\boldsymbol{r}(x,t)$ and $\boldsymbol{T}(x,t)$.

2. Consider the heat propagating in a bimetal strip: 

   To the left of the junction at $x = 0$ the temperature $T(x,t)$ solves the heat equation

$$c_-\rho_-\frac{\partial T}{\partial t} = \kappa_-\frac{\partial^2 T}{\partial x^2}.$$

   To the right of the junction it satisfies

$$c_+\rho_+\frac{\partial T}{\partial t} = \kappa_+\frac{\partial^2 T}{\partial x^2},$$

   where $c_\pm$ are the specific heats of the two materials, $\rho_\pm$ are the densities and $\kappa_\pm$ are heat conductivities. What are the conditions that the temperature field $T(x,t)$ has to satisfy at the junction $x = 0$?

3. Consider the Maxwell system of equations governing propagation of electromagnetic waves in vacuum.

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \quad \nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}, \quad \nabla \cdot \boldsymbol{E} = 0.$$

   Show that each component of the electric field $\boldsymbol{E}$ solves a wave equation with the the speed of propagation c.

4. The "motion" of the electron in a hydrogen atom is governed by Schrödinger's equation

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m_e}\Delta\psi + V(\boldsymbol{x})\psi, \qquad V(\boldsymbol{x}) = -\frac{e^2}{4\pi\varepsilon_0|\boldsymbol{x}|}, \qquad \boldsymbol{x} \in \mathbb{R}^3, \ t > 0.$$

   where $\hbar$ is the Planck constant, m_e is the mass of the electron, e is its electrical charge and ε_0 is the dielectric permittivity of the vacuum. Express solutions $\psi(\boldsymbol{x},t)$ of the Schrödinger's equation in terms of the solutions $\phi(\boldsymbol{\xi},\tau)$ of

$$i\frac{\partial \phi}{\partial \tau} = -\frac{1}{2}\Delta\phi - \frac{\phi}{|\boldsymbol{\xi}|}, \qquad \boldsymbol{\zeta} \in \mathbb{R}^3, \ \tau > 0.$$

   Give expressions of the characteristic time and length scales in terms of the physical parameters $\hbar$, e, m_e and ε_0.

2 Part II: do 2 of 3

1. Find all possible speeds of propagation of acoustic waves in an elastic solid by examining plane wave solutions of

$$\mu\Delta \boldsymbol{u} + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) = \rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}.$$

   Note that plane wave solutions take the form $\boldsymbol{u}(\boldsymbol{x},t) = u_0(\boldsymbol{x}\cdot\boldsymbol{k} - \omega t)\boldsymbol{a}$, where u_0 is smooth scalar function of one variable, $\boldsymbol{a}$ and $\boldsymbol{k}$ are constant vectors in $\mathbb{R}^3$ and ω is a constant scalar.

2. Apply the Poincaré-Linstedt method to compute the correct *asymptotics of the period* of the solution of the differential equation

$$y'' + y = A(1+\epsilon y^2), \quad y(0) = 1,\ y'(0) = 0$$

   to first order in ϵ.

   FYI, this is a rescaled version of the relativistic correction to the equation of motion of Mercury, where $y(\theta) = a/r(\theta)$, $A = GM/av^2$, $\epsilon = 3v^2/c^2$. Here a is the perihelion (smallest distance from the sun), v the speed of Mercury at perihelion, $r(\theta)$—the distance from the sun when Mercury is θ radians along its orbit starting from the perihelion.

3. Consider a dynamical system (coming from a model of a fishery)

$$\dot{x} = x(1-x) - \frac{rx}{x+1}, \qquad x > -1.$$

   (a) Identify all equilibria on the half-line $x > -1$ and determine their stability.

   (b) Sketch the bifurcation diagram. Make sure to indicate stable and unstable branches clearly.

   (c) According to your bifurcation diagram, what is the bifurcation point and what is its type?