Applied Mathematics Qualifying Written Exam

PART I. Do three of the following four problems.

1. Use the Euler-Lagrange equation to determine a minimizer $u(x)$ for the functional

$$J[u] = \int_{x_1}^{x_2} \sqrt{1+u_x^2}\, dx$$

subject to $u(x_1) = u_1$ and $u(x_2) = u_2$. Explain why your solution is actually a minimizer.

2. Find an approximation for the roots of the equation

$$\sin(x^2 + \epsilon \sin x) = 0, \;\; |\epsilon| \ll 1,$$

that is correct to first order in ϵ.

3. Consider the nonlinear oscillator

$$\alpha \ddot{u} + \beta u^3 = 0$$

with $\alpha, \beta > 0$, and $u(0) = A$, $\dot{u}(0) = 0$.

   (a) Find a conserved quantity.

   (b) Nondimensionalize both the original equation and the conserved quantity equation.

   (c) What is the oscillation time scale?

4. Find first order inner, outer, and uniform approximations to the boundary value problem

$$\epsilon u'' - u = 0, \quad u(0) = 1, \;\; u(1) = 0.$$

PART II. Do two of the following three problems.

1. Compute the Green's function for the problem

$$u'' - u = f(x). \quad -\infty < x < \infty,$$

with boundary conditions $\lim_{x\to-\infty} u(x) = \lim_{x\to\infty} u(x) = 0$ and use it to find the solution $u(x)$.

2. Use separation to solve the Laplace equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

in the region $r > 1$ subject to boundary condition $u(1,\theta) = f(\theta)$ for some smooth, 2π-periodic function f.

3. A nonlinear spring has equation of motion

$$m\frac{d^2y}{dt^2} = -\beta y + \gamma y^2 \tag{1}$$

with $m, \beta > 0$ and with initial conditions $y(0) = A$, $\dot{y}(0) = B$. Scale the equation to obtain a non-dimensional version in which equation (1) is parameter free, and identify characteristic system time and length scales. Identify any non-dimensional numbers and interpret their significance. Explain the significance of the scaled version of equation (1) being parameter free.