Applied Mathematics Qualifying Written Exam (August 2019)

1 Part I: do 3 out of 4

1. Consider the equation
$$u' = g(u)u$$
with initial condition $u(0) = u_0$. Set $g(u) := r\left(\frac{1-u}{M}\right)^p$ and $M = \max u$. Scale this equation and explain your choices.

2. Consider the equation
$$\ddot{q} = \alpha(t),$$
where q is the position of an object and α a control.
   (a) Write above equation as a system of first-order equations.
   (b) Compute the controllability matrix G.
   (c) Check whether the system is controllable.

3. Derive the Euler-Lagrange equation for
$$J[u] = \int_V \frac{1}{2}|\nabla u|^2 + r(\vec{x})u \, d\vec{x}.$$

4. Given the system
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$
   (a) compute the eigenvectors and eigenvalues of the system,
   (b) and sketch the phase portrait.

2 Part II: do 2 out of 3

1. Formulate the Hamilton-Jacobi-Bellman equation for the following system:

$$\dot{x}(s) = f(x(s), \alpha(s)), \quad 0 < s < T \tag{1}$$

$$x(0) = x_0 \tag{2}$$

$$J(x,t) = \int_0^T r(x(s), \alpha(s))\, ds + g(x(T)) \tag{3}$$

2. Minimize

$$J[u] := \int_0^1 \frac{1}{2} u(t)^2 - x(t)\, dt,$$

   subject to $\dot{x} = 2(1 - u(t))$, $x(0) = 1$. Use the Hamiltonian and Euler-Lagrange equations to do so.

3. Consider a pendulum with mass m attached to a string of length L. Then one can formulate a force balance

$$mL^2\ddot{\vartheta} + b\dot{\vartheta} + mgL\sin\vartheta = \Gamma,$$

   where Γ is an applied torque.

   (a) Linearize above equation, assuming b is “large”.

   (b) Non-dimensionalize the linearized equation.

   (c) Find all fixed points for $\Gamma = -1$ and classify them.