Comprehensive Examination in Algebra
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Part I. Solve three of the following problems.

I.1 Let $D_4 = \langle r, s | r^4 = s^2 = srsr = e \rangle$ denote the dihedral group of order 8 and $Q_8 = \langle a, b | a^4 = b^2a^2 = b^{-1}aba = e \rangle$ the group of quaternions, also of order 8.

(a) Show that D_4 and Q_8 are not isomorphic as groups.

(b) Show that any nonabelian group of order 8 must be isomorphic to one of D_4 or Q_8.

I.2 Let $A = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 2 & 2 \\ 2 & 0 & 2 \end{smallmatrix}\right) \in M_3(\mathbb{F}_3)$, where $\mathbb{F}_3$ denotes the field of three elements.

(a) Calculate the characteristic polynomial of A.

(b) Find the eigenvalues and eigenspaces (in $\mathbb{F}_3^3$) of A.

(c) What is the minimal polynomial of A? Justify your answer.

(d) Is A diagonalizable over $\mathbb{F}_3$? Over an algebraic closure $\overline{\mathbb{F}}_3$? Justify your answer.

I.3 From the definition of noetherian, show that every principal ideal domain is a noetherian ring. (In particular, you may NOT appeal to the fact that a ring is noetherian if and only if all of its ideals are finitely generated.)

I.4 Let $f = x^4 + 4x^2 + 2 \in \mathbb{Q}[x]$.

(a) Show that f is irreducible over $\mathbb{Q}$.

(b) Calculate the Galois group of the splitting field of f over $\mathbb{Q}$ (up to isomorphism).

Part II. Solve two of the following problems.

II.1 Let p be a prime number, $\mathbb{F}_p$ be the field of p elements, and $G = \mathrm{GL}_2(\mathbb{F}_p)$ be the group of invertible 2×2 matrices with entries in $\mathbb{F}_p$.

(a) Prove that the order of G is $(p^2 - 1)(p^2 - p)$. (Hint: the columns of any element of G form a basis for $\mathbb{F}_p^2$.)

(b) Let $g_0 \in G$ such that the characteristic polynomial $f \in \mathbb{F}_p[x]$ of g_0 is equal to $(x-\alpha)^2$ for some $\alpha \in \mathbb{F}_p^\times$. Write down the possible Jordan forms for such a g_0; justify your answer.

(c) For g_0 as above, calculate the size of the conjugacy class of g_0 in G; justify your answer. (It may be necessary to break your answer into cases.)

II.2 Let $R = \mathbb{Z}[i]$, where i is a root of the polynomial $x^2 + 1$. We recall that R is a principal ideal domain; you do not need to prove this.

(a) Find the units in R and justify your answer. (You may use standard facts about norms without proof.)

(b) Show that both $1 + i$ and 7 are prime in R.

(c) Consider the finitely generated R-module $M = R/\langle 28 \rangle \oplus R^2/\langle (1 + i, 1 - i) \rangle$. Find the rank, invariant factors, and elementary divisors of M; justify your answers.

II.3 Let $K = \mathbb{F}_{11}(x)$, where $\mathbb{F}_{11}$ denotes the field with 11 elements. Let $f = t^{10} - x \in K[t]$, and let F be the splitting field of f over K.

(a) Does K contain a primitive 10-th root of unity? Explain your answer.

(b) Calculate $[F : K]$. Justify your answer.

(c) Explain why F/K is Galois, and calculate its Galois group, up to isomorphism.

(d) Calculate all of the intermediate fields between F and K. Your answers should be given as simple extensions of K, not as fixed fields of subgroups of the Galois group, though you may use Galois theory to justify that you have found all intermediate fields.