Comprehensive Examination in Algebra
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Part I. Do three of these problems.

I.1 Let G be a group (not necessarily finite) and let N be a normal subgroup of G. Prove:

(a) The centralizer $C_G(N) = \{g \in G \mid gn = ng \text{ for all } n \in N\}$ is a normal subgroup of G.

(b) If N is finite then $C_G(N)$ has finite index in G.

(c) If N is finite and G/N is cyclic then the center $Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}$ has finite index in G.

I.2 Let V be a finite dimensional vector space over a field F and let $\langle\,.\,,\,.\,\rangle\colon V \times V \to F$ be a bilinear form which is not necessarily symmetric. The right and left radicals of $\langle\,.\,,\,.\,\rangle$ are the subspaces of V that are defined by

$$\mathcal{R} = \{v \in V \mid \langle V, v\rangle = 0\} \qquad \text{and} \qquad \mathcal{L} = \{v \in V \mid \langle v, V\rangle = 0\}\ ,$$

respectively. (You need not prove that $\mathcal{R}$ and $\mathcal{L}$ are subspaces of V.)

(a) Use the bilinear form $\langle\,.\,,\,.\,\rangle$ to construct a linear transformation f from V to the dual space $(V/\mathcal{R})^*$ of $V/\mathcal{R}$ such that $\operatorname{Ker} f = \mathcal{L}$.

(b) Show that $\dim_F \mathcal{L} = \dim_F \mathcal{R}$, and deduce that the map f is surjective.

I.3 Let R be a commutative ring and let P be a prime ideal of R. If V is a (left) R-module, put

$$W = \{v \in V \mid av = 0 \text{ for some } a \in R, a \notin P\}\ .$$

(a) Show that W is an R-submodule of V.

(b) If V is an irreducible (i.e., simple) R-module and $W = 0$, prove that P is a maximal ideal.

I.4 Let F be a field and let $R \subseteq F$ be a subring of F. Assume that F is *integral* over R, that is, for every $\alpha \in F$, there is a monic polynomial in $f \in R[x]$ (depending on α) such that $f(\alpha) = 0$. Show that R is a field.

Part II. Do two of these problems.

II.1 Let G be a finite group and let $\mathrm{Syl}_p(G)$ denote the set of all Sylow p-subgroups of G. Assume that $\mathrm{Syl}_p(G)$ has exactly n elements for some prime p. Let $\mathcal{S}_n$ denote the symmetric group of degree n and consider the group homomorphism $f\colon G \to \mathcal{S}_n$ that is given by the conjugation action of G on $\mathrm{Syl}_p(G)$. Show that, for any $P, Q \in \mathrm{Syl}_p(G)$ with $P \neq Q$, one has $f(P) \neq f(Q)$.

II.2 Let R be a commutative ring. If $a \in R$, we write $\mathrm{ann}(a) = \{r \in R \mid ar = 0\}$ for the annihilator of a in R; this is an ideal of R. (You need not prove this fact.) Ideals of the form $\mathrm{ann}(a)$ for $0 \neq a \in R$ are referred to as *annihilator ideals* of R. We further let $\mathcal{P} \subseteq R$ denote the set of all elements $a \in R$ such that $\mathrm{ann}(a)$ is a prime ideal of R.

(a) Suppose there exists an annihilator ideal of R, say Q, that is maximal within the set of all annihilator ideals of R. Prove that Q is a prime ideal. (In particular, $\mathcal{P}$ is not empty in this case.)

(b) If $a \in \mathcal{P}$ and $r \in R$, show that either $ar = 0$ or $\mathrm{ann}(ar) = \mathrm{ann}(a)$.

(c) If $a, b \in \mathcal{P}$ and $ab \neq 0$, prove that $\mathrm{ann}(a) = \mathrm{ann}(b)$.

II.3 Let F/K be an extension of fields. We say that an element $\alpha \in F$ is *abelian* if the subfield $K(\alpha) \subseteq F$ is a finite Galois extension of K with abelian Galois group $\mathrm{Gal}(K(\alpha)/K)$. Prove that the set of abelian elements of F is a subfield of F containing K.