Comprehensive Examination in Algebra

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Part I: Do three of the following problems

1. Let G be a nonzero finite abelian group. Show that the following two conditions are equivalent: (1) G is not isomorphic to a direct product of two nonzero subgroups. (2) G is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$, for some prime number p and positive integer n.

2. Let V be a finite dimensional vector space over a field K, and let T denote a K-linear transformation from V to V. Further assume, for some positive integer n, that $T^n(v) = 0$ for all $v \in V$. Prove there exists a basis for V such that the matrix of T with respect to this basis is strictly upper triangular (i.e., the ij-th entry is zero when $i \geq j$).

3. Consider the subring of $\mathbb{Q}[x]$,

$$R = \{a_0 + a_1 x + ... + a_n x^n \ : \ a_0 \in \mathbb{Z},\ a_1, \dots, a_n \in \mathbb{Q}\}.$$

(You do not have to verify that R is a subring of $\mathbb{Q}[x]$.)

(i) Prove that R is an integral domain and that the set of units of R is $\{-1, 1\}$.

(ii) Specifiy which of the following are prime ideals of R and which are maximal:

$$\{\, f(x) \in R \ : \ f(0) = 0 \,\}, \qquad (x), \qquad (p),$$

where $p \in \mathbb{Z}$ is a prime. Be sure to justify your answers.

(iii) Is R a principal ideal domain? If so, prove it. If not, give an example of an ideal of R which is not principal.

4. Let K be a field, let $f(x) \in K[x]$ be a polynomial of degree n, and let F be a splitting field for f over K. Show that if $[F : K] = n!$ then f is irreducible in $K[x]$.

Part II: Do two of the following problems

1. Let G be a group.
   (a) Prove that if $|G| = 36$ then G is not simple.
   (b) Prove that if $|G| = 80$ then G is not simple.

2. Let R be a principal ideal domain, and let M be an R-module. For $m \in M$, let

$$\operatorname{Ann}_R(m) = \{r \in R \ : \ r.m = 0\}.$$

We say that M is *torsion* if $\operatorname{Ann}_R(m) \neq 0$ for all $m \in M$. Now assume that M is torsion, and let p be a prime of R. Set

$$M_p = \{m \in M \ : \ \operatorname{Ann}_R(m) = (p^n) \text{ for some } n \in \mathbb{N}\},$$

and set

$$M'_p = \left\{m \in M \ : \ \big(\operatorname{Ann}_R(m), p\big) = 1\right\}.$$

(i) Show that M_p and M'_p are submodules of M.
(ii) Show that $M = M_p \oplus M'_p$.
(iii) Find M_p for the torsion $\mathbb{Z}$-module $M = \mathbb{Q}/\mathbb{Z}$.

3. Let p be a prime and let K be the splitting field of the polynomial $x^p - 2003$ over $\mathbb{Q}$. (Note that 2003 is prime.)
(i) Prove $K = \mathbb{Q}(\alpha, \xi)$, where α is a pth root of 2003 and ξ is a primitive pth root unity.
(ii) Prove that $[K : \mathbb{Q}] = p(p-1)$.
(iii) Prove that the Galois group of $K/\mathbb{Q}$ is isomorphic to the group of invertible 2×2 matrices of the form

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix},$$

with entries in the field $\mathbb{Z}/p\mathbb{Z}$. (Hint: consider the action of the elements of the Galois group on α and ξ.)