PhD Algebra Exam
Fall 1989

Part I: Do three of these problems.

1. Let G be a group, $I(G)$ the set of inner automorphisms of G.

a) Show $I(G)$ is a group.
b) Show $G/Z(G) \sim I(G)$. ($Z(G) =$ center of G)

2. Let R be a commutative ring, I an ideal of R. Define the annihilator $A(I)$ by $A(I) = \{x \in R : xI = 0\}$. Show $A(I)$ is an ideal of R, and in each of the following cases, find $A(I)$.

a) $R = \mathbb{Z}_{60}$, $I = (18)$
b) $R = \mathbb{Q}[x]/(x^2 - 4)$, $I = (x^2 - 4x - 4)$
c) $R = \mathbb{Q}[x]/(x^2 - 2x + 1)$, $I = (x^4 - 1)$

3. Let V be a finite dimensional vector space over any field, W a subspace. If $n = \dim V$, $r = \dim W$, show that $\dim(V/W) = n - r$.

4. Let F be a finite integral domain.

a) Show that F is a field.
b) Show $|F| = pr$ for some prime p.
c) Show by example that $r > 1$ is possible.

Part II: Do two of these problems.

5. Let G be a finite group such that $|G|$ is the product of distinct primes. Suppose that for each prime dividing $|G|$ the corresponding Sylow subgroup is unique. Prove G is Abelian, and thus cyclic.

6. For each of the following field extensions, determine the group of automorphisms (fixing the ground field) and state (and explain) whether or not the extension is Galois.

a) $\mathbb{Q}(\sqrt[6]{2})/\mathbb{Q}(\sqrt[3]{2})$
b) $\mathbb{Q}(\sqrt[6]{2})/\mathbb{Q}(\sqrt{2})$
c) $\mathbb{Z}_p(x)/\mathbb{Z}_p(x^p)$, p a prime
d) $\mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q}$
e) $\mathbb{Q}(z)/\mathbb{Q}$ where $z =$ primitive 15^{th} root of 1

7. Let R be the subring of $\mathbb{Q}(x)$ consisting of rational functions whose denominators have non-zero constant term, $R = \{f/g : g(0) \neq 0\}$.

a) Find all the units of R
b) Find all the maximal ideals of R
c) Find all the ideals of R