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A MEAN VALUE THEOREM FOR CLASS NUMBERS OF QUADRATIC EXTENSIONS OF FUNCTION FIELDS

A Dissertation
Submitted to
the Temple University Graduate Board

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of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

by
Ibrahim Al-Rasasi
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ABSTRACT

A MEAN VALUE THEOREM FOR CLASS NUMBERS OF QUADRATIC EXTENSIONS OF FUNCTION FIELDS

Ibrahim Al-Rasasi
DOCTOR OF PHILOSOPHY

Temple University, August, 2001

Professor Boris Datskovsky, Chair

In this thesis we study a zeta function associated with the space of binary quadratic forms with coefficients in a function field of characteristic different from two. We establish the convergence, analytic continuation, and the functional equation for this zeta function. The method we use is that of T. Shintani as illustrated in the work of B. Datskovsky and D. J. Wright using adelic analysis.

As an application of studying this adelic zeta function, we obtain a mean value theorem for class numbers of quadratic extensions of a function field. This will be achieved by first conducting some local analysis. This local analysis amounts to studying certain integrals, which we call orbital zeta functions, that appear in a natural way as local factors of the adelic zeta function we started with. Next we put together the global and local information we obtained to construct a sequence of Dirichlet series. Studying some analytic properties of this sequence of Dirichlet series will yield the mean value theorem.

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To

My Parents,

with all my love and gratitude.

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CHAPTER 1

INTRODUCTION

1.1 Objectives

The study of integral binary quadratic forms goes back to Gauss. Gauss [9] conjectured mean value formulas for the number h_d of $SL_2(\mathbb{Z})-$equivalence classes of primitive integral binary quadratic forms of discriminant d. Gauss's formulas were first proved by Lipschitz [15] for $d < 0$ and by Siegel [22] for $d > 0$. Currently, the best result for $d < 0$ is due to Vinogradov [24] and that for $d > 0$ is due to Shintani [21]. Their respective formulas are as follows: For any $\epsilon > 0$,

$$\sum_{0<-d<x} h_d = \frac{4\pi}{21\zeta(3)} x^{\frac{3}{2}} - \frac{2}{3} x + O(x^{\frac{2}{3}+\epsilon}),$$

$$\sum_{0<d<x} h_d \log \epsilon_d = \frac{\pi^2}{18\zeta(3)} x^{\frac{3}{2}} - \frac{1}{2\zeta(2)} x \log x + \frac{1}{2\zeta(2)} (1 - \log(2\pi) + \frac{\zeta'}{\zeta}(2)) x + O(x^{\frac{3}{4}+\epsilon}),$$

where $\epsilon_d = \frac{t+u\sqrt{d}}{2}$ and t and u are the smallest positive integral solutions of $t^2 - du^2 = 4$.

As Gauss noted, integral binary quadratic forms are closely connected to quadratic extensions of $\mathbb{Q}$. Hence similar asymptotic formulas for class numbers of quadratic fields were obtained. In 1985, Goldfeld and Hoffstein [10] gave the following formula: If $\Re(s) \geq 1$, then for any $\epsilon > 0$,

$$\sum_{0<\pm m<x} L(s,\chi_m) = C(s)x + O(x^{\frac{1}{2}+\epsilon}),$$

where $C(s) = \frac{3}{4}\zeta(2s)(1-2^{-2s})\prod_{p\neq 2}(1-p^{-2}-p^{-2s-1}+p^{-2s-2})$, m is square-free, and χ_m is the real primitive Dirichlet character satisfying $\zeta_{\mathbb{Q}(\sqrt{m})}(s) = \zeta(s)L(s,\chi_m)$. When $s = 1$, the above formula gives an asymptotic formula for class numbers of quadratic fields. In 1993, Datskovsky [2] gave a general mean value theorem for class numbers of quadratic extensions of a number field, namely

$$\lim_{x\to\infty}\frac{1}{x}\sum_{D_{L/K}\leq x, L\sim X_S} Res_{s=1}\zeta_L(s) = \frac{Res_{s=1}\zeta_K(s)^2}{2^{r_1(K)+r_2(K)}}\zeta_L(2)R_{X_{S_0}}\prod_{v\notin S}(1-q_v^{-2}-q_v^{-3}+q_v^{-4}).$$

This formula, in turn, can then be translated into a mean value theorem for $h_L R_L$ as $Res_{s=1}\zeta_L(s)$ is given in terms of $h_L R_L$, where h_L and R_L are respectively the class number and the regulator of L.

Our ultimate goal in this thesis is to prove similar results to what Datskovsky did but in the case of quadratic extensions of a function field. Namely, we would like to obtain asymptotic formulas for $\sum_{[L:K]=2, D_{L/K}\leq x} Res_{s=1}\zeta_L(s)$, where L/K is a quadratic extension of a fixed function field K, $D_{L/K}$ is the norm of the relative discriminant of L over K, $\zeta_L(s)$ is the Dedekind zeta function of L. We achieve this via the theory of zeta functions associated with the space of binary quadratic forms with coefficients in a function field. The basic tool we use is adelic analysis in the spirit of Tate's thesis [23].

At this point, it is worth mentioning that the zeta function we will work with, see (1.1) below, is due originally to Shintani [20]. In his study of integral binary cubic forms, Shintani introduced and investigated certain integrals, which he called zeta functions, which enabled him to obtain some density results about the class number of primitive integral binary cubic forms. The results he obtained are improvements of previous results obtained by Davenport [5]. Shintani's work was over $\mathbb{C}$. The adelic version of Shintani's zeta function was later studied by Wright [27], Datskovsky [1] (for binary cubic forms with coefficients in a number field and a function field, respectively), Yukie [28], and Datskovsky [2] (for binary quadratic forms with coefficients in a number field). By further analysis of these zeta functions, Datskovsky and Wright were able to obtain density results for discriminants of cubic extensions of any global field of characteristic other than 2 or 3 (see [3] and [4]). The *first* main objective of this thesis is to study the adelic zeta function associated with the space of binary quadratic forms with coefficients in a function field. The *second* objective is to obtain a mean value theorem for class numbers of quadratic extensions of a function field. We give next a brief description of the content of this thesis.

Let K be a function field in one variable over a finite field of constants F_q, $q \neq 2^n$. Let V be the 3-dimensional affine space. Identify V with the space of binary quadratic forms by means of the correspondence

$$x = (x_1, x_2, x_3) \longleftrightarrow F_x(u, v) = x_1u^2 + x_2uv + x_3v^2.$$

Set $G = Gl_1 \times Gl_2$. Let G act on V as follows:

$$F_{g\cdot x}(u, v) = tF_x(au + cv, bu + dv)$$

for $g = (t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in G$ and $x \in V$. Now V_K is a 3-dimensional vector space over K. Then this action of G on V gives rise to a representation

$$\varrho : G \longrightarrow Gl(V)$$

defined over K. Set $H = \varrho(G)$.

For $x \in V$, let $P(x) = x_2^2 - 4x_1x_3$ denote the discriminant of x. A form x is called nonsingular if $P(x) \neq 0$ and singular otherwise. Denote by V'_K the set of nonsingular forms with coefficients in K. Set $V''_K = \{x \in V'_K : [K_x : K] = 2\}$, where K_x is the splitting field of $F_x(u, 1)$ over K. Let $\mathbf{A}$ be the ring of adeles of K and $\mathbf{A}^*$ be its group of ideles. We adelize G, V, and H so that $H_{\mathbf{A}}$ becomes a subgroup of $Gl(V_{\mathbf{A}})$ and H_K becomes a discrete subgroup of $H_{\mathbf{A}}$. Let Ω be the space of quasicharacters on $\mathbf{A}^*$ that are trivial on K^*, and let $\mathcal{S}(V_{\mathbf{A}})$ be the space of locally constant complex-valued functions with compact support defined on $V_{\mathbf{A}}$. Then for $\omega \in \Omega$ and $f \in \mathcal{S}(V_{\mathbf{A}})$, define

$$Z(\omega, f) = \int_{H_{\mathbf{A}}/H_K} \omega(det(h)) \sum_{x \in V_K''} f(h \cdot x)\, dh, \tag{1.1}$$

where dh is a left Haar measure on $H_{\mathbf{A}}$. This is the *adelic zeta function* associated with the space of binary quadratic forms with coefficients in K.

In Chapter 2, we discuss some aspects of the action of G_K on V_K. In particular, we show how binary quadratic forms are related to quadratic extensions. We also describe the stabilizer group of a nonsingular form and the connected component of the identity inside it. In Chapter 3, we define and establish the convergence of the adelic zeta function $Z(\omega, f)$. Here we show that $Z(\omega, f)$ converges absolutely and locally uniformly for $\Re(\omega) > 1$. A major part of this thesis will be contained in Chapter 4. Here we obtain the analytic continuation and derive the functional equation of $Z(\omega, f)$. This will be achieved by first using an adelic version of the Poisson summation formula. This will reduce finding the analytic continuation of $Z(\omega, f)$ to finding the analytic continuation of a singular integral, denoted by $I(\omega, f)$. The idea of analytically continuing $I(\omega, f)$ is to decompose it into a sum of three integrals depending on the singular G_K-orbits in V_K. For this decomposition to be possible, we use Shintani's idea [20] of first introducing an Eisenstein series in $I(\omega, f)$ to obtain another integral, denoted by $I(\omega, f; w, \phi)$. Next we decompose $I(\omega, f; w, \phi)$ into a sum of three integrals $I^i(\omega, f; w, \phi)$ for $i = 0, 1, 2$, analytically continue each one of them, and then convert this analytic continuation to that of $I(\omega, f)$ and hence $Z(\omega, f)$. In the process we introduce some distributions in terms of which some residues of $Z(\omega, f)$ are given. We summarize the work in this chapter in Theorem 4.1.

As a consequence of our study of $Z(\omega, f)$, we shall obtain the mean value theorem we are after. This will be done in three steps. The first step is to note that $Z(\omega, f)$ in (1.1) can be written in its region of absolute convergence as a sum

$$Z(\omega, f) = \frac{1}{2} \sum_{x \in H_K \backslash V_K''} \int_{H_{\mathbb{A}}/(H_x^\circ)_K} \omega(det(h)) f(h \cdot x)\, dh \tag{1.2}$$

where the sum is over a complete set of representatives of all H_K-orbits in V_K''. Also H_x° is the connected component of the identity in H_x, the stabilizer group of x in H. One important point to note here is that due to the nature of the action of H_K on V_K, the sum in (1.2) is in fact a sum over all quadratic extensions of K. Each integral in (1.2) can be written as a product

$$c_x \mu(x) \int_{H_{\mathbb{A}}/(H_x^\circ)_{\mathbb{A}}} \omega(det(h')) f(h' \cdot x)\, d_x' h' \tag{1.3}$$

where

$$\mu(x) = \int_{(H_x^\circ)_{\mathbb{A}}/(H_x^\circ)_K} d_x'' h'', \tag{1.4}$$

$d_x'h'$ and $d_x''h''$ are measures on $H_{\mathbb{A}}/(H_x^\circ)_{\mathbb{A}}$ and $(H_x^\circ)_{\mathbb{A}}/(H_x^\circ)_K$ respectively, and c_x is a constant given by $dh = c_x d_x'h' d_x''h''$. What makes the sum in (1.2) interesting is that for a suitable choice of the measure $d_x''h''$, $\mu(x)$ will be given in terms of the divisor class number of the quadratic extension K_x of K.

For $f = \prod_{v \in M(K)} f_v$, the integral in (1.3) can be written as a product of local integrals

$$\prod_{v \in M(K)} \int_{H_{K_v}/(H_x^\circ)_{K_v}} \omega_v(det(h_v')) f_v(h_v' \cdot x)\, d_x' h_v'. \tag{1.5}$$

Each local integral in (1.5) will be denoted by $Z_x(\omega_v, f_v)$ and will be called an orbital zeta function. In Chapter 5, we study some properties of $Z_x(\omega_v, f_v)$.

The second step toward the mean value theorem is to put together the global and local information obtained in Chapters 4 and 5 and construct a Dirichlet series $\xi_{x_S}(\omega)$ whose value at $\omega = \omega_s$ is given by

$$\xi_{x_S}(s) = \frac{h_{0,K}}{2} \sum_{[K_x:K]=2, x \sim x_S} \frac{h_{0,K_x}}{D_x^{\frac{3}{2}s}} \frac{\zeta_{K,S}(3s-1)\zeta_{K,S}(3s)^2}{\zeta_{K_x,S}(3s)}$$

where $h_{0,K}$ and h_{0,K_x} are respectively the divisor class numbers of K and K_x, $\zeta_{K,S}(s) = \prod_{v \notin S}(1 - q_v^{-s})^{-1}$ is the truncated Dedekind zeta function of K, D_x

is the absolute norm of the relative discriminant of K_x over K, and finally the symbol $x \sim \mathrm{x}_S$ specifies some local conditions of K_x at the places of S. In Section 6.1 we will study some analytic properties of $\xi_{\mathrm{x}_S}(\omega)$.

Finally, from $\xi_{\mathrm{x}_S}(s)$ we construct a sequence of Dirichlet series $\xi_{\mathrm{x}_S,T_i}(s)$ where $\{T_i\}_{i=1}^{\infty}$ is an increasing sequnce of finite subsets of $M(K)$, the set of all places of K, such that $S \subset T_i$ and $\lim_{i\to\infty} T_i = M(K)$. Denote the residue of $\xi_{\mathrm{x}_S,T_i}(s)$ at $s = 1$ by R_{x_S,T_i}. The important point here is that the limit $\lim_{i\to\infty} R_{\mathrm{x}_S,T_i} = \mathcal{R}_{\mathrm{x}_S}$ exists and nonzero. We can now state the mean value theorem.

Theorem 1.1 *We have*

$$\lim_{n\to\infty} \frac{1}{q^{3n}} \sum_{x\sim \mathrm{x}_S,\, D_x^{\frac{3}{2}}=q^{3n}} h_{0,K_x} = 3\log q \cdot \mathcal{R}_{\mathrm{x}_S}$$

where

$$\mathcal{R}_{\mathrm{x}_S} = \frac{2}{3}\frac{\zeta_K(2)h_{0,K}^2}{(q-1)\log q} \prod_{v\in S} R_{\mathrm{x}_v} \prod_{v\notin S}(1 - q_v^{-2} - q_v^{-3} + q_v^{-4})$$

and

$$R_{\mathrm{x}_v} = \begin{cases} \frac{1-q_v^{-2}}{2} & \text{if } (K_v)_{\mathrm{x}_v} = K_v \\ \frac{(1-q_v^{-1})^2}{2} & \text{if } [(K_v)_{\mathrm{x}_v} : K_v] = 2, \text{ unramified} \\ \frac{q_v^{-1}(1-q_v^{-1})^2(1+q_v^{-1})}{2} & \text{if } [(K_v)_{\mathrm{x}_v} : K_v] = 2, \text{ ramified, } q_v \neq 2^n. \end{cases}$$

For a proof, see the proof of Theorem 6.1 in Section 6.2.

1.2 Notations

For any set A, we let $|A|$ denote the cardinality of A. We denote by $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers, respectively. For a ring R, we denote by R^* the group of invertible elements of R. We use the symbol $\bigsqcup$ to indicate disjoint union. For two expressions E_1 and E_2, we write $E_1 \ll E_2$ to mean $E_1 \leq cE_2$ for some constant c. For $z \in \mathbb{C}$, we denote by $\Re(z)$ and $\Im(z)$ respectively the real and imaginary parts of z.

Suppose G is a locally compact topological group. Let Γ be a discrete subgroup of G contained in the maximal unimodular subgroup of G. Then for any left invariant measure dg on G, there exists a left invariant measure $d\bar{g}$ on G/Γ such that if f is an integrable function on G, then

$$\int_G f(g)\,dg = \int_{G/\Gamma} \sum_{\gamma\in\Gamma} f(g\gamma)\,d\bar{g}.$$

In what follows, we will denote both measures on G and on G/Γ by the same symbol, namely dg.

Throughout this thesis, K will denote a function field in one variable over a finite field of constants F_q, $q \neq 2^n$. The genus of K will be denoted by $\mathbf{g}$. Let $M(K)$ denote the set of all places (equivalence classes of absolute values) of K. For $v \in M(K)$, let K_v be the completion of K at v, O_v be the ring of integers in K_v, O_v^* be the group of invertible elements in O_v, π_v be a fixed uniformizer of K_v, q_v be the cardinality of the residue field $O_v/\pi_v O_v$, and $|\cdot|_v$ be the absolute value of K_v normalized so that $|\pi_v|_v = q_v^{-1}$. The ring of adeles of K will be denoted by $\mathbf{A}$; explicitly, $\mathbf{A} = \{x = (x_v) \in \prod_{v\in M(K)} K_v :$ $x_v \in K_v$ and $x_v \in O_v$ for all but finitely many $v\}$. The group of ideles of K will be denoted by $\mathbf{A}^*$; explicitly, $\mathbf{A}^* = \{x = (x_v) \in \prod_{v\in M(K)} K_v^* : x_v \in$ K_v^* and $x_v \in O_v^*$ for all but finitely many $v\}$. Endowed with the restricted product topology, $\mathbf{A}$ and $\mathbf{A}^*$ become a locally compact topological ring and group, respectively. We denote by $|\cdot|_{\mathbf{A}}$ the adelic absolute value on $\mathbf{A}^*$ given by $|x|_{\mathbf{A}} = \prod_{v\in M(K)} |x_v|_v$ for $x = (x_v) \in \mathbf{A}^*$. We set $\mathbf{A}^1 = \{x \in \mathbf{A}^* : |x|_{\mathbf{A}} = 1\}$.

Let P be a finite subset of $M(K)$. We set $\mathbf{A}_P = \prod_{v\in P} K_v \prod_{v\notin P} O_v$ and $\mathbf{A}_P^* = \prod_{v\in P} K_v^* \prod_{v\notin P} O_v^*$. We also define $\mathbf{A}(\emptyset) = \prod_{v\in M(K)} O_v$ and $\mathbf{A}^*(\emptyset) =$ $\prod_{v\in M(K)} O_v^*$.

Let G be a locally compact topological group. A quasicharacter of G is a continuous homomorphism of G into $\mathbb{C}^*$. We let $\Omega = \Omega(\mathbf{A}^*/K^*)$ denote the group of quasicharacters of $\mathbf{A}^*/K^*$, which will be identified with the group of quasicharacters of $\mathbf{A}^*$ that are trivial on K^*. The principal quasicharacters are those given by $\omega_s(x) = |x|_{\mathbf{A}}^s$ for $s \in \mathbb{C}$. For $\omega \in \Omega$, define the symbol $\delta(\omega)$

as follows:

$$\delta(\omega) = \begin{cases} 1 & \text{If } \omega(t) = 1 \ \forall t \in \mathbf{A}^1 \\ 0 & \text{otherwise.} \end{cases}$$

V will denote the 3-dimensional affine space. We let $G = Gl_1 \times Gl_2$, $B = \{g \in G : g = (t, \begin{pmatrix} a & 0 \\ c & d \end{pmatrix})\}$, and $T = \{g \in G : g = (t, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix})\}$.The following notations will be used to denote some particular elements of G:

$$d(t, t_1) = (t, \begin{pmatrix} t_1 & 0 \\ 0 & t_1 \end{pmatrix}), \ a(\tau) = (1, \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}), \ n(u) = (1, \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}).$$

If X is an algebraic variety and R is a ring, then we let X_R be the set of points of X with coordinates in R. If G is a group acting on X on the right, we let $G\backslash X$ denote the orbit space of this action. For $x \in X$, the stabilizer group of x will be denoted by G_x and the connected component of the identity in G_x will be denoted by G_x°.

Other notations will be mentioned as the need arises.

CHAPTER 2

THE SPACE OF BINARY QUADRATIC FORMS WITH COEFFICIENTS IN A FUNCTION FIELD

2.1 Binary Quadratic Forms

Let K be a function field in one variable having as constants the finite field F_q, where $q \neq 2^n$. Let V be the 3-dimensional affine space. Identify V with the space of binary quadratic forms by means of the correspondence

$$x = (x_1, x_2, x_3) \longleftrightarrow F_x(u,v) = x_1u^2 + x_2uv + x_3v^2. \tag{2.1}$$

Set $G = Gl_1 \times Gl_2$. Let G act on V as follows:

$$F_{g\cdot x}(u,v) = tF_x(au+cv, bu+dv), \tag{2.2}$$

for $g = (t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in G$ and $x \in V$. Now V_K is a 3-dimensional vector space over K. Then this action of G on V gives rise to a representation

$$\varrho : G \longrightarrow Gl(V) \tag{2.3}$$

defined over K. The kernel of ϱ is the 1-dimensional torus $T_\varrho = \{(t^{-2}, \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix})\}$ lying in the center of G. Set $H = \varrho(G)$. Then H is a closed subgroup of $Gl(V)$.

For $x \in V$, denote by $P(x)$ the discriminant of x : $P(x) = x_2^2 - 4x_1x_3$. For $g = (t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in G$, set $\chi(g) = t(ad - bc)$. Then $P(g \cdot x) = \chi(g)^2 P(x)$.

We call $x \in V$ nonsingular if $P(x) \neq 0$, and singular if $P(x) = 0$. The set of all nonsingular forms in V_K will be denoted by V'_K. By the splitting field of a form $x \in V_K$ we shall mean the splitting field of the polynomial $F_x(u,1) = x_1u^2 + x_2u + x_3$ over K and will be denoted by K_x.

Proposition 2.1 *Two nonsingular forms in V'_K are G_K-equivalent (i.e, they lie in the same G_K-orbit) if and only if their splitting fields over K are the same.*

Proof : Let $x, y \in V'_K$ be such that $y = g \cdot x$ for some $g \in G_K$. As the splitting field of y is $K_y = K(\sqrt{P(y)})$ and that of x is $K_x = K(\sqrt{P(x)})$ and $P(y) = \chi(y)^2 P(x)$, then $K_y = K_x$ ($\chi(y) \in K$). Conversely, suppose $x, y \in V'_K$ have the same splitting field over K. If this splitting field is K, then x and y are G_K-equivalent to the form uv and hence x and y are G_K-equivalent. Suppose this splitting field is quadratic. Write

$$F_x(u,v) = x_1u^2 + x_2uv + x_3v^2 = x_1(u + \theta v)(u + \theta' v)$$
$$F_y(u,v) = y_1u^2 + y_2uv + y_3v^2 = y_1(u + \alpha v)(u + \alpha' v)$$

where θ' and α' are the Galois conjugates of θ and α, respectively, and $K_x = K(\theta)$ and $K_y = K(\alpha)$. Since the splitting field of x and y is quadratic, then $x_1 \neq 0$ and $y_1 \neq 0$. Now if $K_x = K_y$, then $\alpha = a + b\theta$ for some $a, b \in K$. Suppose first $b \neq 0$. Then $F_y(u,v) = F_{g \cdot x}(u,v)$, where $g = (\frac{y_1}{x_1}, \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix})$. This implies $y = g \cdot x$ and hence x and y lie in the same orbit.

Next suppose $b = 0$. Then $\alpha \in K$ and hence $\theta \in K$ too. So if we put $\alpha = c\theta$ for some $c \in K^*$, then $F_y(u,v) = F_{g \cdot x}(u,v)$, where $g = (\frac{y_1}{x_1}, \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix})$.

This implies $y = g \cdot x$ and hence x and y lie in the same orbit. This completes the proof. ■

Corollary 2.1 *Nonsingular G_K-orbits are in one-to-one correspondence with extensions of K of degree less than or equal to 2.*

Corollary 2.1 is the reason why studying the space of binary quadratic forms over K and the zeta function associated with it will lead to information about the quadratic extensions of K.

2.2 The Stabilizer of a Nonsingular Form

In this section we will describe the stabilizer group G_x of a nonsingular form $x \in V'_K$ and the connected component of the identity G_x° in G_x. This information will be useful to us in Chapters 5 and 6. Without loss of generality, we assume x is monic, i.e, $x = (1, x_2, x_3)$, and consider two cases.

Assume first that x splits over K. By Proposition 2.1, x is G_K-equivalent to the form uv and hence G_x is conjugate to G_{uv}. Straightforward calculations yield

$$G_{uv} = \{((ad)^{-1}, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}), ((bc)^{-1}, \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}) \in G : a, b, c, d \in K^*\}.$$

Now G_{uv} is a closed subgroup of the linear algebraic group G, and hence G_{uv} is also an algebraic group. Note that $N = \{((ad)^{-1}, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}) \in G : a, d \in K^*\}$ is a closed subgroup of G_{uv} and the cosets of N in G_{uv} are N and $(1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})N$. Thus N is a closed normal subgroup of G_{uv} of index 2 and hence $N \supseteq G_{uv}^\circ$, the connected component of the identity in G_{uv}. As $\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in K^*\} \cong K^* \times K^*$ as groups and varieties and the variety $K^* \times K^*$ is connected (irreducible), then N is an irreducible subset of G_{uv}. Thus N is contained in some connected component of G_{uv}. But as the identity is in N, then we must have $N \subseteq G^{\circ}_{uv}$ (every element of G_{uv} belongs to exactly one irreducible component). Hence $N = G^{\circ}_{uv}$ and $|G_{uv}/G^{\circ}_{uv}| = 2$.

Next we assume x does not split over K, i.e, the splitting field K_x of x is a quadratic extension of K. If we write $F_x(u,v) = (u+\theta v)(u+\theta' v)$, then $K_x = K(\theta)$ and θ' is the Galois conjugate of θ over K. Thus we can also write $F_x(u,v) = N_{K_x/K}(u+\theta v)$, where $N_{K_x/K}(\cdot)$ stands for the norm function in K_x over K. Let us describe G_x. For $g = (t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in G$, we have

$$F_{g\cdot x}(u,v) = tN_{K_x/K}(a+b\theta)N_{K_x/K}(u+\sigma v)$$

where $\sigma = \frac{c+d\theta}{a+b\theta}$. If $F_{g\cdot x}(u,v) = F_x(u,v)$, then on comparing the coefficient of u^2, we get $tN_{K_x/K}(a+b\theta) = 1$ and hence $\sigma = \theta$ or $\sigma = \theta'$. Thus we have found that

$$G_x = \{(t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in G : tN_{K_x/K}(a+b\theta) = 1 \quad \text{and } \sigma = \theta \quad \text{or } \theta'\}.$$

As in the splitting case, G_x is algebraic group. To describe G°_x, we utilize Proposition 2.1. Let $\bar{K}$ be an algebraic closure of K. Then the two forms uv and $F_x(u,v) = (u+\theta v)(u+\theta' v)$ both split in $\bar{K}$ and hence they are $G_{\bar{K}}$-equivalent and consequently their stablizer groups are conjugate: i.e, there exists $g \in G_{\bar{K}}$ such that $G^{\circ}_x = gG^{\circ}_{uv}g^{-1}$. Since for $h \in G^{\circ}_{uv}$, we have $\chi(h) = 1$, then we may describe G°_x as $G^{\circ}_x = \{g \in G_x : \chi(g) = 1\}$. It turns out that $G^{\circ}_x = \{g \in G : tN_{K_x/K}(a+b\theta) = 1 \text{ and } \sigma = \theta\}$. Furthermore, $|G_x/G^{\circ}_x| = 2$ since the cosets of G°_x in G_x are given by G°_x and $(\frac{1}{n}, \begin{pmatrix} 0 & 1 \\ n & 0 \end{pmatrix})G^{\circ}_x$ where $n = N_{K_x/K}(\theta)$.

Consider the map $\phi : G^{\circ}_x(K) \longrightarrow Gl_1(K_x)$ given by $\phi(g) = a + b\theta$, for $g = (t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in G^{\circ}_x(K)$. Then ϕ is a morphism of linear algebraic groups defined over K_x. Further, since ϕ is one-to-one and onto, then we have $G^{\circ}_x(K) \cong Gl_1(K_x)$ as groups. This in turn implies that $G^{\circ}_x(K)$ is isomorphic to the base restriction $R_{K_x/K}(Gl_1)$ of Gl_1 from K_x to K.[For given $Gl_1(K_x)$, we find the base restriction $G'(K) = R_{K_x/K}(Gl_1)$ so that $G'(K) \cong Gl_1(K_x)$ as groups. But since ϕ implies $G_x^\circ(K) \cong Gl_1(K_x)$ as groups, then we have $G_x^\circ(K) \cong G'(K) = R_{K_x/K}(Gl_1)$.] For more information about base restriction, see [18] and [25]. We summarize the above discussion in the following proposition.

Proposition 2.2 *Let $x \in V'_K$. Then*

1. $|G_x/G_x^\circ| = 2$.

2. If $[K_x : K] = 2$, *then* $G_x^\circ(K) = R_{K_x/K}(Gl_1)$.

CHAPTER 3

THE ADELIC ZETA FUNCTION: DEFINITION AND CONVERGENCE

3.1 Definition of the Adelic Zeta Function

In this section we will define the adelic zeta function that we will work with. For this, we first introduce some notations. As in Chapter 2, let K be a function field in one variable with field of constants F_q, where $q \neq 2^n$. Denote by $M(K)$ the complete set of absolute values defined on K. For $v \in M(K)$, let K_v be the completion of K at v, O_v be the ring of integers in K_v, O_v^* be the group of invertible elements in O_v, π_v be a fixed uniformizer of K_v, q_v be the cardinality of the residue field $O_v/\pi_v O_v$, and $|\cdot|_v$ be the absolute value of K_v normalized so that $|\pi_v|_v = q_v^{-1}$. Denote by $\mathbf{A}$ and $\mathbf{A}^*$ the ring of adeles and the group of ideles of K, respectively. K can be identified with a discrete subgroup of $\mathbf{A}$ by the diagonal embedding. Let $V_{\mathbf{A}}$ be the space of binary quadratic forms with coefficients in $\mathbf{A}$. Then V_K is discrete in $V_{\mathbf{A}}$. Set $V_K'' = \{x \in V_K' : [K_x : K] = 2\}$.

In Chapter 2, we set $H = \varrho(G) \subset Gl(V)$. We adelize H and so $H_{\mathbf{A}}$ becomes a subgroup of $Gl(V_{\mathbf{A}})$ and H_K becomes a discrete subgroup of $H_{\mathbf{A}}$. We observe here that V''_K is H_K-invariant as any two H_K-equivalent forms have the same splitting field.

Let Ω be the space of quasicharacters on $\mathbf{A}^*/K^*$, which will be identified in a natural way with the space of quasicharacters on $\mathbf{A}^*$ that are trivial on K^*. Let $\mathcal{S}(V_{\mathbf{A}})$ be the space of locally constant complex-valued functions with compact support defined on $V_{\mathbf{A}}$. For $\omega \in \Omega$ and $f \in \mathcal{S}(V_{\mathbf{A}})$, define

$$Z(\omega, f) = \int_{H_{\mathbf{A}}/H_K} \omega(det(h)) \sum_{x \in V''_K} f(h \cdot x)\, dh \tag{3.1}$$

where dh is a left invariant Haar measure on $H_{\mathbf{A}}$. This is the *adelic zeta function* associated with the space of binary quadratic forms. Of course, the definition makes sense provided the integral converges. Our first objective in this work is to study the convergence and analytic continuation of $Z(\omega, f)$.

Denote by $|\cdot|_{\mathbf{A}}$ the adelic absolute value on $\mathbf{A}^*$ given by $|x|_{\mathbf{A}} = \prod_{v \in M(K)} |x_v|_v$ for $x = (x_v) \in \mathbf{A}^*$. Let $\mathbf{A}^1 = \{x \in \mathbf{A}^* : |x|_{\mathbf{A}} = 1\}$. Then $K^* \subset \mathbf{A}^1$, by the product formula. Further, $\mathbf{A}^1/K^*$ is compact. Since $\mathbf{A}^*/K^* \cong \mathbf{A}^1/K^* \times \mathbb{Z}$, then every quasicharacter ω on $\mathbf{A}^*/K^*$ can be written as a product $\tilde{\omega}\omega_s$, $s \in \mathbb{C}$, where ω_s is the principal quasicharacter, $\omega_s(x) = |x|^s_{\mathbf{A}}$, and $\tilde{\omega}$ is a character on $\mathbf{A}^1/K^*$. In the decomposition $\omega = \tilde{\omega}\omega_s$, s is unique modulo $\frac{2\pi i}{\log q}\mathbb{Z}$. To be more precise, the morphism $s \longrightarrow \omega_s$ of $\mathbb{C}$ onto the subgroup of principal quasicharacters has kernel $\frac{2\pi i}{\log q}\mathbb{Z}$. We also note that while s in the decomposition $\omega = \tilde{\omega}\omega_s$ is not unique, $\Re(s)$ is unique. So we set $\Re(\omega) = \Re(s)$. Thus the decomposition $\omega = \tilde{\omega}\omega_s$ implies that Ω is isomorphic to the direct product of the dual of $\mathbf{A}^1/K^*$ and $\mathbb{C}^*$. Since $\mathbf{A}^1/K^*$ is compact, and so its dual is discrete, we may view Ω as a discrete union of copies of $\mathbb{C}^*$. So by analytic continuation of a function on Ω, we shall mean its analytic continuation on each copy of $\mathbb{C}^*$. For proofs of the above facts, see [26].

3.2 Description of the Haar Measures

To study the convergence of $Z(\omega, f)$, we need three things: (i) a description of a Haar measure dh on $H_{\mathbf{A}}$, (ii) a description of a fundamental domain for

H_K in $H_{\mathbf{A}}$ or a slightly bigger set that contains a fundamental domain, and (iii) a bound for the integrand. We start first with (i).

To describe a Haar measure on $H_{\mathbf{A}}$, we make use of the Iwasawa decomposition $G_{\mathbf{A}} = \mathcal{K}B_{\mathbf{A}}$, where $\mathcal{K} = \prod_{v\in M(K)} \mathcal{K}_v$, $\mathcal{K}_v = G_{O_v}$, and $B_{\mathbf{A}} = \{(t, \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}) \in G_{\mathbf{A}}\}$. $\mathcal{K}$ is the maximal compact subgroup of $G_{\mathbf{A}}$. To prove this decomposition, it is enough to prove it locally, i.e, $G_{K_v} = \mathcal{K}_v B_{K_v}$ for $v \in M(K)$. Let $|\cdot|_v$ be the absolute value on K_v. Let $(t, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}) \in G_{K_v}$. If $\beta = 0$, then we are done. So suppose $\beta \neq 0$. If $|\beta|_v \leq |\delta|_v$, then $\delta \neq 0$ and $|\frac{\beta}{\delta}|_v \leq 1$, i.e, $\frac{\beta}{\delta} \in O_v$ and we have

$$(1, \begin{pmatrix} 1 & \frac{-\beta}{\delta} \\ 0 & 1 \end{pmatrix})(t, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}) = (t, \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}) \in B_{K_v}$$

If $|\beta|_v > |\delta|_v$, then $|\frac{\delta}{\beta}|_v < 1$, i.e, $\frac{\delta}{\beta} \in O_v$ and we have

$$(1, \begin{pmatrix} \frac{-\delta}{\beta} & 1 \\ 1 & 0 \end{pmatrix})(t, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}) = (t, \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}) \in B_{K_v}$$

For uniqueness, suppose $\kappa_1 b_1 = \kappa_2 b_2$, where $\kappa_i \in \mathcal{K}_v$ and $b_i \in B_{K_v}$. Then $\kappa_2^{-1}\kappa_1 = b_2 b_1^{-1} \in \mathcal{K}_v \cap B_{K_v} = B_{O_v}$. Thus $\kappa_1 \in \kappa_2 B_{O_v}$, i.e, κ_1 is unique modulo B_{O_v}. Also $b_1 b_2^{-1} = (b_2 b_1^{-1})^{-1} \in B_{O_v}$ implies $b_1 \in B_{O_v} b_2$ and so b_1 is unique modulo B_{O_v}. Thus the decomposition is locally unique modulo $\backslash B_{O_v}/$. (So if we write $\kappa_1 = \kappa_2 b$ and $b_1 = b' b_2$, $b, b' \in B_{O_v}$, then $\kappa_1 b_1 = \kappa_2 b b' b_2$, $bb' \in B_{O_v}$).

We also note that every element b of $B_{\mathbf{A}}$ can be written uniquely as $b = d(t,t_1)n(u)a(\tau)$, where $d(t,t_1) = (t, \begin{pmatrix} t_1 & 0 \\ 0 & t_1 \end{pmatrix})$, $n(u) = (1, \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix})$, $a(\tau) = (1, \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix})$ and $u \in \mathbf{A}$, $t, t_1, \tau \in \mathbf{A}^*$. This follows from the following identity:

$$(t, \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}) = (t, \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix})(1, \begin{pmatrix} 1 & 0 \\ \frac{\gamma}{\alpha} & 1 \end{pmatrix})(1, \begin{pmatrix} 1 & 0 \\ 0 & \frac{\delta}{\alpha} \end{pmatrix}).$$

The uniqueness part is trivial.

Now we are ready to describe a Haar measure on $H_{\mathbf{A}}$. Let du and d^*t be respectively an additive and multiplicative Haar measures on $\mathbf{A}$ and $\mathbf{A}^*$. Let d^1t be a Haar measure on $\mathbf{A}^1$. Normalize du and d^1t so that

$$\int_{\mathbf{A}/K} du = 1 \text{ and } \int_{\mathbf{A}^1/K^*} d^1t = 1$$

By the Iwasawa decomposition, the map $\rho : \mathcal{K} \times B_{\mathbf{A}} \longrightarrow G_{\mathbf{A}}$, given by $\rho(\kappa, b) = \kappa b$, is surjective. We define a Haar measure dg on $G_{\mathbf{A}}$ by setting $dg = d\kappa db$, where db is a Haar measure on $B_{\mathbf{A}}$ and $d\kappa$ is a Haar measure on $\mathcal{K}$ normalized so that

$$\int_{\mathcal{K}} d\kappa = 1.$$

Since, as we pointed out above, every element b of $B_{\mathbf{A}}$ can be written uniquely as $b = d(t, t_1)n(u)a(\tau)$, where $u \in \mathbf{A}$, $t, t_1, \tau \in \mathbf{A}^*$, then it follows that the map $\phi : \mathbf{A}^* \times \mathbf{A}^* \times \mathbf{A} \times \mathbf{A}^* \longrightarrow B_{\mathbf{A}}$, given by $\phi(t, \alpha, \gamma, \delta) = (t, \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix})$, is surjective. We define a Haar measure db on $B_{\mathbf{A}}$ by setting $db = d^*td^*t_1dud^*\tau$.

At this point we will mention one observation that we will use later. Note that b can also be written uniquely as $b = d(t, t_1)a(\tau)n(u)$. If we write b this way, then the measure db takes a slightly different form. Since $a(\tau)n(u) = n(u\tau)a(\tau)$, then $b = d(t, t_1)n(u\tau)a(\tau)$ and the measure becomes $db = |\tau|_{\mathbf{A}} d^*td^*t_1dud^*\tau$.

Finally, since $H \cong G/T_\varrho$, where $T_\varrho = \{d(t_1^{-2}, t_1) \in G\}$, then we define a measure dh on H by setting $dg = d^*t_1dh$. Explicitly, if we write $h = \varrho(\kappa d(t, 1)n(u)a(\tau))$ then $dh = d\kappa d^*tdud^*\tau$. While if we write $h = \varrho(\kappa d(t, 1)a(\tau)n(u))$ then $dh = |\tau|_{\mathbf{A}} d\kappa d^*tdud^*\tau$.

We close this section by an observation about the uniqueness of the Iwasawa decomposition. When we write $g = \kappa d(t, t_1)n(u)a(\tau)$ or $g = \kappa d(t, t_1)a(\tau)n(u)$, then in both cases κ is unique modulo B_{O_v}, as we observed above, and t, t_1, and τ are unique up to multiplication by $\mathbf{A}^*(\emptyset) = \prod_{v \in M(K)} O_v^*$. However, u is unique up to multiplication by $\mathbf{A}^*(\emptyset)$ followed by a translation by $\mathbf{A}(\emptyset) = \prod_{v \in M(K)} O_v$ in the first, while it is unique modulo $\frac{1}{\tau}\mathbf{A}(\emptyset)$ in the second.

3.3 Convergence of the Adelic Zeta Function

We start by quoting a result that describes a set that contains a fundamental domain for H_K in $H_{\mathbf{A}}$. Namely, we summarize Lemma 2.1 and Lemma 2.2 of [1], modified to fit our situation, in the following lemma:

Lemma 3.1 *Every element of $H_{\mathbf{A}}$ is right H_K-equivalent to an element of the set*

$$\mathcal{S} = \bigcup_u \bigcup_{t,\tau} \varrho(\kappa n(u)d(t,1)a(\tau))$$

where t and τ run over a set of representatives of $\mathbf{A}^*/K^*\mathbf{A}^*(\emptyset)$ in $\mathbf{A}^*$, $|\tau|_{\mathbf{A}} \leq q^{2\mathbf{g}}$, and u runs over a finite set in $\mathbf{A}$. (Here, $\mathbf{g}$ is the genus of K and $\mathbf{A}^*(\emptyset) = \prod_{v\in M(K)} O_v^*$).

Next we give a bound for the integrand of $Z(\omega, f)$.

Lemma 3.2 *Let $h = \varrho(\kappa n(u)d(t,1)a(\tau)) \in \mathcal{S}$. Then*
1. There exists an integer c such that $\sum_{x\in V_K''} |f(h\cdot x)| = 0$ if $|t|_{\mathbf{A}} > q^c$.
2. For any h with $|t|_{\mathbf{A}} \leq q^c$, we have $\sum_{x\in V_K} |f(h\cdot x)| = O(|t\tau|_{\mathbf{A}}^{-3})$.

Proof : Let h be as in the lemma. Let $x \in V_K''$. Then

$$h\cdot x = \varrho(\kappa n(u)d(t,1)a(\tau))\cdot x = \kappa n(u)d(t,1)a(\tau)\cdot x$$

Now $h \cdot x \in Supp(f)$ iff $d(t,1)a(\tau)\cdot x \in n(-u)\kappa^{-1} \circ Supp(f) \subset n(-u)\mathcal{K} \circ Supp(f)$. Let $S(f) = \bigcup_u n(-u)\mathcal{K}\circ Supp(f) \subseteq V_{\mathbf{A}}$. As $\mathcal{K}\circ Supp(f)$ is compact and the number of $u^{'s}$ is finite, then $S(f)$ is compact.

For $x = (x_1, x_2, x_3)$, we have

$$d(t,1)a(\tau)\cdot x = (tx_1, t\tau x_2, t\tau^2 x_3).$$

Since $d(t,1)a(\tau)\cdot x \in S(f)$, then the adelic absolute value of the first and the second coordinate of $d(t,1)a(\tau)\cdot x$ are bounded, i.e, there exists an integer $c > 0$ such that

$$\max(|tx_1|_{\mathbf{A}}, |t\tau x_2|_{\mathbf{A}}) \leq q^c.$$

Since x is nonsingular, then x_1 and x_2 cannot both simultaneously be zero. Since $[K_x : K] = 2$, then certainly $x_1 \neq 0$. If $x_2 = 0$, then the above inequality reduces to $|t|_{\mathbf{A}} \leq q^c$. If $x_2 \neq 0$, then the above inequality implies $|t|_{\mathbf{A}} \leq |t|_{\mathbf{A}} \max(1, |\tau|_{\mathbf{A}}) \leq q^c$. Thus we have shown if $h \cdot x \in Supp(f)$, then $|t|_{\mathbf{A}} \leq q^c$. This proves (1) of the lemma.

To prove (2), note first that there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in V_{\mathbf{A}}$. So we have

$$\begin{aligned} \textstyle\sum_{x \in V_K} |f(h \cdot x)| &\leq M|\{x \in V_K : h \cdot x \in Supp(f)\}| \\ &\leq M|\{x \in V_K : d(t,1)a(\tau) \cdot x \in S(f)\}| \end{aligned}$$

Since $S(f)$ is compact, then there is $\alpha \in \mathbf{A}^*$ such that $S(f) \subseteq (\alpha\mathbf{A}(\emptyset))^3$, where $\mathbf{A}(\emptyset) = \prod_{v \in M(K)} O_v$. So we need to find a bound for the cardinality of the set

$$R = \{x \in V_K : d(t,1)a(\tau) \cdot x \in (\alpha\mathbf{A}(\emptyset))^3\}.$$

But $x \in R$ iff $x_1 \in t^{-1}\alpha\mathbf{A}(\emptyset)$, $x_2 \in t^{-1}\tau^{-1}\alpha\mathbf{A}(\emptyset)$, and $x_3 \in t^{-1}\tau^{-2}\alpha\mathbf{A}(\emptyset)$. We estimate the number of such $x_i^{\prime s}$. By Riemann-Roch theorem, see [14], we deduce $|K \cap \beta\mathbf{A}(\emptyset)| = O(\max(1, |\beta|_{\mathbf{A}}))$ for any $\beta \in \mathbf{A}^*$. Thus the number of $x_i^{\prime s}$, $i = 1,2,3$, are respectively $O(\max(|t^{-1}|_{\mathbf{A}}, 1))$, $O(\max(|t^{-1}\tau^{-1}|_{\mathbf{A}}, 1))$, and $O(\max(|t^{-1}\tau^{-2}|_{\mathbf{A}}, 1))$. Note that $|\alpha|_{\mathbf{A}}$ is constant. But $O(\max(|t^{-1}|_{\mathbf{A}}, 1)) = O(|t^{-1}|_{\mathbf{A}} \max(1, |t|_{\mathbf{A}})) = O(|t^{-1}|_{\mathbf{A}})$, since $|t|_{\mathbf{A}} \leq q^c$. Similarly we have $O(\max(|t^{-1}\tau^{-1}|_{\mathbf{A}}, 1)) = O(|t^{-1}\tau^{-1}|_{\mathbf{A}})$ and $O(\max(|t^{-1}\tau^{-2}|_{\mathbf{A}}, 1)) = O(|t^{-1}\tau^{-2}|_{\mathbf{A}})$. This implies the cardinality of R is $O(|t\tau|_{\mathbf{A}}^{-3})$ and hence the lemma follows. ■

We are now ready to state and prove the main theorem of this chapter.

Theorem 3.1 *The integral defining $Z(\omega, f)$ converges absolutely and locally uniformly in ω for $\Re(\omega) > 1$. Thus $Z(\omega, f)$ is analytic in the region $\Re(\omega) > 1$.*

Proof : Let $\omega = \tilde{\omega}\omega_s$ and $\Re(s) = \sigma$. Then for $h = \varrho(\kappa d(t,1)n(u)a(\tau))$, we have

$$\begin{aligned} |Z(\omega, f)| &\leq \textstyle\int_{H_{\mathbf{A}}/H_K} |\omega(det(h))| \sum_{x \in V_K''} |f(h \cdot x)|\, dh \\ &\leq \textstyle\int_S \omega_\sigma(det(h)) \sum_{x \in V_K''} |f(h \cdot x)|\, dh \\ &\ll \textstyle\sum_{|t|_{\mathbf{A}} \leq q^c} \sum_{|\tau|_{\mathbf{A}} \leq q^{2c}} |t\tau|_{\mathbf{A}}^{3\sigma} |t\tau|_{\mathbf{A}}^{-3} \\ &= \textstyle\sum_{|t|_{\mathbf{A}} \leq q^c} |t|_{\mathbf{A}}^{3\sigma-3} \sum_{|\tau|_{\mathbf{A}} \leq q^{2c}} |\tau|_{\mathbf{A}}^{3\sigma-3} \end{aligned}$$

Here we have used the fact that the number of u'^s is finite and that $\int_K d\kappa = 1$. Also t and τ run over a complete set of representatives of $\mathbf{A}^*/K^*\mathbf{A}^*(\emptyset)$. Since the set $\{t \in \mathbf{A}^*/K^*\mathbf{A}^*(\emptyset) : |t|_\mathbf{A} = q^{-n}\}$ is finite, its cardinality is $|\mathbf{A}^1/K^*\mathbf{A}^*(\emptyset)|$, say N, then we have

$$\sum_{|t|_\mathbf{A}\le q^c} |t|_\mathbf{A}^{3\sigma-3} \sum_{|\tau|_\mathbf{A}\le q^{2g}} |\tau|_\mathbf{A}^{3\sigma-3} = N^2 \sum_{n=-c}^{\infty} q^{-n(3\sigma-3)} \sum_{m=-2g}^{\infty} q^{-m(3\sigma-3)}$$

and both series converge absolutely and locally uniformly for $\sigma > 1$. The theorem now follows. ■

3.4 $Z_-(\omega, f)$ and $Z_+(\omega, f)$

For analytic continuation purposes, we make some definitions. Set

$$H_\mathbf{A}^- = \{h \in H_\mathbf{A} : |det(h)|_\mathbf{A} \le 1\},$$

and

$$H_\mathbf{A}^+ = \{h \in H_\mathbf{A} : |det(h)|_\mathbf{A} \ge 1\}.$$

Set

$$\lambda_-(h) = \begin{cases} 1 & \text{If } |det(h)|_\mathbf{A} < 1 \\ \frac{1}{2} & \text{If } |det(h)|_\mathbf{A} = 1 \\ 0 & \text{If } |det(h)|_\mathbf{A} > 1 \end{cases},$$

and

$$\lambda_+(h) = \begin{cases} 0 & \text{If } |det(h)|_\mathbf{A} < 1 \\ \frac{1}{2} & \text{If } |det(h)|_\mathbf{A} = 1 \\ 1 & \text{If } |det(h)|_\mathbf{A} > 1 \end{cases}.$$

Define

$$Z_-(\omega, f) = \int_{H_\mathbf{A}/H_K} \omega(det(h))\lambda_-(h) \sum_{x\in V_K''} f(h\cdot x)\, dh, \tag{3.2}$$

$$Z_+(\omega, f) = \int_{H_\mathbf{A}/H_K} \omega(det(h))\lambda_+(h) \sum_{x\in V_K''} f(h\cdot x)\, dh. \tag{3.3}$$

Proposition 3.1 *$Z_+(\omega, f)$ is an entire function of ω. Further, it is a polynomial in q^s, $\omega = \tilde{\omega}\omega_s$.*

Proof : Let $\mathcal{S}^+ = \{h \in \mathcal{S} : |det(h)|_{\mathbf{A}} \geq 1\}$. Then $\mathcal{S}^+$ contains a fundamental domain of H_K in $H^+_{\mathbf{A}}$. Let c be the constant found in Lemma 3.2. Set $\mathcal{S}^+(c) = \{h \in \mathcal{S}^+ : |t(h)|_{\mathbf{A}} \leq q^c\}$. Then the integral $Z_+(\omega, f)$ is an integral over a subset of $\mathcal{S}^+(c)$. We describe $\mathcal{S}^+(c)$ and show that it is compact. Observe that $\mathcal{S}^+(c) = \cup_u \cup_{t,\tau} \varrho(\kappa n(u) d(t,1) a(\tau))$, where the number of u'^s is finite, t and τ run over a complete set of representatives of $\mathbf{A}^*/K^*\mathbf{A}^*(\emptyset)$ such that $|t|_{\mathbf{A}} \leq q^c$ and $|\tau|_{\mathbf{A}} \leq q^{2\mathbf{g}}$, and $|det(h)|_{\mathbf{A}} = |t\tau|^3_{\mathbf{A}} \geq 1$. The inequality $|t\tau|^3_{\mathbf{A}} \geq 1$ implies

1. $|t|^3_{\mathbf{A}} \geq |\tau|^{-3}_{\mathbf{A}} \geq q^{-6\mathbf{g}}$ and hence $q^{-2\mathbf{g}} \leq |t|_{\mathbf{A}} \leq q^c$, and

2. $|\tau|^3_{\mathbf{A}} \geq |t|^{-3}_{\mathbf{A}} \geq q^{-3c}$ and hence $q^{-c} \leq |\tau|_{\mathbf{A}} \leq q^{2\mathbf{g}}$

This shows that the number of t'^s and τ'^s is finite, and hence $\mathcal{S}^+(c)$ is compact. So we have

$$\begin{aligned} |Z_+(\omega, f)| &\leq \int_{\mathcal{S}^+(c)} \omega_\sigma(det(h)) \sum_{x \in V''_K} |f(h \cdot x)|\, dh \\ &\ll \sum_{q^{-2\mathbf{g}} \leq |t|_{\mathbf{A}} \leq q^c} |t|^{3\sigma-3}_{\mathbf{A}} \sum_{q^{-c} \leq |\tau|_{\mathbf{A}} \leq q^{2\mathbf{g}}} |\tau|^{3\sigma-3}_{\mathbf{A}} \end{aligned}$$

Since the two sums on the right are finite, they converge absolutely and locally uniformly for all σ. Thus $Z_+(\omega, f)$ is entire.

Let $H^n_{\mathbf{A}} = \{h \in H_{\mathbf{A}} : |det(h)|_{\mathbf{A}} = q^n\}$. Then $H^+_{\mathbf{A}} = \bigsqcup_{n=0}^\infty H^n_{\mathbf{A}}$. Also for $h \in H^n_{\mathbf{A}}$, $\omega(det(h)) = \tilde{\omega}(det(h))|det(h)|^s_{\mathbf{A}} = q^{ns}\tilde{\omega}(det(h))$. So we may write

$$\begin{aligned} Z_+(\omega, f) &= \tfrac{1}{2} \int_{H^0_{\mathbf{A}}/H_K} \tilde{\omega}(det(h)) \sum_{x \in V''_K} f(h \cdot x)\, dh \\ &+ \sum_{n=1}^\infty q^{ns} \int_{H^n_{\mathbf{A}}/H_K} \tilde{\omega}(det(h)) \sum_{x \in V''_K} f(h \cdot x)\, dh \end{aligned}$$

Let $\mathcal{S}^n = \{h \in \mathcal{S} : |det(h)|_{\mathbf{A}} = q^n\}$. Then $\mathcal{S}^n$ contains a fundamental domain of H_K in $H^n_{\mathbf{A}}$. Again for c as above, let $\mathcal{S}^n(c) = \{h \in \mathcal{S}^n : |t(h)|_{\mathbf{A}} \leq q^c\}$. Then each integral over $H^n_{\mathbf{A}}/H_K$ is an integral over a subset of $\mathcal{S}^n(c)$. As above, it is easy to show that for $h \in \mathcal{S}^n(c)$, we have $q^{\frac{n}{3}-2\mathbf{g}} \leq |t(h)|_{\mathbf{A}} \leq q^c$ and $q^{\frac{n}{3}-c} \leq |\tau(h)|_{\mathbf{A}} \leq q^{2\mathbf{g}}$. So the number of t'^s and τ'^s is finite and hence $\mathcal{S}^n(c)$ is compact. Observe also that $\mathcal{S}^+ = \bigsqcup_{n=0}^\infty \mathcal{S}^n$ and $\mathcal{S}^+(c) = \bigsqcup_{n=0}^\infty \mathcal{S}^n(c)$. By the finiteness of the number of t'^s and τ'^s in $\mathcal{S}^+(c)$ and $\mathcal{S}^n(c)$, it follows that $\mathcal{S}^+(c) = \bigsqcup_{n=0}^N \mathcal{S}^n(c)$, a finite union. Thus the above sum is in fact finite. As above, each of the integrals converges absolutely and hence converges. Also each integral is independent of s. This completes the proof of the proposition. ■

Remark 3.1 *1. As $Z_-(\omega, f)$ is dominated by $Z(\omega, f)$, then $Z_-(\omega, f)$ converges absolutely and locally uniformly for $\Re(\omega) > 1$, and hence it is analytic in this region.*

2. We also note that $Z(\omega, f) = Z_+(\omega, f) + Z_-(\omega, f)$.

CHAPTER 4

THE ADELIC ZETA FUNCTION: ANALYTIC CONTINUATION AND FUNCTIONAL EQUATION

4.1 Poisson Summation Formula

Our goal in this chapter is to obtain the analytic continuation and the functional equation of $Z(\omega, f)$. To achieve this, we first use the Poisson summation formula.

Let $\psi : \mathbf{A} \longrightarrow \mathbf{C}^*$ be a nontrivial additive character of $\mathbf{A}$ which is trivial on K. Identify $\mathbf{A}$ with its dual by means of the correspondence $y \longrightarrow \psi_y$, where $\psi_y(x) = \psi(xy)$. Under this identification, K becomes self-dual.

Define a symmetric bilinear form $[*, *]$ on $V_{\mathbf{A}}$ by

$$[x, y] = x_1y_3 - \tfrac{1}{2}x_2y_2 + x_3y_1.$$

Identify $V_{\mathbf{A}}$ with its dual by means of the correspondence $y \longrightarrow \psi_y$, where $\psi_y(x) = \psi([x, y])$. Again, V_K is self-dual under this identification. We remark here that the bilinear form $[*,*]$ is chosen so that $[x,y]=[g\cdot x, g'\cdot y]$, where $g=(t, \begin{pmatrix} a & b \\ c & d \end{pmatrix})$ and $g'=(t^{-1}, \frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix})$.

For an additive self-dual Haar measure dy on $V_{\mathbf{A}}$, we take $dy=dy_1dy_2dy_3$, where dy_i is the additive normalized Haar measure on $\mathbf{A}$ given in Chapter 3. For $f\in\mathcal{S}(V_{\mathbf{A}})$, the Fourier transform $\hat{f}$ of f is defined by

$$\hat{f}(x)=\int_{V_{\mathbf{A}}} f(y)\psi([x,y])\,dy.$$

It follows that $\hat{f}\in\mathcal{S}(V_{\mathbf{A}})$ and $\hat{\hat{f}}=f$. Applying the Poisson summation formula to the lattice V_K yields

$$\sum_{x\in V_K} f(x)=\sum_{x\in V_K}\hat{f}(x) \tag{4.1}$$

For our purposes, we use a different version of the Poisson summation formula from that given by equation (4.1). Let $h=\varrho(g)$. Set $f_h(x)=f(h\cdot x)$ and $h'=\varrho(g')$. It is easy to show that $\hat{f}_h=|det(h)|_{\mathbf{A}}^{-1}\hat{f}_{h'}$. With this, equation (4.1) becomes

$$\sum_{x\in V_K} f(h\cdot x)=\omega_{-1}(det(h))\sum_{x\in V_K}\hat{f}(h'\cdot x) \tag{4.2}$$

Let $S_K=V_K-V''_K$. By Remark (3.1), we have

$$Z(\omega,f)=Z_+(\omega,f)+Z_-(\omega,f)$$

Let us apply equation (4.2) to the sum in $Z_-(\omega,f)$:

$$\begin{aligned} Z_-(\omega,f) &= \textstyle\int_{H_{\mathbf{A}}/H_K}\omega(det(h))\lambda_-(h)\sum_{x\in V''_K} f(h\cdot x)\,dh \\ &= \textstyle\int_{H_{\mathbf{A}}/H_K}\omega(det(h))\lambda_-(h)[\sum_{x\in V_K} f(h\cdot x)-\sum_{x\in S_K} f(h\cdot x)]\,dh \\ &= \textstyle\int_{H_{\mathbf{A}}/H_K}\omega(det(h))\lambda_-(h)[\omega_{-1}(det(h))\sum_{x\in V_K}\hat{f}(h'\cdot x) \\ &\quad \textstyle-\sum_{x\in S_K} f(h\cdot x)]\,dh \\ &= \textstyle\int_{H_{\mathbf{A}}/H_K}\lambda_-(h)\omega(det(h))\omega_{-1}(det(h))\sum_{x\in V''_K}\hat{f}(h'\cdot x) \\ &\quad \textstyle+\omega(det(h))\lambda_-(h)\sum_{x\in S_K}[\omega_{-1}(det(h))\hat{f}(h'\cdot x)-f(h\cdot x)]\,dh \end{aligned}$$

Now consider the integral

$$\textstyle\int_{H_{\mathbf{A}}/H_K}\lambda_-(h)\omega(det(h))\omega_{-1}(det(h))\sum_{x\in V''_K}\hat{f}(h'\cdot x)\,dh$$

Making the change of variable $h \longrightarrow h'$, we get $dh \longrightarrow dh' = dh$. Since $h' = \varrho((t^{-2}, \frac{1}{ad-bc}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})h$, then $det(h') = det(h)^{-1}$. So if $|det(h)|_{\mathbf{A}} \leq 1$, then $|det(h')|_{\mathbf{A}} \geq 1$. So the last integral reduces to

$$\int_{H_{\mathbf{A}}/H_K} \lambda_+(h')\omega^{-1}(det(h'))\omega_1(det(h')) \sum_{x \in V_K''} \hat{f}(h' \cdot x)\, dh'$$

which is nothing but $Z_+(\omega_1\omega^{-1}, \hat{f})$. We summarize all of this in the following proposition.

Proposition 4.1 *We have*

$$Z(\omega, f) = Z_+(\omega, f) + Z_+(\omega_1\omega^{-1}, \hat{f}) + I(\omega, f),$$

where

$$I(\omega, f) = \int_{H_{\mathbf{A}}/H_K} \omega(det(h))\lambda_-(h) \sum_{x \in S_K} [\omega_{-1}(det(h))\hat{f}(h' \cdot x) - f(h \cdot x)]\, dh$$

We note that by Theorem 3.1 and Proposition 3.1, it follows that $I(\omega, f)$ converges absolutely and locally uniformly in the region $\Re(\omega) > 1$. Further, since $Z_+(\omega, f)$ and $Z_+(\omega_1\omega^{-1}, \hat{f})$ are entire functions of ω, then finding the analytic continuation of $Z(\omega, f)$ amounts to finding the analytic continuation of $I(\omega, f)$. For this, we first describe the set S_K.

Every element of S_K is H_K-equivalent to one of the following forms: 0, v^2, uv. We describe the stabilizers of the sets $\bigcup_{a \in K^*}(0, 0, a)$ and $\bigcup_{a \in K^*}(0, a, 0)$. Let $x = (x_1, x_2, x_3)$ and $g = (t, \begin{pmatrix} a & b \\ c & d \end{pmatrix})$. It is easy to show that $F_{g \cdot x}(u, v) = F_x(u, v)$ if and only if

$$\begin{aligned} x_1 &= t(a^2x_1 + abx_2 + b^2x_3), \\ x_2 &= t(2acx_1 + (ad + bc)x_2 + 2bdx_3), \\ x_3 &= t(c^2x_1 + cdx_2 + d^2x_3). \end{aligned}$$

If $x = (0, 0, x_3)$, $x_3 \neq 0$, then $g \cdot x$ has the same form as x if and only if $b = 0$. So the set stabilizer of $\bigcup_{a \in K^*}(0, 0, a)$ is B_K. If $x = (0, x_2, 0)$, $x_2 \neq 0$, then $g \cdot x$ has the same form as x if and only if either $a = d = 0$ or $b = c = 0$. Let

$T_K = \{g \in G_K : g = (t, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix})\}$. Then the group $\mathcal{T}_K$ generated by T_K and the involution $i = (1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ is the set stabilizer of the set $\bigcup_{a \in K^*} (0, a, 0)$.

We also note that, by definition of the action of H on V, if $h = \varrho(g)$ for some $g \in G$, then $h \cdot x = x$ if and only if $g \cdot x = x$. We summarize the above discussion in the following proposition.

Proposition 4.2 *We have* $S_K = \bigcup_{i=0}^{2} S_K^i$, *where*

1. $S_K^0 = \{0\}$
2. $S_K^1 = \bigcup_{\gamma \in H_K/\mathcal{B}_K} \bigcup_{a \in K^*} \gamma \cdot (0, 0, a)$, *where* $\mathcal{B}_K = \varrho(B_K)$
3. $S_K^2 = \bigcup_{\gamma \in H_K/\widehat{\mathcal{T}}_K} \bigcup_{a \in K^*} \gamma \cdot (0, a, 0)$, *where* $\widehat{\mathcal{T}}_K$ *is the subgroup generated by* $\varrho(T_K)$ *and* $\varrho(i) = \varrho((1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}))$

Proposition 4.2 hints that to analytically continue $I(\omega, f)$, we first break up the sum in $I(\omega, f)$ into sums over each S_K^i and then analytically continue each resulting integral. Unfortunately, the resulting integrals do not converge. To remedy this problem, we use Shintani's idea of introducing an Eisenstein series to make each integral of the sum over S_K^i converge. So our next task is to describe the Eisenstein series introduced by Shintani and show how to use it to recover the convergence we want. This will be done in the next two sections.

Later in this chapter we will need to integrate over $H_{\mathbf{A}}/\mathcal{B}_K$, $H_{\mathbf{A}}/\varrho(T_K)$, and $H_{\mathbf{A}}/\widehat{\mathcal{T}}_K$. Such integrals are given explicitly as follows: For $f_1 \in L^1(H_{\mathbf{A}}/\mathcal{B}_K)$ and $f_2 \in L^1(H_{\mathbf{A}}/\varrho(T_K))$, we have

$$\int_{H_{\mathbf{A}}/\mathcal{B}_K} f_1(h)\, dh = \int_K \int_{\mathbf{A}^*/K^*} \int_{\mathbf{A}^*/K^*} \int_{\mathbf{A}/K} f_1(\kappa d(t,1) a(\tau) n(u))\, |\tau|_{\mathbf{A}} d\kappa d^*t d^*\tau du,$$

$$\int_{H_{\mathbf{A}}/\varrho(T_K)} f_2(h)\, dh = \int_K \int_{\mathbf{A}^*/K^*} \int_{\mathbf{A}^*/K^*} \int_{\mathbf{A}} f_2(\kappa d(t,1) a(\tau) n(u))\, |\tau|_{\mathbf{A}} d\kappa d^*t d^*\tau du.$$

For the integral over $H_{\mathbf{A}}/\widehat{\mathcal{T}}_K$, we first describe $G_{\mathbf{A}}/\mathcal{T}_K$. For $u = (u_v) \in \mathbf{A}$, define $\alpha(u) = \prod_{v \in M(K)} \sup(1, |u_v|_v)$. Then for $g = \kappa d(t,1) n(u) a(\tau) \in G_{\mathbf{A}}$, we have $|\tau(gi)|_{\mathbf{A}} = \alpha(u)^2|\tau(g)|_{\mathbf{A}}^{-1}$. To prove this, we need to find the Iwasawa decomposition of gi. Note that $a(\tau)i = id(1,\tau)a(\tau^{-1})$ and $n(u)i = in(u)^t$, where $n(u)^t$ is the transpose of $n(u)$. Then we get $gi = \kappa d(t,1)n(u)a(\tau)i = \kappa' d(t\tau,1)n(u)^t a(\tau^{-1})$, where $\kappa' = \kappa i$. If $u \in \mathbf{A}(\emptyset)$, then $n(u)^t \in \mathcal{K}$ and hence $gi = \kappa'' d(t\tau,1)a(\tau^{-1})$, where $\kappa'' = \kappa' n(u)^t$. This implies that $\alpha(u) = 1$ and $\tau(gi) = \tau(g)^{-1}$. If $u \notin \mathbf{A}(\emptyset)$, then there is a finite subset P of $M(K)$ such that $u_v \notin O_v$ if and only if $v \in P$. Write $n(u)^t = (1, \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}) = \prod_{v \in M(K)}(1, \begin{pmatrix} 1 & u_v \\ 0 & 1 \end{pmatrix})$. If $v \notin P$, then $(1, \begin{pmatrix} 1 & u_v \\ 0 & 1 \end{pmatrix}) \in \mathcal{K}_v$. If $v \in P$, then $(1, \begin{pmatrix} 1 & u_v \\ 0 & 1 \end{pmatrix}) = (1, \begin{pmatrix} 0 & 1 \\ -1 & u_v^{-1} \end{pmatrix})(1, \begin{pmatrix} u_v^{-1} & 0 \\ 1 & u_v \end{pmatrix})$. In this decompsition, the first factor lies in $\mathcal{K}_v$, and the τ−part of the second factor (in its local Iwasawa decomposition) is $\frac{u_v}{u_v^{-1}} = u_v^2$. Thus $\tau(gi) = w\tau(g)^{-1}$, where $w = (w_v)$ and $w_v = u_v^2$ if $v \in P$ and 1 otherwise. Since $|w|_{\mathbf{A}} = \alpha(u)^2$, the result follows.

Since $|\tau(gi)|_{\mathbf{A}}|\tau(g)|_{\mathbf{A}} = \alpha(u)^2$, then either $|\tau(g)|_{\mathbf{A}} \leq \alpha(u)$ or $|\tau(gi)|_{\mathbf{A}} \leq \alpha(u)$. Set $F = \{g = \kappa d(t,1)n(u)a(\tau) \in G_{\mathbf{A}} : |\tau(g)|_{\mathbf{A}} \leq \alpha(u)\}$. Then $Fi = \{g \in G_{\mathbf{A}} : |\tau(g)|_{\mathbf{A}} \geq \alpha(u)\}$. Note that $G_{\mathbf{A}} = F \bigcup Fi$ and $F \bigcap Fi = \{g \in G_{\mathbf{A}} : |\tau(g)|_{\mathbf{A}} = \alpha(u)\}$. Set $F^* = \{g = \kappa d(t,1)n(u)a(\tau) \in G_{\mathbf{A}} : |\tau(g)|_{\mathbf{A}} \leq^* \alpha(u)\}$; more precisely, F^* consists of all elements $g \in G_{\mathbf{A}}$ such that $|\tau(g)|_{\mathbf{A}} < \alpha(u)$ and "half" the elements in $F \bigcap Fi$. (For our purposes, we really only need half the measure of $F \bigcap Fi$.) Thus F^*/T_K is a fundamental domain of $\mathcal{T}_K$ in $G_{\mathbf{A}}$. So for any $f \in L^1(H_{\mathbf{A}}/\widehat{\mathcal{T}}_K)$, we have

$$\int_{H_{\mathbf{A}}/\widehat{\mathcal{T}}_K} f(h)\,dh = \int_{\mathcal{K}} \int_{\mathbf{A}^\bullet/K^\bullet} \int_{\mathbf{A}} \int_{\mathbf{A}^\bullet/K^\bullet, |\tau|_{\mathbf{A}} \leq^* \alpha(u)} f(\varrho(\kappa d(t,1)n(u)a(\tau))\,d\kappa d^*t du d^*\tau.$$

4.2 Shintani's Eisenstein Series

In this section we collect the basic properties of Eisenstein series that are relevant to our work. We first define the Eisenstein series as given in [1] as

$$E(g,z) = \sum_{\gamma \in G_K/B_K} |\tau(g\gamma)|_{\mathbf{A}}^{-\frac{z}{2}-\frac{1}{2}}$$

for $g \in G_{\mathbf{A}}$ and $z \in \mathbb{C}$. $E(g,z)$ converges absolutely and locally uniformly for $\Re(z) > 1$, and it is a rational function in q^z. Note that $E(g,z)$ is invariant under multiplication by $\mathcal{K}Z(G)$ on the left and by G_K on the right ($Z(G)$ is the center of G). Also the summand is right B_K-invariant, so the sum over G_K/B_K is well-defined. For our purposes, we modify $E(g,z)$. Let $h = \varrho(g)$ for some $g \in G_{\mathbf{A}}$. Define $\tau(h) = \tau(g)$. This definition is well-defined since $Ker(\varrho) \subset Z(G)$. We define

$$E(h,z) = \sum_{\gamma \in H_K/B_K} |\tau(h\gamma)|_{\mathbf{A}}^{-\frac{z}{2}-\frac{1}{2}} \tag{4.3}$$

Note that essentially $E(h,z) = E(g,z)$. So $E(h,z)$ converges absolutely and locally uniformly for $\Re(z) > 1$.

Write $h = \varrho(\kappa d(t,1)a(\tau)n(u))$. Since $E(h,z)$ is right H_K-invariant, then in particular $E(h\varrho(n(a)),z) = E(h,z)$, i.e, $E(\varrho(\kappa d(t,1)a(\tau)n(u+a)),z) = E(\varrho(\kappa d(t,1)a(\tau)n(u)),z)$ for all $a \in K$. So thinking of $E(h,z)$ as a function of u, we see that it is invariant under the map $u \longrightarrow u+a$ for all $a \in K$. Thus $E(h,z)$ has a Fourier expansion:

$$E(h,z) = C_0(\tau,z) + \sum_{a \in K^\bullet} C_a(\tau,z)\psi(au) \tag{4.4}$$

For the properties of this Fourier expansion, we quote Proposition 3.2 of [1] in the following lemma.

Lemma 4.1 *For $E(h,z)$ given in (3.4), we have*

1. $C_0(\tau,z) = |\tau|_{\mathbf{A}}^{-\frac{z}{2}-\frac{1}{2}} + |\tau|_{\mathbf{A}}^{\frac{z}{2}-\frac{1}{2}} q^{1-g}\frac{\zeta_K(z)}{\zeta_K(z+1)}$, where $\zeta_K(z)$ is the Dedekind zeta function of the field K.

2. Let $[\tau] = \sum_{v \in M(K)}(ord_v(\tau))v$ denote the divisor of τ and let $[\psi]$ stand for the canonical divisor associated with the character ψ. Then $C_a(\tau,z) = 0$ for all $a \notin L([\psi]-[\tau])$, the linear system of the divisor $[\psi]-[\tau]$. If $a \in L([\psi]-[\tau])$, $a \neq 0$, then

$$C_a(\tau,z) = |\tau|_{\mathbf{A}}^{\frac{z}{2}-\frac{1}{2}} \frac{P_a(\tau,z)}{\zeta_K(z+1)},$$

where $P_a(\tau, z)$ is a polynomial in q^{-z}. In particular, $C_a(\tau, z)$ is a holomorphic function of q^{-z} in the half-plane $\Re(z) > -\frac{1}{2}$.

Let ϕ be an entire function on $\mathbb{C}$ such that for any $c_1, \ c_2 \in \mathbb{R}$ and any $N > 0$,

$$\sup_{c_1 < \Re(w) < c_2} (1 + |w|)^N |\phi(w)| < \infty.$$

The Smoothed Eisenstein series of Shintani [20] is defined by

$$\mathcal{E}(w, \phi, h) = \frac{1}{2\pi i} \int_{\Re(z) = x_0, 1 < x_0 < \Re(w)} \frac{E(h, z)}{w - z} \phi(z)\, dz. \tag{4.5}$$

Here the orientation of the contour $\Re(z) = x_0$ is taken from $x_0 - i\infty$ to $x_0 + i\infty$. Like $E(h, z)$, $\mathcal{E}(w, \phi, h)$ has a Fourier expansion

$$\mathcal{E}(w, \phi, h) = \sum_{a \in K} \mathcal{C}_a(w, \phi, \tau) \psi(au) \tag{4.6}$$

where

$$\mathcal{C}_a(w, \phi, \tau) = \frac{1}{2\pi i} \int_{1 < \Re(z) = x_0 < \Re(w)} \frac{C_a(\tau, z)}{w - z} \phi(z)\, dz. \tag{4.7}$$

It follows from Lemma 4.1 that for all but finitely many a'^s, $\mathcal{C}_a(w, \phi, \tau) = 0$ and for $a \neq 0$, $\mathcal{C}_a(w, \phi, \tau)$ is an analytic function of w in the region $\Re(w) > -\frac{1}{2}$. Set

$$\mathcal{E}''(w, \phi, h) = \frac{1}{2\pi i} \int_{1 < \Re(z) = x_0 < \Re(w)} \frac{|\tau|_{\mathbf{A}}^{-\frac{z}{2} - \frac{1}{2}}}{w - z} \phi(z)\, dz + \sum_{a \in K^*} \mathcal{C}_a(w, \phi, \tau) \psi(au). \tag{4.8}$$

We summarize the properties of $\mathcal{E}(w, \phi, h)$ in the following lemma.

Lemma 4.2 *We have*

1. $\mathcal{E}(w, \phi, h)$ is an analytic function of w in the half-plane $\Re(w) > 1$
2. For $h \in \mathcal{S}$, $\mathcal{E}(w, \phi, h) = O(|\tau(h)|_{\mathbf{A}}^{\frac{\Re(w) - 1}{2}})$.
3. $\lim_{w \to 1}(1 - q^{1-w}) \mathcal{E}(w, \phi, h) = \phi(1) q^{1-\mathbf{g}} \frac{Res_q \zeta_K}{\zeta_K(2)}$, *where* $Res_q \zeta_K = \lim_{w \to 1}(1 - q^{1-w}) \zeta_K(w)$. *The limit converges uniformly in h, $h \in \mathcal{S}^-$.*
4. For any $F \in L^1(H_{\mathbf{A}}^- / H_K)$,

$\lim_{w\to 1}(1-q^{1-w})\int_{H_{\mathbf{A}}^-/H_K} F(h)\mathcal{E}(w,\phi,h)\,dh = \phi(1)q^{1-g}\frac{Res_q\zeta_K}{\zeta_K(2)}\int_{H_{\mathbf{A}}^-/H_K} F(h)\,dh$

5. $\mathcal{E}''(w,\phi,h)$ is an analytic function of w in the region $\Re(w) > -\frac{1}{2}$. Moreover, for all $h \in H_{\mathbf{A}}$ and w with $\Re(w) > 0$,

$$\mathcal{E}''(w,\phi,h) = O(|\tau(h)|_{\mathbf{A}}^{3/2}).$$

Proof : The first four statements form the content of Lemma 3.2 of [1]. The last statement follows by first explicitly evaluating $C_a(\tau, z)$, $a \neq 0$, of the previous lemma and then applying the Riemann-Roch theorem. ■

4.3 The Integral $I(\omega, f; w, \phi)$

Recall the integral $I(\omega, f)$ of Proposition 4.1 in Section 4.1. We pointed out in that section that the analytic continuation of $Z(\omega, f)$ is equivalent to to the analytic continuation of $I(\omega, f)$. Due to the structure of the set S_K, see Proposition 4.2, we pointed out that to analytically continue $I(\omega, f)$, it is natural first to decompose $I(\omega, f)$ into a sum of three similar integrals each over S_K^i, $i = 0, 1, 2$, and then analytically continue each resulting integral. But unfortunately, each of the resulting integrals does not converge. And this is what brought us to Shintani's Eisenstein series in the previous section. In this section we show how we can use Shintani's Eisenstein series to recover the convergence of the integrals over S_K^i for $i = 0, 1, 2$.

Let $\mathcal{S}^- = \mathcal{S} \cap H_{\mathbf{A}}^-$. Let $I(\mathcal{S}^-)$ be the space of functions defined on $\mathcal{S}^-$. Define a seminorm $N_{\sigma\alpha}$ on $I(\mathcal{S}^-)$ as follows: For $F \in I(\mathcal{S}^-)$,

$$N_{\sigma\alpha}(F) = \int_{\mathcal{S}^-} |det(h)|_{\mathbf{A}}^{\sigma} |F(h)| |\tau(h)|_{\mathbf{A}}^{\alpha}\,dh.$$

Set $I(\mathcal{S}^-, \sigma, \alpha) = \{F \in I(\mathcal{S}^-) : N_{\sigma\alpha}(F) < \infty\}$. Then we have the following lemma:

Lemma 4.3 *If $f \in \mathcal{S}(V_{\mathbf{A}})$, then both $\sum_{x\in V_K} f(h\cdot x)$ and $\omega_{-1}(det(h))\sum_{x\in V_K}\hat{f}(h'\cdot x)$ are in $I(\mathcal{S}^-, \sigma, \alpha)$ for any $\sigma > 1$ and $\alpha > 0$.*

Proof : In the proof of Lemma 3.2(2), we have found that for any $h \in \mathcal{S}$,

$$\sum_{x \in V_K} |f(h \cdot x)| \ll \max(|t|_{\mathbf{A}}^{-1}, 1) \max(|t\tau|_{\mathbf{A}}^{-1}, 1) \max(|t\tau^2|_{\mathbf{A}}^{-1}, 1). \tag{4.9}$$

But for $h \in \mathcal{S}^-$, we have $|det(h)|_{\mathbf{A}} = |t\tau|_{\mathbf{A}}^3 \leq 1$, i.e, $|t\tau|_{\mathbf{A}} \leq 1$. So we get $\max(|t\tau|_{\mathbf{A}}^{-1}, 1) = |t\tau|_{\mathbf{A}}^{-1}$ and $\max(|t\tau^2|_{\mathbf{A}}^{-1}, 1) = |t\tau^2|_{\mathbf{A}}^{-1} \max(1, |t\tau^2|_{\mathbf{A}}) \ll |t\tau^2|_{\mathbf{A}}^{-1}$, as we also have $|\tau|_{\mathbf{A}} \leq q^{2\mathbf{g}}$ for $h \in \mathcal{S}^-$. Thus we get the following bound

$$\textstyle\sum_{x \in V_K} |f(h \cdot x)| \ll \max(|t|_{\mathbf{A}}^{-1}, 1)|t\tau|_{\mathbf{A}}^{-2}|\tau|_{\mathbf{A}}^{-1}.$$

With $h = \varrho(\kappa d(t,1)a(\tau)n(u))$, this bound implies

$$N_{\sigma\alpha}(\sum_{x \in V_K} f(h \cdot x)) \ll \int_{|t\tau|_{\mathbf{A}} \leq 1, |\tau|_{\mathbf{A}} \leq q^{2\mathbf{g}}} |t\tau|_{\mathbf{A}}^{3\sigma} \max(|t|_{\mathbf{A}}^{-1}, 1)|t\tau|_{\mathbf{A}}^{-2}|\tau|_{\mathbf{A}}^{\alpha}\, d^*t d^*\tau.$$

We break up this integral into two integrals: the first is over $|t\tau|_{\mathbf{A}} \leq 1$, $|\tau|_{\mathbf{A}} \leq q^{2\mathbf{g}}$, $|t|_{\mathbf{A}} < 1$; and the second is over $|t\tau|_{\mathbf{A}} \leq 1$, $|\tau|_{\mathbf{A}} \leq q^{2\mathbf{g}}$, $|t|_{\mathbf{A}} \geq 1$. The first integral is dominated by

$$\int_{|\tau|_{\mathbf{A}} \leq q^{2\mathbf{g}}, |t|_{\mathbf{A}} \leq 1} |t\tau|_{\mathbf{A}}^{3\sigma - 3}|\tau|_{\mathbf{A}}^{\alpha+1}\, d^*t d^*\tau = \sum_{n=0}^{\infty} q^{-n(3\sigma-3)} \sum_{m=-2\mathbf{g}}^{\infty} q^{-m(3\sigma+\alpha-2)}$$

which converges provided $\sigma > 1$ and $3\sigma + \alpha > 2$.

The second integral is equal to

$$\int_{|t\tau|_{\mathbf{A}} \leq 1} \int_{|\tau|_{\mathbf{A}} \leq |t\tau|_{\mathbf{A}}} |t\tau|_{\mathbf{A}}^{3\sigma-2}|\tau|_{\mathbf{A}}^{\alpha}\, d^*\tau d^*t\tau = \sum_{n=0}^{\infty}(q^{-n(3\sigma-2)} \sum_{m=n}^{\infty} q^{-m\alpha})$$

$$= \sum_{n=0}^{\infty} q^{-n(3\sigma-2)} \frac{q^{-n\alpha}}{1-q^{-\alpha}} = \frac{1}{1-q^{-\alpha}} \frac{1}{1-q^{-(3\sigma+\alpha-2)}}$$

provided $\alpha > 0$ and $3\sigma + \alpha > 2$. This proves the lemma for $\sum_{x \in V_K} f(h \cdot x)$.

For the second part of the lemma, note that, as in (4.9), we also have $(h' \in \mathcal{S})$

$$\textstyle\sum_{x \in V_K} |\hat{f}(h' \cdot x)| \ll \max(|t'|_{\mathbf{A}}^{-1}, 1) \max(|t'\tau'|_{\mathbf{A}}^{-1}, 1) \max(|t'\tau'^2|_{\mathbf{A}}^{-1}, 1).$$

Since $det(h') = (det(h))^{-1}$, then $|t'\tau'|_{\mathbf{A}} = |t\tau|_{\mathbf{A}}^{-1}$. Since $|t\tau|_{\mathbf{A}} \leq 1$, then $\max(|t'\tau'|_{\mathbf{A}}^{-1}, 1) = 1$. Also since $\tau(h) = \tau(h')$, then $\tau = \tau'$. So $|t'\tau'|_{\mathbf{A}} = |t\tau|_{\mathbf{A}}^{-1}$ implies $|t'|_{\mathbf{A}}^{-1} = |t\tau^2|_{\mathbf{A}} \leq q^{2\mathbf{g}}$ and $|t'\tau'^2|_{\mathbf{A}}^{-1} = |t|_{\mathbf{A}}$. All this yield the bound

$$\sum_{x\in V_K} |\hat{f}(h'\cdot x)| \ll \max(|t|_{\mathbf{A}}, 1)$$

So with $h = \varrho(\kappa d(t,1)a(\tau)n(u))$, this bound implies

$$N_{\sigma\alpha}(\omega_{-1}(det(h))\sum_{x\in V_K}\hat{f}(h'\cdot x)) \ll \int_{|t\tau|_{\mathbf{A}}\leq 1, |\tau|_{\mathbf{A}}\leq q^{2\mathbf{s}}} |t\tau|_{\mathbf{A}}^{3\sigma-3} \max(|t|_{\mathbf{A}}, 1)|\tau|_{\mathbf{A}}^{\alpha+1}\, d^*t d^*\tau$$

$$= \int_{|t\tau|_{\mathbf{A}}\leq 1, |\tau|_{\mathbf{A}}\leq q^{2\mathbf{s}}, |t|_{\mathbf{A}}<1} |t\tau|_{\mathbf{A}}^{3\sigma-3}|\tau|_{\mathbf{A}}^{\alpha+1}\, d^*t d^*\tau + \int_{|t\tau|_{\mathbf{A}}\leq 1, |\tau|_{\mathbf{A}}\leq q^{2\mathbf{s}}, |t|_{\mathbf{A}}\geq 1} |t\tau|_{\mathbf{A}}^{3\sigma-2}|\tau|_{\mathbf{A}}^{\alpha}\, d^*t d^*\tau$$

But these integrals are exactly the ones we considered in the proof of the first part of the lemma. So the proof of the lemma is complete. ■

Now Lemma 4.3 and Lemma 4.2(2) imply

$$\omega(det(h))\sum_{x\in S_K} f(h\cdot x)\mathcal{E}(w,\phi,h) \in L^1(H_{\mathbf{A}}^-/H_K) \tag{4.10}$$

and

$$\omega(det(h))\omega_{-1}(det(h))\sum_{x\in S_K}\hat{f}(h'\cdot x)\mathcal{E}(w,\phi,h) \in L^1(H_{\mathbf{A}}^-/H_K). \tag{4.11}$$

provided $\Re(\omega) > 1$ and $\Re(w) > 1$.

Set

$$I(\omega, f; w, \phi) \tag{4.12}$$
$$= \int_{H_{\mathbf{A}}/H_K} \omega(det(h))\lambda_-(h)\sum_{x\in S_K}[\omega_{-1}(det(h))\hat{f}(h'\cdot x) - f(h\cdot x)]\mathcal{E}(w,\phi,h)\, dh$$

Then by (4.10) and (4.11), $I(\omega, f; w, \phi)$ is well-defined provided $\Re(\omega) > 1$ and $\Re(w) > 1$. We may thus write

$$I(\omega, f; w, \phi) = \sum_{i=0}^{2} I^i(\omega, f; w, \phi)$$

where

$$I^i(\omega, f; w, \phi) \tag{4.13}$$
$$= \int_{H_{\mathbf{A}}/H_K} \omega(det(h))\lambda_-(h)\sum_{x\in S_K^i}[\omega_{-1}(det(h))\hat{f}(h'\cdot x) - f(h\cdot x)]\mathcal{E}(w,\phi,h)\, dh$$

and again, by (4.10) and (4.11), each $I^i(\omega, f; w, \phi)$ is well-defined provided $\Re(\omega) > 1$ and $\Re(w) > 1$.

We point out here that by Lemma 4.2(4), we have

$$\lim_{w\to 1}(1-q^{1-w})I(\omega,f;w,\phi)=\phi(1)q^{1-g}\frac{Res_q\zeta_K}{\zeta_K(2)}I(\omega,f).$$

Since we are interested in computing $\lim_{w\to 1}(1-q^{1-w})I(\omega,f;w,\phi)$, we introduce the following notation:

Notation 4.1 *For two meromorphic functions $f(w)$ and $g(w)$, we say $f(w)$ is equivalent to $g(w)$, denoted*

$$f(w)\sim g(w),$$

if $f(w)-g(w)$ is analytic in a region $\sigma_1<\Re(w)<\sigma_2$ for some σ_1 and σ_2 satisfying $\sigma_1<1<\sigma_2$.

The objective of the following three sections is to evaluate $I^i(\omega,f;w,\phi)$, for $i=0,\ 1,\ 2$. Before we start the computation, we give one more definition. For $\omega\in\Omega$, define the symbols $\eta(\omega)$ and $\delta(\omega)$ by

$$\eta(\omega)=\begin{cases}1 & \text{If } \omega(t)=1\ \forall t\in\mathbf{A}^*(\emptyset)\\ 0 & \text{otherwise;}\end{cases}$$

$$\delta(\omega)=\begin{cases}1 & \text{If } \omega(t)=1\ \forall t\in\mathbf{A}^1\\ 0 & \text{otherwise.}\end{cases}$$

Since $\mathbf{A}^*(\emptyset)\subset\mathbf{A}^1$, then $\eta(\omega)\delta(\omega)=\delta(\omega)$.

4.4 Evaluation of $I^0(\omega,f;w,\phi)$

Proposition 4.3 *We have*

$$I^0(\omega,f;w,\phi)\sim\delta(\omega^3)\phi(w)\frac{q^{-(\frac{w-1}{2})}}{1-q^{-(\frac{w-1}{2})}}\left(\frac{\hat{f}(0)}{1-q^{-(3s-3)}}-\frac{f(0)}{1-q^{-3s}}+\frac{f(0)-\hat{f}(0)}{2}\right)$$

Proof : We first consider the integral

$$\int_{H_{\mathbf{A}}/H_K}\omega(det(h))\lambda_-(h)f(0)\mathcal{E}(w,\phi,h)\,dh=$$

$$\int_{H_{\mathbf{A}}/H_K} \omega(det(h))\lambda_-(h)f(0)\frac{1}{2\pi i}\int_{1<\Re(z)=x_0<\Re(w)} \frac{\sum_{\gamma\in H_K/\mathcal{B}_K}|\tau(h\gamma)|_{\mathbf{A}}^{-\frac{z}{2}-\frac{1}{2}}}{w-z}\phi(z)\,dz\,dh \tag{4.14}$$

As ω is trivial on K^*, we have $\omega(det(\gamma)) = 1$ for all $\gamma \in H_K$. Also by absolute convergence of (4.14), integral (4.14) reduces to

$$\sum_{\gamma\in H_K/\mathcal{B}_K}\int_{(H_{\mathbf{A}}/H_K)\gamma} \omega(det(h))\lambda_-(h)f(0)\frac{1}{2\pi i}\int_{1<\Re(z)=x_0<\Re(w)} \frac{|\tau(h)|_{\mathbf{A}}^{-\frac{z}{2}-\frac{1}{2}}}{w-z}\phi(z)\,dz\,dh$$

As $\mathcal{B}_K$ is a subgroup of H_K, we have $H_K = \bigsqcup \gamma\mathcal{B}_K$, a coset decomposition. Let F be a fundamental domain for H_K in $H_{\mathbf{A}}^-$; so $H_{\mathbf{A}}^- = FH_K$. This gives $H_{\mathbf{A}}^- = (\bigsqcup_{\gamma\in H_K/\mathcal{B}_K} F\gamma)\mathcal{B}_K$. Thus $\bigsqcup_{\gamma\in H_K/\mathcal{B}_K} F\gamma$ is a fundamental domain for $\mathcal{B}_K$ in $H_{\mathbf{A}}^-$. So the last integral becomes

$$\int_{H_{\mathbf{A}}/\mathcal{B}_K} \omega(det(h))\lambda_-(h)f(0)\frac{1}{2\pi i}\int_{1<\Re(z)=x_0<\Re(w)} \frac{|\tau(h)|_{\mathbf{A}}^{-\frac{z}{2}-\frac{1}{2}}}{w-z}\phi(z)\,dz\,dh$$

With $h = \varrho(\kappa d(t,1)a(\tau)n(u))$, $dh = |\tau|_{\mathbf{A}} d\kappa d^*t du d^*\tau$ and $det(h) = (det(\kappa)\,t\tau)^3$. So the last integral reduces to

$$\int_K \omega^3(det(\kappa))\,d\kappa \int_{\mathbf{A}/K} f(0)\,du \underbrace{\int_{\mathbf{A}^*/K^*}\int_{\mathbf{A}^*/K^*}}_{|t\tau|_{\mathbf{A}}\leq^* 1} \omega^3(t\tau)\,\frac{1}{2\pi i}\int_{\Re(z)=x_0} \frac{|\tau|_{\mathbf{A}}^{-\frac{z}{2}+\frac{1}{2}}}{w-z}\phi(z)\,dz\,d^*t\,d^*\tau \tag{4.15}$$

where $\leq^*$ signifies that when equality is satisfied, we multiply the integral by the factor of $\frac{1}{2}$. Making the change of variable $t \longrightarrow \frac{t}{\tau}$, integral (4.15) becomes

$$\eta(\omega^3)f(0)\int_{\mathbf{A}^*/K^*,|t|_{\mathbf{A}}\leq^* 1} \omega^3(t)\,d^*t \int_{\mathbf{A}^*/K^*} \frac{1}{2\pi i}\int_{1<\Re(z)=x_0<\Re(w)} \frac{|\tau|_{\mathbf{A}}^{-\frac{z}{2}+\frac{1}{2}}}{w-z}\phi(z)\,dz\,d^*\tau \tag{4.16}$$

Now the first integral in (4.16) equals ($|\pi|_{\mathbf{A}} = q$)

$$\begin{aligned}\frac{1}{2}\int_{\mathbf{A}^1/K^*}\omega^3(t)\,d^1t \quad &+ \sum_{n=1}^{\infty}\int_{\mathbf{A}^1/K^*}\omega^3(\pi^{-n}t)\,d^1t\\ &= \frac{1}{2}\delta(\omega^3) + \sum_{n=1}^{\infty}|\pi^{-n}|_{\mathbf{A}}^{3s}\int_{\mathbf{A}^1/K^*}\tilde{\omega}^3(t)\,d^1t\\ &= \frac{1}{2}\delta(\omega^3) + \delta(\omega^3)\sum_{n=1}^{\infty}q^{-3ns}\\ &= \delta(\omega^3)(-\tfrac{1}{2} + \tfrac{1}{1-q^{-3s}})\end{aligned}$$

Also since $\delta(\omega^3)\eta(\omega^3) = \delta(\omega^3)$, then (4.16) becomes

$$\delta(\omega^3)f(0)(-\tfrac{1}{2}+\tfrac{1}{1-q^{-3s}})[(\textstyle\int_{\mathbf{A}^-/K^*}+\int_{\widetilde{\mathbf{A}^+}/K^*})(\tfrac{1}{2\pi i}\int_{1<\Re(z)=x_0<\Re(w)} \tfrac{|\tau|_{\mathbf{A}}^{-\frac{z}{2}+\frac{1}{2}}}{w-z}\phi(z)\, dz)\, d^*\tau] \tag{4.17}$$

where $\mathbf{A}^- = \{\tau \in \mathbf{A}^* : |\tau|_{\mathbf{A}} \leq 1\}$ and $\widetilde{\mathbf{A}^+} = \{\tau \in \mathbf{A}^* : |\tau|_{\mathbf{A}} > 1\}$.

Next we consider the integrals over $\mathbf{A}^-/K^*$ and $\widetilde{\mathbf{A}^+}/K^*$ in (4.17) separately. The integral over $\mathbf{A}^-/K^*$ in (4.17) equals

$$\sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{1<\Re(z)=x_0<\Re(w)} \frac{q^{-n(-\frac{z}{2}+\frac{1}{2})}}{w-z}\phi(z)\, dz$$

But the integral in the above sum is entire and it is of order $O(q^{-nm})$ for any $m > \frac{1}{2}$: Just shift the contour of integration to the left to $\Re(z) = -\alpha = 1-2m < 0$. This implies the above sum is an entire function of w and hence we can disregard its contribution to $I^0(\omega, f; w, \phi)$.

For the integral over $\widetilde{\mathbf{A}^+}/K^*$ in (4.17), it equals

$$\sum_{n=1}^{\infty} \frac{1}{2\pi i}\int_{1<\Re(z)=x_o<\Re(w)} \frac{q^{n(-\frac{z}{2}+\frac{1}{2})}}{w-z}\phi(z)\, dz$$

By shifting the contour of integration to the right to $\Re(z) = \alpha > \Re(w)$, the integral in the above sum equals

$$q^{\frac{n}{2}(1-w)}\phi(w) + \frac{1}{2\pi i}\int_{\Re(z)=\alpha>\Re(w)} \frac{q^{\frac{n}{2}(1-z)}}{w-z}\phi(z)\, dz$$

So the integral over $\widetilde{\mathbf{A}^+}/K^*$ equals

$$\frac{q^{-(\frac{w-1}{2})}}{1-q^{-(\frac{w-1}{2})}}\phi(w) + \sum_{n=1}^{\infty} \frac{1}{2\pi i}\int_{\Re(z)=\alpha>\Re(w)} \frac{q^{\frac{n}{2}(1-z)}}{w-z}\phi(z)\, dz$$

But the complex integral in the above sum is entire and it is of order $O(q^{-nm})$ for any $m > 0$. Hence the above series is an entire function of w and so its contribution to $I^0(\omega, f; w, \phi)$ can be disregarded. Combining the above calculations gives

$$\int_{H_{\mathbf{A}}/H_K} \omega(det(h))\lambda_-(h)f(0)\mathcal{E}(w,\phi,h)\, dh \sim \delta(\omega^3)\phi(w) \quad f(0)\frac{q^{-(\frac{w-1}{2})}}{1-q^{-(\frac{w-1}{2})}} (-\tfrac{1}{2}+\tfrac{1}{1-q^{-3s}})$$

Similarly, we get

$$\int_{H_{\mathbf{A}}/H_K} \omega(det(h))\omega_{-1}(det(h))\lambda_-(h)\hat{f}(0)\mathcal{E}(w,\phi,h) \quad dh \sim \delta(\omega^3)\phi(w)\hat{f}(0) \frac{q^{-(\frac{w-1}{2})}}{1-q^{-(\frac{w-1}{2})}}\left(-\frac{1}{2}+\frac{1}{1-q^{-(3s-3)}}\right)$$

This completes the proof of the proposition. ■

4.5 Evaluation of $I^1(\omega, f; w, \phi)$

In this section we compute $I^1(\omega, f; w, \phi)$. But first, following Wright [27], we introduce a certain distribution that will come up in the course of computing $I^1(\omega, f; w, \phi)$.

For $f \in \mathcal{S}(V_{\mathbf{A}})$ and $\omega \in \Omega$, define $M_\omega(f)$ by

$$M_\omega(f)(x) = \int_{\mathcal{K}} \omega(det(\kappa))f(\kappa \cdot x)\, d\kappa.$$

Since $\mathcal{K}$ is compact, then $M_\omega(f) \in \mathcal{S}(V_{\mathbf{A}})$. The following lemma states some of the properties of $M_\omega(f)$, see [27].

Lemma 4.4 *For $f \in \mathcal{S}(V_{\mathbf{A}})$ and $\omega \in \Omega$, we have*

1. $M_\omega(f) = M_{\tilde{\omega}}(f)$
2. *For any $\kappa \in \mathcal{K}$, we have* $M_\omega(f)(\kappa \cdot x) = \omega^{-1}(det(\kappa))M_\omega(f)$
3. $M_\omega(M_\omega(f)) = M_\omega(f)$
4. $\widehat{M_\omega(f)} = M_{\tilde{\omega}}(\hat{f})$

Because of Lemma 4.4(1), when $\tilde{\omega} = 1$, we write $M(f)$ for $M_\omega(f)$.

We also need to recall Tate's zeta functin, first introduced by Tate in his thesis [23]. For $F \in \mathcal{S}(\mathbf{A})$, $\omega \in \Omega$, Tate's zeta function is defined by

$$\zeta(\omega, F) = \int_{\mathbf{A}^*} \omega(t)F(t)\, d^*t$$

We collect its basic properties in the following lemma. For proofs, see [23] or [26].

Lemma 4.5 *$\zeta(\omega, F)$ satisfies the following*

1. $\zeta(\omega, F)$ converges absolutely and locally uniformly for $\Re(\omega) > 1$.

2. $\zeta(\omega, F)$ can be analytically continued to a meromorphic function defined on all of Ω. Further, it satisfies the functional equation $\zeta(\omega, F) = \zeta(\omega_1\omega^{-1}, \widehat{F})$, where $\widehat{F}$ is the Fourier transform of F.

3. $\zeta(\omega, F)$ is analytic everywhere in Ω except for simple poles at ω_0 and ω_1 with respective residues $\frac{-F(0)}{\log q}$ and $\frac{\widehat{F}(0)}{\log q}$.

For $f \in \mathcal{S}(V_{\mathbf{A}})$, define $T_1(f) \in \mathcal{S}(\mathbf{A})$ by $T_1(f)(t) = f(0, 0, t)$. Define

$$\Sigma_1(z, \omega, f) = \zeta(\omega_z, T_1(M_\omega(f))) = \int_{\mathbf{A}^*} |t|_{\mathbf{A}}^z T_1(M_\omega(f))(t)\, d^*t$$

When $\tilde{\omega} = 1$, we write $\Sigma_1(z, f)$ for $\Sigma_1(z, \omega, f)$. Lemma 4.5 implies the following proposition.

Proposition 4.4 *$\Sigma_1(z, \omega, f)$ satisfies the following*

1. $\Sigma_1(z, \omega, f)$ converges absolutely and locally uniformly for $\Re(z) > 1$, for all $\omega \in \Omega$ and $f \in \mathcal{S}(V_{\mathbf{A}})$.

2. $\Sigma_1(z, \omega, f)$ has a meromorphic continuation to all of $\mathbb{C}$. Further, it is analytic everywhere in $\mathbb{C}$ except for simple poles at $z = 0$ and $z = 1$, with respective residues $\frac{-T_1(M_\omega(f))(0)}{\log q}$ and $\frac{T_1(\widehat{M_\omega}(f))(0)}{\log q}$, where $T_1(M_\omega(f))(0) = f(0)\delta(\omega)$, and $T_1(\widehat{M_\omega}(f))(0) = \int_{\mathbf{A}} M_\omega(f)(0, 0, t)\, dt$

Now we are ready to state the main result of this section.

Proposition 4.5

$$\begin{aligned} I^1(\omega, f; w, \phi) \ \sim \ & \delta(\omega^3)\frac{\zeta_K(w)}{\zeta_K(w+1)} q^{1-\mathbf{g}}\phi(w) \\ & \left[\frac{\Sigma_1(\frac{w+1}{2}, f)}{1-q^{-(3s-\frac{w}{2}-\frac{1}{2})}} - \frac{\Sigma_1(\frac{w+1}{2}, \hat{f})}{1-q^{-(3s-3+\frac{w}{2}+\frac{1}{2})}} + \frac{\Sigma_1(\frac{w+1}{2}, \hat{f})}{2} - \frac{\Sigma_1(\frac{w+1}{2}, f)}{2}\right] \end{aligned}$$

Proof : We first consider the integral

$$\int_{H_{\mathbf{A}}/H_K} \omega(det(h))\lambda_-(h) \sum_{x \in S_K^1} f(h \cdot x)\mathcal{E}(w, \phi, h)\, dh \qquad (4.18)$$

Since $S_K^1 = \bigcup_{\gamma\in H_K/B_K}\bigcup_{a\in K^\bullet}\gamma\cdot(0,0,a)$ and $\omega(det(\gamma)) = 1$ for all $\gamma \in H_K$, then reasoning as in the proof of Propsitin 4.3, the integral (4.18) becomes

$$\int_{H_{\mathbf{A}}/B_K} \omega(det(h))\lambda_-(h) \sum_{a\in K^\bullet} f(h\cdot(0,0,a))\mathcal{E}(w,\phi,h)\,dh \tag{4.19}$$

With $h = \varrho(\kappa d(t,1)a(\tau)n(u))$, we have $dh = |\tau|_{\mathbf{A}} d\kappa d^*td^*\tau du$, $det(h) = (det(\kappa)t\tau)^3$, and $h\cdot(0,0,a) = \varrho(\kappa)\cdot(0,0,t\tau^2 a) = \kappa\cdot(0,0,t\tau^2a)$. So integral (4.19) becomes

$$\underbrace{\int_{\mathbf{A}^\bullet/K^\bullet}\int_{\mathbf{A}^\bullet/K^\bullet}}_{|t\tau|\leq^\bullet 1} \omega^3(t\tau)\textstyle\sum_{a\in K^\bullet}\int_K \omega^3(det(\kappa))f(\kappa\cdot(0,0,t\tau^2a)\,d\kappa \int_{\mathbf{A}/K}\mathcal{E}(w,\phi,h)\,du$$
$$|\tau|_{\mathbf{A}}\,d^*\tau d^*t \tag{4.20}$$

Using the Fourier expansion of $\mathcal{E}(w,\phi,h)$ and the fact that $\mathbf{A}/K$ is compact, we conclude that the integral over $\mathbf{A}/K$ in (4.20) is $\mathcal{C}_0(w,\phi,h)$. So integral (4.20) reduces to

$$\underbrace{\int_{\mathbf{A}^\bullet/K^\bullet}\int_{\mathbf{A}^\bullet/K^\bullet}}_{|t\tau|_{\mathbf{A}}\leq^\bullet 1} \omega^3(t\tau)\textstyle\sum_{a\in K^\bullet} \quad M_{\omega^3}(f)(0,0,t\tau^2a)\,\frac{1}{2\pi i}\int_{1<\Re(z)=x_0<\Re(w)}$$
$$\frac{|\tau|_{\mathbf{A}}^{-\frac{z}{2}-\frac{1}{2}}+|\tau|_{\mathbf{A}}^{\frac{z}{2}-\frac{1}{2}}\frac{\zeta_K(z)}{\zeta_K(z+1)}q^{1-g}}{w-z}\phi(z)\,dz\,|\tau|_{\mathbf{A}}d^*\tau d^*t$$

Making the change of variables $\tau \longrightarrow \frac{\tau}{t}$ and $t \longrightarrow \frac{t^2}{\tau}$, we get

$$\textstyle\int_{\mathbf{A}^\bullet/K^\bullet,|t|_{\mathbf{A}}\leq^\bullet 1}\int_{\mathbf{A}^\bullet/K^\bullet}\omega^3(t)\sum_{a\in K^\bullet} \quad M_{\omega^3}(f)(0,0,\tau a)\,\frac{1}{2\pi i}\int_{1<\Re(z)=x_0<\Re(w)}$$
$$\frac{|\frac{\tau}{t}|_{\mathbf{A}}^{-\frac{z}{2}+\frac{1}{2}}+|\frac{\tau}{t}|_{\mathbf{A}}^{\frac{z}{2}+\frac{1}{2}}\frac{\zeta_K(z)}{\zeta_K(z+1)}q^{1-g}}{w-z}\phi(z)\,dz\,d^*\tau d^*t$$

By absorbing the sum over K^* with the integral over τ, we get

$$\textstyle\int_{\mathbf{A}^\bullet/K^\bullet,|t|_{\mathbf{A}}\leq^\bullet 1}\int_{\mathbf{A}^\bullet}\omega^3(t)M_{\omega^3}(f)(0,0,\tau) \quad \frac{1}{2\pi i}\int_{1<\Re(z)=x_0<\Re(w)}$$
$$\frac{|\frac{\tau}{t}|_{\mathbf{A}}^{-\frac{z}{2}+\frac{1}{2}}+|\frac{\tau}{t}|_{\mathbf{A}}^{\frac{z}{2}+\frac{1}{2}}\frac{\zeta_K(z)}{\zeta_K(z+1)}q^{1-g}}{w-z}\phi(z)\,dz\,d^*\tau d^*t \tag{4.21}$$

We break up the integral (4.21) into two parts. The first one is

$$\int_{\mathbf{A}^\bullet/K^\bullet,|t|_{\mathbf{A}}\leq^\bullet 1}\int_{\mathbf{A}^\bullet}\omega^3(t)M_{\omega^3}(f)(0,0,\tau)\frac{1}{2\pi i}\int_{1<\Re(z)=x_0<\Re(w)}\frac{|\frac{\tau}{t}|_{\mathbf{A}}^{-\frac{z}{2}+\frac{1}{2}}}{w-z}\phi(z)\,dz\,d^*\tau d^*t \tag{4.22}$$

To evaluate (4.22), we need to switch the order of integration. For this, it turns out that we need first to shift the contour of integration in the complex integral in (4.22) to the line $\Re(z) = x_1 < -1$. This is possible as we did in the calculation of $I^0(\omega, f; w, \phi)$. Then it follows that the integral (4.22) converges absolutely provided $3\Re(s) > -\frac{z}{2} + \frac{1}{2}$. Thus by Fubini-Tonelli's theorem, integral (4.22) equals

$$\frac{1}{2\pi i}\int_{\Re(z)=x_1<-1,\Re(z)<\Re(w)} \frac{\phi(z)}{w-z}\int_{\mathbf{A}^\bullet} M_{\omega^3}(f)(0,0,\tau)|\tau|_{\mathbf{A}}^{-\frac{z}{2}+\frac{1}{2}}\, d^*\tau \int_{\mathbf{A}^\bullet/K^\bullet,|t|_{\mathbf{A}}\leq^\bullet 1} \omega^3(t)\, |t|_{\mathbf{A}}^{\frac{z}{2}-\frac{1}{2}} d^*t\, dz$$

$$= \frac{1}{2\pi i}\int_{\Re(z)=x_1<-1,\Re(z)<\Re(w)} \frac{\phi(z)}{w-z}\Sigma_1\left(-\frac{z}{2}+\frac{1}{2},\omega^3,f\right)\delta(\omega^3)\left[-\frac{1}{2}+\frac{1}{1-q^{-(3s+\frac{z}{2}-\frac{1}{2})}}\right] dz$$

This last integral is analytic in the region $\Re(w) > -1$, and so its contribution can be disregarded.

Next we consider the second part of integral (4.21), namely

$$\int_{\mathbf{A}^\bullet/K^\bullet,|t|_{\mathbf{A}}\leq^\bullet 1}\int_{\mathbf{A}^\bullet} \omega^3(t) M_{\omega^3}(f)(0,0,\tau)\frac{1}{2\pi i}\int_{1<\Re(z)=x_0<\Re(w)} \frac{|\frac{\tau}{t}|_{\mathbf{A}}^{\frac{z}{2}+\frac{1}{2}}\frac{\zeta_K(z)}{\zeta_K(z+1)}}{w-z} q^{1-\mathbf{g}}\, \phi(z)\, dz\, d^*\tau d^*t \tag{4.23}$$

Again, to evaluate (4.23), we need to switch the order of integration. It turns out that integral (4.23) converges absolutely provided $3\Re(s) > \frac{\Re(z)}{2} + \frac{1}{2}$. So under this assumption, by Fubini-Tonelli's theorem, integral (4.23) equals

$$\int_{\mathbf{A}^\bullet} M_{\omega^3}(f)(0,0,\tau)\frac{q^{1-\mathbf{g}}}{2\pi i}\int_{1<\Re(z)=x_0<\Re(w)} \frac{|\tau|_{\mathbf{A}}^{\frac{z}{2}+\frac{1}{2}}\frac{\zeta_K(z)}{\zeta_K(z+1)}}{w-z}\phi(z)\int_{\mathbf{A}^\bullet/K^\bullet,|t|_{\mathbf{A}}\leq^\bullet 1}\omega^3(t)\, |t|_{\mathbf{A}}^{-\frac{z}{2}-\frac{1}{2}} d^*t dz d^*\tau$$

which in turn equals

$$\delta(\omega^3)\int_{\mathbf{A}^\bullet} M_{\omega^3}(f)(0,0,\tau)\frac{q^{1-\mathbf{g}}}{2\pi i}\int_{1<\Re(z)=x_0<\Re(w)} \frac{|\tau|_{\mathbf{A}}^{\frac{z}{2}+\frac{1}{2}}\frac{\zeta_K(z)}{\zeta_K(z+1)}}{w-z}\phi(z)\left[-\frac{1}{2}+\frac{1}{1-q^{-(3s-\frac{z}{2}-\frac{1}{2})}}\right] dz d^*\tau \tag{4.24}$$

Because of $\delta(\omega^3)$, we may replace $M_{\omega^3}(f)$ by $M(f)$. Also, in the complex integral in (4.24), we shift the contour to the right to $\Re(z) = x_1$ so that

$x_0 < \Re(w) < x_1 < 6\Re(s) - 1$. Then the complex integral in (4.24) is equal to

$$-|\tau|_{\mathbf{A}}^{\frac{w}{2}+\frac{1}{2}} \frac{\zeta_K(w)}{\zeta_K(w+1)} q^{1-\mathbf{g}} \phi(w)[-\frac{1}{2} + \frac{1}{1-q^{-(3s-\frac{w}{2}-\frac{1}{2})}}] +$$

$$+\frac{q^{1-\mathbf{g}}}{2\pi i} \int_{\Re(z)=x_1>\Re(w)} \frac{|\tau|_{\mathbf{A}}^{\frac{z}{2}+\frac{1}{2}} \frac{\zeta_K(z)}{\zeta_K(z+1)}}{w-z} \phi(z)[-\frac{1}{2} + \frac{1}{1-q^{-(3s-\frac{z}{2}-\frac{1}{2})}}] \, dz$$

So integral (4.24) reduces to

$$-\delta(\omega^3)\Sigma_1(\tfrac{w}{2}+\tfrac{1}{2}, f) \tfrac{\zeta_K(w)}{\zeta_K(w+1)} q^{1-\mathbf{g}} \phi(w)[-\tfrac{1}{2} + \tfrac{1}{1-q^{-(3s-\frac{w}{2}-\frac{1}{2})}}]$$
$$+\tfrac{\delta(\omega^3)}{2\pi i} \int_{\mathbf{A}^*} M(f)(0,0,\tau) \int_{\Re(z)=x_1>\Re(w)} \tfrac{|\tau|_{\mathbf{A}}^{\frac{z}{2}+\frac{1}{2}} \frac{\zeta_K(z)}{\zeta_K(z+1)}}{w-z} q^{1-\mathbf{g}} \phi(z)[-\tfrac{1}{2} + \tfrac{1}{1-q^{-(3s-\frac{z}{2}-\frac{1}{2})}}] \, dz d^*\tau \tag{4.25}$$

We show the second term of (4.25) is analytic in the region $\Re(w) < 6\Re(s) - 1$ and hence its contribution can be disregarded as $6\Re(s) - 1 > 5$ for $\Re(s) > 1$. First note that the integral over $\Re(z) = x_1$ is $O(|\tau|_{\mathbf{A}}^{\alpha})$ for any $\alpha > \Re(w) + \frac{1}{2}$.(Just push the contour to the right to $\Re(z) = 2\alpha - 1$ where $2\Re(w) < 2\alpha - 1 < 6\Re(s) - 1$.) It follows that the absolute value of the integral in (4.25) is majorized by $\sum_1(\alpha, f)$, which is convergent since $\alpha > 1$. So the double integral in (4.25) converges absolutely, and hence we can interchange the order of the integrals to get

$$\int_{\Re(z)=x_1, \Re(w)<x_1<6\Re(s)-1} \Sigma_1(\frac{z}{2}+\frac{1}{2}, f) \frac{q^{1-\mathbf{g}}\phi(z)}{w-z} \frac{\zeta_K(z)}{\zeta_K(z+1)} [-\frac{1}{2} + \frac{1}{1-q^{-(3s-\frac{z}{2}-\frac{1}{2})}}] \, dz$$

This last integral is analytic in the region $\Re(w) < 6\Re(s) - 1$. So we have

$$\int_{H_{\mathbf{A}}/H_K} \omega(det(h))\lambda_-(h) \sum_{x\in S_K^1} f(h\cdot x)\mathcal{E}(w,\phi,h)\, dh \sim -\delta(\omega^3)\Sigma_1(\tfrac{w+1}{2}, f)$$
$$\tfrac{\zeta_K(w)}{\zeta_K(w+1)} q^{1-\mathbf{g}} \phi(w)[-\tfrac{1}{2} + \tfrac{1}{1-q^{-(3s-\frac{w}{2}-\frac{1}{2})}}]$$

Similarly, we get

$$\int_{H_{\mathbf{A}}/H_K} \omega(det(h))\omega_{-1}(det(h))\lambda_-(h) \sum_{x\in S_K^1} \hat{f}(h'\cdot x)\mathcal{E}(w,\phi,h)\, dh \sim -\delta(\omega^3)$$
$$\Sigma_1(\tfrac{w+1}{2}, \hat{f}) \tfrac{\zeta_K(w)}{\zeta_K(w+1)} q^{1-\mathbf{g}} \phi(w)[-\tfrac{1}{2} + \tfrac{1}{1-q^{-(3s-3+\frac{w}{2}+\frac{1}{2})}}]$$

Basically, we make the change of variable $h \longrightarrow h'$, then $\lambda_-(h) \longrightarrow \lambda_+(h')$. We proceed exactly as above, and at one point we need to calculate the following integral

$$\int_{\mathbf{A}^\bullet/K^\bullet, |t|_{\mathbf{A}^\bullet} \geq 1} \omega^{-3}(t)\omega_3(t)|t|_{\mathbf{A}}^{-\frac{z}{2}-\frac{1}{2}} = \delta(\omega^3)[-\frac{1}{2} + \frac{1}{1-q^{-(3s-3+\frac{z}{2}+\frac{1}{2})}}]$$

This completes the proof of the proposition. ■

Because of $\zeta_K(w)$ and $\Sigma_1(\frac{w}{2}+\frac{1}{2}, f)$, it follows that $I^1(\omega, f; w, \phi)$ has a double pole at $w = 1$. We calculate some Laurent expansions at $w = 1$. Let the first two terms in the Laurent expansion of $\Sigma_1(w, *)$ be $\frac{R(*)}{w-1} + \Gamma(*)$. Then

$$\Sigma_1(\frac{w}{2}+\frac{1}{2}, f) = \frac{2R(f)}{w-1} + \Gamma(f), \quad \Sigma_1(\frac{w}{2}+\frac{1}{2}, \hat{f}) = \frac{2R(\hat{f})}{w-1} + \Gamma(\hat{f}).$$

We also have

$$\frac{1}{1-q^{-(3s-\frac{w}{2}-\frac{1}{2})}} - \frac{1}{2} = A + B(w-1) + \cdots, \quad \frac{1}{2} - \frac{1}{1-q^{-(3s-3+\frac{w}{2}+\frac{1}{2})}} = A' + B'(w-1) + \cdots,$$

where

$$A = \frac{1}{1-q^{-(3s-1)}} - \frac{1}{2}, \quad B = \frac{(\frac{1}{2}\log q)q^{-(3s-1)}}{[1-q^{-(3s-1)}]^2},$$
$$A' = \frac{1}{2} - \frac{1}{1-q^{-(3s-2)}}, \quad B' = \frac{(\frac{1}{2}\log q)q^{-(3s-2)}}{[1-q^{-(3s-2)}]^2}.$$

The first two terms of the Laurent expansion of $\Sigma_1(\frac{w}{2}+\frac{1}{2}, \hat{f})[\frac{1}{2} - \frac{1}{1-q^{-(3s-3+\frac{w}{2}+\frac{1}{2})}}] + \Sigma_1(\frac{w}{2}+\frac{1}{2}, f)[\frac{1}{1-q^{-(3s-\frac{w}{2}-\frac{1}{2})}} - \frac{1}{2}]$ are

$$\frac{2AR(f) + 2A'R(\hat{f})}{w-1} + [A\Gamma(f) + 2BR(f) + A'\Gamma(\hat{f}) + 2B'R(\hat{f})].$$

Thus we get

$$\begin{aligned} I^1(\omega, f; w, \phi) \sim\ & \delta(\omega^3)\frac{\zeta_K(w)}{\zeta_K(w+1)} q^{1-\mathbf{g}}\phi(w)\frac{2AR(f)+2A'R(\hat{f})}{w-1} \\ & + \delta(\omega^3)\frac{\zeta_K(w)}{\zeta_K(w+1)} q^{1-\mathbf{g}}\phi(w)[A\Gamma(f) + 2BR(f) + A'\Gamma(\hat{f}) + 2B'R(\hat{f})]. \end{aligned}$$

We note here that by Proposition 4.4, we have

$$R(f) = \frac{1}{\log q}\int_{\mathbf{A}} M(f)(0,0,t)\, dt \text{ and } R(\hat{f}) = \frac{1}{\log q}\int_{\mathbf{A}} M(\hat{f})(0,0,t)\, dt.$$

4.6 Evaluation of $I^2(\omega, f; w, \phi)$

In this section we compute $I^2(\omega, f; w, \phi)$. But first, we introduce another distribution that will come up in the course of computing $I^2(\omega, f; w, \phi)$.

Let v be an absolute value of K, $\Psi_v \in \mathcal{S}(K_v^2)$, and $\Psi \in \mathcal{S}(\mathbf{A}^2)$. For $\omega \in \Omega$ and $w \in \mathbb{C}$, define

$$T_v(\omega_v, w, \Psi_v) = \int_{K_v^*} \int_{K_v} \omega_v(t_v)\Psi_v(t_v, u_v)\alpha_v(u_v t_v^{-1})^w \, du_v \, d^*t_v$$

$$T(\omega, w, \Psi) = \int_{\mathbf{A}^*} \int_{\mathbf{A}} \omega(t)\Psi(t, u)\alpha(ut^{-1})^w \, du \, d^*t$$

$$T^-(\omega, w, \Psi) = \int_{\mathbf{A}^*, |t|_{\mathbf{A}} \leq^* 1} \int_{\mathbf{A}} \omega(t)\Psi(t, u)\alpha(ut^{-1})^w \, du \, d^*t$$

$$T^+(\omega, w, \Psi) = \int_{\mathbf{A}^*, |t|_{\mathbf{A}} \, ^*\geq 1} \int_{\mathbf{A}} \omega(t)\Psi(t, u)\alpha(ut^{-1})^w \, du \, d^*t$$

where

$$\alpha(u) = \prod_{v \in M(K)} \sup(1, |u_v|_v) \quad \textit{and} \quad \alpha_v(u_v) = \sup(1, |u_v|_v).$$

Also the star in the inequalities $\leq^*$ and $^* \geq$ signifies that when equality occurs, the integral will be multiplied by the factor of $\frac{1}{2}$.

If $\Psi = \prod_{v \in M(K)} \Psi_v$, then $T(\omega, w, \Psi)$ has the decomposition:

$$T(\omega, w, \Psi) = q^{1-g}\left(\frac{q-1}{h_{0,K}}\right) \prod_{v \in M(K)} T_v(\omega_v, w, \Psi_v),$$

where ω_v is the restriction of ω to K_v and $h_{0,K}$ is the divisor class number of K.

We point out here that the above distributions are special cases of those given in Definition 2.7 of [28]. We state some properties of the above distributions in the following lemmas.

Lemma 4.6 *1. If $\omega_v(t_v) = |t_v|_v^s$ and Ψ_v is the characteristic function of O_v^2, then*

$$T_v(\omega_v, w, \Psi_v) = \frac{1 - q_v^{-(s-w+1)}}{(1 - q_v^{-(s+1)})(1 - q_v^{-(s-w)})}$$

2. $T_v(\omega_v, w, \Psi_v)$ is a rational function in q^{-s} and q^{-w} which is analytic in the region $\Re(\omega_v) - \Re(w) > 0$ and $\Re(\omega_v) > -1$.

Proof : The proof is a special case of the proof of Propositions 2.8 and 2.9 of [28]. ■

Suppose $\Psi = \prod_{v\in M(K)} \Psi_v$. Write $\omega = \tilde{\omega}|\cdot|_{\mathbf{A}}^{s} = \prod_{v\in M(K)} \tilde{\omega}_v|\cdot|_v^{s}$, where $\tilde{\omega} = \prod_{v\in M(K)} \tilde{\omega}_v$ is a character on $\mathbf{A}^1/K^*$, and hence $\tilde{\omega}_v$ is trivial on O_v for all but finitely many v. Let P be a finite set of places of K such that if $v \notin P$, then $\tilde{\omega}_v$ is trivial on O_v and Ψ_v is the characteristic function of O_v^2. Write

$$T_P(\omega, w, \Psi) = \prod_{v\in P} T_v(\omega_v, w, \Psi_v).$$

Then Lemma 4.6(1) gives

$$T(\omega, w, \Psi) = q^{1-g}(\frac{q-1}{h_{0,K}})T_P(\omega, w, \Psi)\frac{\zeta_{K,P}(s+1)\ \zeta_{K,P}(s-w)}{\zeta_{K,P}(s-w+1)}$$

where $\zeta_{K,P}(z) = \prod_{v\notin P}(1-q_v^{-z})^{-1}$ is the truncated Dedekind zeta function.

The above observation and Lemma 4.6 now imply the following lemma.

Lemma 4.7 *1. $T(\omega, w, \Psi)$ is a rational function in q^{-s} and q^{-w} which is analytic in the region $\Re(\omega) - \Re(w) > 1$ and $\Re(\omega) > 0$.*
2. The derivative of $T(\omega, w, \Psi)$ at $w = 0$ is given by

$$\begin{aligned}\tfrac{d}{dw}T(\omega, w, \Psi)|_{w=0} &= q^{1-g}(\tfrac{q-1}{h_{0,K}})\tfrac{d}{dw}T_P(\omega, w, \Psi)|_{w=0}\zeta_{K,P}(s)\\ &+q^{1-g}(\tfrac{q-1}{h_{0,K}})T_P(\omega, 0, \Psi)[\zeta'_{K,P}(s) - \tfrac{\zeta_{K,P}(s)\zeta'_{K,P}(s+1)}{\zeta_{K,P}(s+1)}].\end{aligned}$$

This lemma, in turn, implies the following lemma.

Lemma 4.8 *$T^+(\omega, w, \Psi)$ is an entire function of ω and w.*

For $f \in \mathcal{S}(V_{\mathbf{A}})$, define $T_2(f)(t,u) = f(0,t,u)$. Then $T_2(f) \in \mathcal{S}(\mathbf{A}^2)$. Define

$$\Sigma_2(\omega, w, f) = T(\omega, w, T_2(M_\omega(f)))$$

$$\Sigma_2^+(\omega, w, f) = T^+(\omega, w, T_2(M_\omega(f)))$$

$$\Sigma_2^-(\omega, w, f) = T^-(\omega, w, T_2(M_\omega(f))).$$

We now state and prove the main result of this section.

Proposition 4.6

$$I^2(\omega, f; w, \phi) \sim \frac{\zeta_K(w)}{\zeta_K(w+1)} q^{1-\mathbf{g}} \phi(w) \left[\frac{q^{-(\frac{w}{2}-\frac{1}{2})}}{1-q^{-(\frac{w}{2}-\frac{1}{2})}} + \frac{1}{2}\right] \left(\Sigma_2^-(\omega^3\omega_{-1}, \tfrac{w}{2} - \tfrac{1}{2}, f) - \Sigma_2^+(\omega^{-3}\omega_2, \tfrac{w}{2} - \tfrac{1}{2}, \hat{f})\right)$$

Proof : We first consider the integral

$$\int_{H_{\mathbf{A}}/H_K} \omega(det(h))\lambda_-(h) \sum_{x \in S_K^2} f(h \cdot x)\mathcal{E}(w, \phi, h)\, dh \tag{4.26}$$

Since $S_K^2 = \bigcup_{\gamma \in H_K/\widehat{T}_K} \bigcup_{a \in K^\bullet} \gamma \cdot (0, a, 0)$, then reasoning as in I^0 and I^1, integral (4.26) can be rewritten as

$$\int_{H_{\mathbf{A}}/\widehat{T}_K} \omega(det(h))\lambda_-(h) \sum_{a \in K^\bullet} f(h \cdot (0, a, 0))\mathcal{E}(w, \phi, h)\, dh$$

or, equivalently, with $h = \varrho(\kappa d(t, 1)n(u)a(\tau))$,

$$\int_{\mathcal{K}} \int_{\mathbf{A}^\bullet/K^\bullet} \int_{\mathbf{A}^\bullet/K^\bullet, |\tau|_{\mathbf{A}} \leq^\bullet \alpha(u)} \int_{\mathbf{A}} \omega(det(h))\lambda_-(h) \sum_{a \in K^\bullet} f(h \cdot (0, a, 0))\mathcal{E}(w, \phi, h)\, dud^*\tau d^*t d\kappa \tag{4.27}$$

Now for the above given h, $det(h) = (det(\kappa)\ t\tau)^3$, $f(h \cdot (0, a, 0)) = f(\kappa \cdot (0, at\tau, at\tau u))$, and

$$\int_{\mathcal{K}} \omega^3(det(\kappa)) f(\kappa \cdot (0, at\tau, at\tau u))\, d\kappa = M_{\omega^3}(f)(0, at\tau, at\tau u).$$

So integral (4.27) becomes

$$\underbrace{\int_{\mathbf{A}^\bullet/K^\bullet} \int_{\mathbf{A}^\bullet/K^\bullet}}_{|t\tau|_{\mathbf{A}} \leq^\bullet 1, |\tau|_{\mathbf{A}} \leq^\bullet \alpha(u)} \int_{\mathbf{A}} \omega^3(t\tau) \sum_{a \in K^\bullet} M_{\omega^3}(f)(0, at\tau, at\tau u)\mathcal{E}(w, \phi, h)\, dud^*\tau d^*t \tag{4.28}$$

Since

$$\mathcal{E}(w, \phi, h) = \frac{1}{2\pi i} \int_{1 < \Re(z) = x_0 < \Re(w)} \frac{|\tau|_{\mathbf{A}}^{\frac{z}{2}-\frac{1}{2}} \frac{\zeta_K(z)}{\zeta_K(z+1)}}{w - z} q^{1-\mathbf{g}} \phi(z)\, dz + \mathcal{E}''(w, \phi, h),$$

then by integral (4.28), we are led to two integrals:

$$\underbrace{\int_{\mathbf{A}^\bullet/K^\bullet}\int_{\mathbf{A}^\bullet/K^\bullet}}_{|t\tau|_{\mathbf{A}}\leq^\bullet 1, |\tau|_{\mathbf{A}}\leq^\bullet \alpha(u)} \int_{\mathbf{A}} \omega^3(t\tau) \sum_{a\in K^\bullet} M_{\omega^3}(f)(0, at\tau, at\tau u) \frac{1}{2\pi i} \int_{1<\Re(z)=x_0<\Re(w)} \frac{|\tau|_{\mathbf{A}}^{\frac{z}{2}-\frac{1}{2}} \frac{\zeta_K(z)}{\zeta_K(z+1)}}{w-z} q^{1-g}\phi(z)\, dz du d^*\tau d^*t \tag{4.29}$$

and

$$\underbrace{\int_{\mathbf{A}^\bullet/K^\bullet}\int_{\mathbf{A}^\bullet/K^\bullet}}_{|t\tau|_{\mathbf{A}}\leq^\bullet 1, |\tau|_{\mathbf{A}}\leq^\bullet \alpha(u)} \int_{\mathbf{A}} \omega^3(t\tau) \sum_{a\in K^\bullet} M_{\omega^3}(f)(0, at\tau, at\tau u) \mathcal{E}''(w, \phi, h)\, du d^*\tau d^*t \tag{4.30}$$

We first consider integral (4.30). Making the change of variables $t \longrightarrow \frac{t}{\tau}$ and $u \longrightarrow \frac{u}{t\tau}$, then $d^*t \longrightarrow d^*t$ and $du \longrightarrow |t\tau|_{\mathbf{A}}^{-1} du$. So integral (4.30) becomes

$$\int_{\mathbf{A}^\bullet/K^\bullet, |t|_{\mathbf{A}}\leq^\bullet 1} \int_{\mathbf{A}^\bullet/K^\bullet, |\tau|_{\mathbf{A}}\leq^\bullet \alpha(ut^{-1})} \int_{\mathbf{A}} \omega^3\omega_{-1}(t) \sum_{a\in K^\bullet} M_{\omega^3}(f)(0, at, au) \mathcal{E}''(w, \phi, h) du d^*\tau d^*t$$

By absorbing the sum with the integral over t, the last integral becomes

$$\int_{\mathbf{A}^\bullet, |t|_{\mathbf{A}}\leq^\bullet 1} \int_{\mathbf{A}^\bullet/K^\bullet, |\tau|_{\mathbf{A}}\leq^\bullet \alpha(ut^{-1})} \int_{\mathbf{A}} \omega^3\omega_{-1}(t) M_{\omega^3}(f)(0, t, u) \mathcal{E}''(w, \phi, h)\, du d^*\tau d^*t$$

By Lemma 4.2(5), $\mathcal{E}''(w, \phi, h)$ is a holomorphic function of w in the region $\Re(w) > -\frac{1}{2}$ and it is of order $O(|\tau|_{\mathbf{A}}^{3/2})$ in the region $\Re(w) > 0$. So in this region, $\Re(w) > 0$, the last integral is dominated in absolute value by

$$\int_{\mathbf{A}^\bullet, |t|_{\mathbf{A}}\leq^\bullet 1} \int_{\mathbf{A}^\bullet/K^\bullet, |\tau|_{\mathbf{A}}\leq^\bullet \alpha(ut^{-1})} \int_{\mathbf{A}} \omega_{3\Re(s)-1}(t) |M_{\omega^3}(f)(0, t, u)| |\tau|^{3/2}\, du d^*\tau d^*t$$

$$= [\frac{q^{-l}}{1-q^{-l}} + \frac{1}{2}] \int_{\mathbf{A}^\bullet, |t|_{\mathbf{A}}\leq^\bullet 1} \int_{\mathbf{A}} \omega_{3\Re(s)-1}(t) |M_{\omega^3}(f)(0, t, u)|\ \alpha(ut^{-1})^{3/2}\, du d^*t$$

But the last double integral is $T^-(\omega_{3\Re(s)-1}, \frac{3}{2}, |T_2(M_{\omega^3}(f))|)$ which converges provided $\Re(s) > \frac{7}{6}$ by Lemma 4.7. So the integral (4.30) converges absolutely and locally uniformly in w in the region $\Re(w) > 0$, and so it is analytic there and hence its contribution to $I^2(\omega, f; w, \phi)$ can be disregarded.

Next, we consider integral (4.29). As we did with integral (4.30), by making the change of variables $t \longrightarrow \frac{t}{\tau}$ and $u \longrightarrow \frac{u}{t\tau}$, and absorbing the sum with the integral over t, integral (4.29) reduces to

$$\int_{\mathbf{A}^\bullet,|t|_{\mathbf{A}}\leq^\bullet 1}\int_{\mathbf{A}}\omega^3\omega_{-1}(t)M_{\omega^3}(f)(0,t,u)\Big(\frac{1}{2\pi i}\int_{\mathbf{A}^\bullet/K^\bullet,|\tau|_{\mathbf{A}}\leq^\bullet\alpha(ut^{-1})}\int_{1<\Re(z)=x_0<\Re(w)} \frac{|\tau|_{\mathbf{A}}^{\frac{z}{2}-\frac{1}{2}}\frac{\zeta_K(z)}{\zeta_K(z+1)}}{w-z}q^{1-\mathbf{g}}\phi(z)\,dz d^*\tau\Big)dud^*t \tag{4.31}$$

By shifting the complex integral to $\Re(z)=x_1=2l+1>2$, $x_1>x_0$, $x_1>\Re(w)$, this complex integral equals

$$-|\tau|_{\mathbf{A}}^{\frac{w}{2}-\frac{1}{2}}\frac{\zeta_K(w)}{\zeta_K(w+1)}q^{1-\mathbf{g}}\phi(w)+\frac{1}{2\pi i}\int_{\Re(z)=x_1>\Re(w)}\frac{|\tau|_{\mathbf{A}}^{\frac{z}{2}-\frac{1}{2}}\frac{\zeta_K(z)}{\zeta_K(z+1)}}{w-z}q^{1-\mathbf{g}}\phi(z)\,dz$$

As in I^0 and I^1, the integral over $\Re(z)=x_1$ is an entire function of w and it is of order $O(|\tau|_{\mathbf{A}}^{l})$ for any $l>\frac{1}{2}$. So its contribution to integral (4.31) is an entire function of w. Thus integral (4.31) is equivalent to

$$-q^{1-\mathbf{g}}\frac{\zeta_K(w)}{\zeta_K(w+1)}\phi(w)\int_{\mathbf{A}^\bullet,|t|_{\mathbf{A}}\leq^\bullet 1}\int_{\mathbf{A}}\omega^3\omega_{-1}(t)M_{\omega^3}(f)(0,t,u)\Big(\int_{|\tau|_{\mathbf{A}}\leq^\bullet\alpha(ut^{-1})}|\tau|_{\mathbf{A}}^{\frac{w}{2}-\frac{1}{2}}d^*\tau\Big)dud^*t$$

$$=-q^{1-\mathbf{g}}\frac{\zeta_K(w)}{\zeta_K(w+1)}\phi(w)\Big[\frac{q^{-(\frac{w}{2}-\frac{1}{2})}}{1-q^{-(\frac{w}{2}-\frac{1}{2})}}+\frac{1}{2}\Big]\int_{\mathbf{A}^\bullet,|t|_{\mathbf{A}}\leq^\bullet 1}\int_{\mathbf{A}}\omega^3\omega_{-1}(t)M_{\omega^3}(f)(0,t,u)\,\alpha(ut^{-1})^{\frac{w}{2}-\frac{1}{2}}dud^*t$$

$$=-q^{1-\mathbf{g}}\frac{\zeta_K(w)}{\zeta_K(w+1)}\phi(w)\Big[\frac{q^{-(\frac{w}{2}-\frac{1}{2})}}{1-q^{-(\frac{w}{2}-\frac{1}{2})}}+\frac{1}{2}\Big]\Sigma_2^-(\omega^3\omega_{-1},\tfrac{w}{2}-\tfrac{1}{2},f)$$

Similarly

$$\int_{H_{\mathbf{A}}/H_K}\omega(det(h))\omega_{-1}(det(h))\lambda_-(h)\sum_{x\in S_K^2}\hat{f}(h'\cdot x)\mathcal{E}(w,\phi,h)\,dh \sim -q^{1-\mathbf{g}}\frac{\zeta_K(w)}{\zeta_K(w+1)}\phi(w)\Big[\frac{q^{-(\frac{w}{2}-\frac{1}{2})}}{1-q^{-(\frac{w}{2}-\frac{1}{2})}}+\frac{1}{2}\Big]\Sigma_2^+(\omega^{-3}\omega_2,\tfrac{w}{2}-\tfrac{1}{2},\hat{f})$$

Basically, we make the change of variable $h\longrightarrow h'$, then $\lambda_-(h)\longrightarrow\lambda_+(h')$ and proceed exactly as above, and make use of Lemma 4.8. This completes the proof of the proposition. ■

Because of $\zeta_K(w)$ and $[\frac{q^{-(\frac{w}{2}-\frac{1}{2})}}{1-q^{-(\frac{w}{2}-\frac{1}{2})}}+\frac{1}{2}]$, $I^2(\omega,f;w,\phi)$ has a double pole at $w=1$. As we did with $I^1(\omega,f;w,\phi)$, we calculate some Laurent expansions.

Let the first two terms of the Laurent expansion of $\Sigma_2(*, w, *)$ at $w = 0$ be $\Sigma_2(*, 0, *) + \frac{d}{dw}\Sigma_2(*, w, *)|_{w=0}w$. Then we have

$$\Sigma_2^-(\omega^3\omega_{-1}, \frac{w}{2}-\frac{1}{2}, f) = \Sigma_2^-(\omega^3\omega_{-1}, 0, f)+\frac{d}{dw}\Sigma_2^-(\omega^3\omega_{-1}, w, f)|_{w=0}(\frac{w-1}{2})+\cdots,$$

$$\Sigma_2^+(\omega^{-3}\omega_2, \frac{w}{2}-\frac{1}{2}, \hat{f}) = \Sigma_2^+(\omega^{-3}\omega_2, 0, \hat{f})+\frac{d}{dw}\Sigma_2^+(\omega^{-3}\omega_2, w, \hat{f})|_{w=0}(\frac{w-1}{2})+\cdots.$$

We also have

$$\frac{q^{-(\frac{w}{2}-\frac{1}{2})}}{1-q^{-(\frac{w}{2}-\frac{1}{2})}}+\frac{1}{2}=\frac{\frac{2}{\log q}}{w-1}+0+\cdots.$$

So the first two terms of the Laurent expansion at $w = 1$ of $(\frac{q^{-(\frac{w}{2}-\frac{1}{2})}}{1-q^{-(\frac{w}{2}-\frac{1}{2})}} + \frac{1}{2})(\Sigma_2^-(\omega^3\omega_{-1}, \frac{w}{2} - \frac{1}{2}, f) - \Sigma_2^+(\omega^{-3}\omega_2, \frac{w}{2} - \frac{1}{2}, \hat{f}))$ are

$$\frac{2}{\log q}\frac{(\Sigma_2^-(\omega^3\omega_{-1},0,f)-\Sigma_2^+(\omega^{-3}\omega_2,0,\hat{f}))}{w-1}+$$
$$\frac{1}{\log q}\frac{d}{dw}(\Sigma_2^-(\omega^3\omega_{-1}, w, f) - \Sigma_2^+(\omega^{-3}\omega_2, w, \hat{f}))|_{w=0}.$$

Thus we get

$$I^2(\omega, f; w, \phi) \sim q^{1-g}\frac{\zeta_K(w)}{\zeta_K(w+1)}\phi(w)\frac{2}{\log q}\frac{(\Sigma_2^-(\omega^3\omega_{-1},0,f)-\Sigma_2^+(\omega^{-3}\omega_2,0,\hat{f}))}{w-1} + q^{1-g}\frac{\zeta_K(w)}{\zeta_K(w+1)}$$
$$\phi(w)[\frac{1}{\log q}\frac{d}{dw}(\Sigma_2^-(\omega^3\omega_{-1}, w, f) - \Sigma_2^+(\omega^{-3}\omega_2, w, \hat{f}))|_{w=0}].$$

4.7 Cancellation of the Double Pole

We pointed out after the proofs of Proposition 4.5 and Proposition 4.6 that both $I^1(\omega, f; w, \phi)$ and $I^2(\omega, f; w, \phi)$ have a double pole at $w = 1$. In this section we will show how the double pole of $I^2(\omega, f; w, \phi)$ cancels the double pole of $I^1(\omega, f; w, \phi)$.

For $f \in \mathcal{S}(V_{\mathbf{A}})$, define

$$T_3(f)(t) = \int_{\mathbf{A}} f(0, t, u)\, du.$$

We first prove the following lemma.

Lemma 4.9 *For $f \in \mathcal{S}(V_{\mathbf{A}})$, $\widehat{T_3(f)}(t) = T_3(\hat{f})(-2t)$.*

Proof : First we have

$$\begin{aligned}\widehat{T_3(f)}(t) &= \int_{\mathbf{A}} T_3(f)(x)\psi(xt)\,dx \\ &= \int_{\mathbf{A}}\int_{\mathbf{A}} f(0,x,u)\psi(xt)\,dudx\end{aligned}$$

Set $F_t(y) = \int_{\mathbf{A}}\int_{\mathbf{A}} f(y,x,u)\psi(xt)\,dudx$. Then $\widehat{T_3(f)}(t) = F_t(0)$. Since

$$\begin{aligned}\widehat{F}_t(v) &= \int_{\mathbf{A}} F_t(y)\psi(vy)\,dy \\ &= \int_{\mathbf{A}}\int_{\mathbf{A}}\int_{\mathbf{A}} f(y,x,u)\psi(xt)\psi(vy)\,dudxdy \\ &= \int_{V_{\mathbf{A}}} f(y,x,u)\psi([(0,-2t,v),(y,x,u)])\,dudxdy \\ &= \hat{f}(0,-2t,v),\end{aligned}$$

then by the Fourier inversion formula, we have

$$F_t(y) = \int_{\mathbf{A}} \widehat{F}_t(v)\psi(vy)\,dv$$

and hence

$$F_t(0) = \int_{\mathbf{A}} \widehat{F}_t(v)\,dv = \int_{\mathbf{A}} \hat{f}(0,-2t,v)\,dv = T_3(\hat{f})(-2t).$$

This completes the proof of the lemma. ■

Now it is easy to see that

$$\Sigma_2^-(\omega^3\omega_{-1},0,f) = \zeta^-(\omega^3\omega_{-1},T_3(M_{\omega^3}(f))),$$

$$\Sigma_2^+(\omega^{-3}\omega_2,0,\hat{f}) = \zeta^+(\omega^{-3}\omega_2,T_3(M_{\bar{\omega}^3}(\hat{f}))),$$

where ζ is Tate's zeta function. Thus the double pole term of $I^2(\omega,f;w,\phi)$ equals

$$q^{1-g}\frac{\zeta_K(w)}{\zeta_K(w+1)}\phi(w)\frac{2}{\log q}\,\frac{\zeta^-(\omega^3\omega_{-1},T_3(M_{\omega^3}(f))) - \zeta^+(\omega^{-3}\omega_2,T_3(M_{\bar{\omega}^3}(\hat{f})))}{w-1} \tag{4.32}$$

Applying the Poisson summation formula (as in Tate's thesis) to ζ^- and using Lemma 4.9, we can show

$$\zeta^-(\omega^3\omega_{-1},T_3(M_{\omega^3}(f))) - \zeta^+(\omega^{-3}\omega_2,T_3(M_{\bar{\omega}^3}(\hat{f})))$$

$$= \delta(\omega^3)[T_3(M(\hat{f})(0)(\frac{1}{1-q^{-(3s-2)}} - \frac{1}{2}) - T_3(M(f))(0)(\frac{1}{1-q^{-(3s-1)}} - \frac{1}{2})].$$

By the note at the end of Section 4.5, we have

$$T_3(M(\hat{f}))(0) = \int_{\mathbf{A}} M(\hat{f})(0,0,u)\,du = \log q\ R(\hat{f}),$$

$$T_3(M(f))(0) = \int_{\mathbf{A}} M(f)(0,0,u)\,du = \log q\ R(f).$$

So we get

$$\begin{aligned} \zeta^- - \zeta^+ &= \delta(\omega^3)\log q[R(\hat{f})(\tfrac{1}{1-q^{-(3s-2)}} - \tfrac{1}{2}) - R(f)(\tfrac{1}{1-q^{-(3s-1)}} - \tfrac{1}{2})] \\ &= \delta(\omega^3)\log q[-A'R(\hat{f}) - AR(f)] \end{aligned} \tag{4.33}$$

where A and A' are as defined at the end of Section 4.5. Now plugging 4.33 in 4.32 and comparing with the double pole term of $I^1(\omega, f; w, \phi)$, we see at once that these terms cancel out.

4.8 The Functional Equation

Now the work in the previous four sections gives the following proposition.

Proposition 4.7

$$\begin{aligned} I(\omega, f; w, \phi) \sim\ & \delta(\omega^3)\phi(w)\frac{q^{-(\frac{w}{2}-\frac{1}{2})}}{q^{-(\frac{w}{2}-\frac{1}{2})}}[\tfrac{\hat{f}(0)}{1-q^{-(3s-3)}} - \tfrac{f(0)}{1-q^{-3s}} + \tfrac{f(0)-\hat{f}(0)}{2}] \\ & + \tfrac{\zeta_K(w)}{\zeta_K(w+1)}q^{1-\mathbf{g}}\phi(w) \\ & + O(\omega)\tfrac{\zeta_K(w)}{\zeta_K(w+1)}q^{1-\mathbf{g}}\phi(w)[A\Gamma(f) + 2BR(f) + A'\Gamma(\hat{f}) + 2B'R(\hat{f})] \\ & + \tfrac{\zeta_K(w)}{\zeta_K(w+1)}q^{1-\mathbf{g}}\phi(w) \\ & [\tfrac{1}{\log q}\tfrac{d}{dw}(\Sigma_2^-(\omega^3\omega_{-1}, w, f) - \Sigma_2^+(\omega^{-3}\omega_2, w, \hat{f}))|_{w=0}]. \end{aligned}$$

As we pointed out in Section 4.3, we have, by Lemma 4.2(4),

$$\lim_{w\to 1}(1 - q^{1-w})I(\omega, f; w, \phi) = \phi(1)q^{1-\mathbf{g}}\frac{Res_q\zeta_K}{\zeta_K(2)}I(\omega, f).$$

So equipped with this observation, Proposition 4.1, and Proposition 4.7, we get the main theorem of this chapter.

Theorem 4.1 *We have*

1. $Z(\omega, f) = Z_+(\omega, f) + Z_+(\omega_1\omega^{-1}, \hat{f}) + I(\omega, f)$, *where* $Z_+(\omega, f)$ *and* $Z_+(\omega_1\omega^{-1}, \hat{f})$ *are entire functions of* ω, *and for* $\omega = \tilde{\omega}\omega_s$

$$\begin{aligned} I(\omega, f) \; &= \tfrac{2\delta(\omega^3)\zeta_K(2)}{q^{1-g} Res_q \zeta_K}\left[\tfrac{\hat{f}(0)}{1-q^{-(3s-3)}} - \tfrac{f(0)}{1-q^{-3s}} + \tfrac{f(0)-\hat{f}(0)}{2}\right] \\ &+\delta(\omega^3)[A\Gamma(f) + 2BR(f) + A'\Gamma(\hat{f}) + 2B'R(\hat{f})] \\ &+\tfrac{1}{\log q}\tfrac{d}{dw}(\Sigma_2^-(\omega^3\omega_{-1}, w, f) - \Sigma_2^+(\omega^{-3}\omega_2, w, \hat{f}))|_{w=0}, \end{aligned}$$

where

$$\begin{aligned} A &= \tfrac{1}{1-q^{-(3s-1)}} - \tfrac{1}{2}, \quad B = \tfrac{(\frac{1}{2}\log q)q^{-(3s-1)}}{[1-q^{-(3s-1)}]^2} \\ A' &= \tfrac{1}{2} - \tfrac{1}{1-q^{-(3s-2)}}, \quad B' = \tfrac{(\frac{1}{2}\log q)q^{-(3s-2)}}{[1-q^{-(3s-2)}]^2} \end{aligned}$$

2. Let $\mathcal{Z}(\omega, f) = Z(\omega, f) - \frac{1}{\log q}\frac{d}{dw}\Sigma_2(\omega^3\omega_{-1}, w, f)|_{w=0}$. *Then we have*

$$\mathcal{Z}(\omega, f) = \mathcal{Z}(\omega_1\omega^{-1}, \hat{f}).$$

We now describe the poles of $Z(\omega, f)$. By Lemma 4.7(2), Lemma 4.8, and Theorem 4.1, we conclude that the poles of $Z(\omega, f)$ occur at $s = 0 + \frac{2\pi i n}{3\log q}$, $1 + \frac{2\pi i n}{3\log q}$, $\frac{1}{3} + \frac{2\pi i n}{3\log q}$, $\frac{2}{3} + \frac{2\pi i n}{3\log q}$ $(n \in \mathbb{Z})$ and at the poles of $\Sigma_2^-(\omega^3\omega_{-1}, 0, f)$. The exact poles of $\Sigma_2^-(\omega^3\omega_{-1}, 0, f)$ depend on the choice of f. For example, if we choose f so that its support lies in $V''_{\mathbb{A}}$, then all the f−terms in $I(\omega, f)$ will disappear. Hence the only poles of $Z(\omega, f)$ in this case are simple poles at $s = 1 + \frac{2\pi i n}{3\log q}$ and double poles at $s = \frac{2}{3} + \frac{2\pi i n}{3\log q}$.

CHAPTER 5

SOME LOCAL ANALYSIS

5.1 Decomposition of $Z(\omega, f)$ and the Orbital Zeta Functions

In this chapter we conduct some local analysis that will be crucial to obtain the mean value theorem we are after. The local analysis amounts to studying certain integrals, which we will call orbital zeta functions, that appear in a natural way as local factors of the adelic zeta function $Z(\omega, f)$. In this section we introduce these orbital zeta functions.

By the absolute convergence of $Z(\omega, f)$ in the region $\Re(\omega) > 1$, we may interchange the sum and the integral and rearrange the sum orbitwise to get

$$\begin{aligned} Z(\omega, f) &= \int_{H_{\mathbf{A}}/H_K} \omega(det(h)) \sum_{x \in V''_K} f(h \cdot x)\, dh \\ &= \sum_{x \in H_K \backslash V''_K} \sum_{\gamma \in H_K/(H_x)_K} \int_{H_{\mathbf{A}}/H_K} \omega(det(h)) f(h\gamma \cdot x)\, dh \\ &= \sum_{x \in H_K \backslash V''_K} \int_{H_{\mathbf{A}}/(H_x)_K} \omega(det(h)) f(h \cdot x)\, dh \\ &= \frac{1}{2} \sum_{x \in H_K \backslash V''_K} \int_{H_{\mathbf{A}}/(H_x^\circ)_K} \omega(det(h)) f(h \cdot x)\, dh, \end{aligned} \tag{5.1}$$

where the sum is over a complete set of representatives of all H_K-orbits in V''_K. Also note that the last equality follows from the fact that H_x° is of index 2 in H_x, by Proposition 2.2, and the invariance of the measure dh. Since $H_{\mathbf{A}}/(H_x^\circ)_K \cong H_{\mathbf{A}}/(H_x^\circ)_{\mathbf{A}} \times (H_x^\circ)_{\mathbf{A}}/(H_x^\circ)_K$, then every $h \in H_{\mathbf{A}}/(H_x^\circ)_K$ can be written as a product $h = h'h''$, where $h' \in H_{\mathbf{A}}/(H_x^\circ)_{\mathbf{A}}$ and $h'' \in (H_x^\circ)_{\mathbf{A}}/(H_x^\circ)_K$.

Then we get the following:

$$\int_{H_{\mathbf{A}}/(H_x^\circ)_K} \omega(det(h)) f(h\cdot x)\, dh = c_x\ \mu(x) \int_{H_{\mathbf{A}}/(H_x^\circ)_{\mathbf{A}}} \omega(det(h')) f(h'\cdot x)\, d_x'h', \tag{5.2}$$

where

$$\mu(x) = \int_{(H_x^\circ)_{\mathbf{A}}/(H_x^\circ)_K} d_x''h'', \tag{5.3}$$

$d_x'h'$ and $d_x''h''$ are measures on $H_{\mathbf{A}}/(H_x^\circ)_{\mathbf{A}}$ and $(H_x^\circ)_{\mathbf{A}}/(H_x^\circ)_K$ respectively, and c_x is a constant given by $dh = c_x d_x'h' d_x''h''$.

We also note that for $f = \prod_{v\in M(K)} f_v$, we have the decomposition

$$\int_{H_{\mathbf{A}}/(H_x^\circ)_{\mathbf{A}}} \omega(det(h')) f(h'\cdot x)\, d_x'h' = \prod_{v\in M(K)} \int_{H_{K_v}/(H_x^\circ)_{K_v}} \omega_v(det(h_v')) f_v(h_v'\cdot x)\, d_x'h_v', \tag{5.4}$$

where ω_v is the restriction of ω to K_v. So combining all of the above equations yields the decomposition

$$Z(\omega, f) = \frac{1}{2} \sum_{x\in H_K\backslash V_K''} c_x\mu(x) \prod_{v\in M(K)} \int_{H_{K_v}/(H_x^\circ)_{K_v}} \omega_v(det(h_v')) f_v(h_v'\cdot x)\, d_x'h_v'. \tag{5.5}$$

It is worth observing here that since H_K-orbits in V_K'' are in one-to-one correspondence with quadratic extensions of K, by Corollary 2.1, then the sum in (5.5) is in fact a sum over the quadratic extensions of K.

The objective of this chapter is to study the integrals in (5.5). We mention one observation about these integrals. Since H_x° is of index 2 in H_x, then the map $H_{K_v}/(H_x^\circ)_{K_v} \longrightarrow H_{K_v}\cdot x$ given by $h_v' \longrightarrow h_v'\cdot x$ defines a double cover (i,e. a 2-to-1 and onto continuous map) of the open orbit $H_{K_v}\cdot x \subset V_{K_v}$. Define the left H_{K_v}-invariant measure $\frac{dx_v}{|P(x_v)|_v^{\frac{3}{2}}}$ on V_{K_v}:

$$\frac{dh_v\cdot x_v}{|P(h_v\cdot x_v)|_v^{\frac{3}{2}}} = \frac{|det(h_v)|_v dx_v}{|(det(h_v))^{\frac{2}{3}} P(x_v)|_v^{\frac{3}{2}}} = \frac{dx_v}{|P(x_v)|_v^{\frac{3}{2}}}.$$

Then the double cover map induces an invariant measure $d_x'h_v'$ on $H_{K_v}/(H_x^\circ)_{K_v}$

which we normalize so that the integral in (5.5) equals

$$\begin{aligned}
&\int_{H_{K_v}/(H_x^\circ)_{K_v}} \omega_v(det(h_v'))f_v(h_v'\cdot x)\,d_x'h_v' \\
&= \int_{H_{K_v}\cdot x} \omega_v(\frac{P(h_v'\cdot x)}{P(x)})^{3/2} f_v(h_v'\cdot x)\,\frac{dh_v'\cdot x}{|P(h_v'\cdot x)|_v^{\frac{3}{2}}} \\
&= \frac{1}{\omega_v(P(x))^{3/2}} \int_{H_{K_v}\cdot x} \omega_v(P(x_v))^{3/2} f_v(x_v)\,\frac{dx_v}{|P(x_v)|_v^{3/2}}.
\end{aligned} \tag{5.6}$$

Before we continue our analysis, we find it convenient at this point to introduce some definitions and notations that will simplify our exposition.

Throughout the remaining of this chapter, we will let $\mathbf{K} = K_v$ for some $v \in M(K)$. We will denote the absolute value $|\cdot|_v$ on $K_v = \mathbf{K}$ simply by $|\cdot|$. The cardinality of the residue field of $\mathbf{K}$ will be denoted by q. Let dx be the additive Haar measure on $\mathbf{K}$ normalized so that $\int_{\mathbf{O}} dx = 1$, where $\mathbf{O}$ is the ring of integers in $\mathbf{K}$. Let d^*x be the multiplicative Haar measure on $\mathbf{K}^*$ normalized so that $\int_{\mathbf{O}^*} d^*x = 1$.

Let $x = (x_1, x_2, x_3) \in V_{\mathbf{K}}$. Set $dx = dx_1 dx_2 dx_3$. Then $\frac{dx}{|P(x)|^{3/2}}$ is a left $H_{\mathbf{K}}$-invariant measure on $V_{\mathbf{K}}'$. For $\mathbf{x} \in V_{\mathbf{K}}'$, denote the orbit of $\mathbf{x}$ by $V_{\mathbf{x}} = H_{\mathbf{K}}\cdot\mathbf{x}$. For a quasicharacter ω on $\mathbf{K}^*$ satisfying $\omega(-1) = 1$ and for $f \in \mathcal{S}(V_{\mathbf{K}})$, define two integrals $Z_{\mathbf{x}}(\omega, f)$ and $\mathcal{Z}_{\mathbf{x}}(\omega, f)$ by

$$Z_{\mathbf{x}}(\omega, f) = \int_{H_{\mathbf{K}}/(H_{\mathbf{x}}^\circ)_{\mathbf{K}}} \omega(det(h'))f(h'\cdot\mathbf{x})\,d_{\mathbf{x}}'h', \tag{5.7}$$

and

$$\mathcal{Z}_{\mathbf{x}}(\omega, f) = \int_{V_{\mathbf{x}}} \omega(P(x))^{3/2} f(x)\,\frac{dx}{|P(x)|^{3/2}}. \tag{5.8}$$

We call these integrals the *orbital zeta functions* associated with the space of binary quadratic forms. By (5.6), we note that $Z_{\mathbf{x}}(\omega, f)$ is a constant multiple of $\mathcal{Z}_{\mathbf{x}}(\omega, f)$. This constant depends on $\mathbf{x}$, ω, and the normalization of $d_{\mathbf{x}}'h'$. Note that the condition $\omega(-1) = 1$ was stipulated since without it $Z_{\mathbf{x}}(\omega, f)$ reduces to zero as every nonsingular form has a stabilizer of determinant -1.

Proposition 5.1 *$Z_{\mathbf{x}}(\omega, f)$ and $\mathcal{Z}_{\mathbf{x}}(\omega, f)$ converge absolutely and locally uniformly for $\Re(\omega) > 1$, and hence they represent analytic functions of ω in that* *region. Furthermore, if f has compact support contained in $V'_{\mathbf{K}}$, then $Z_{\mathbf{x}}(\omega, f)$ and $\mathcal{Z}_{\mathbf{x}}(\omega, f)$ become entire functions of ω.*

Proof : Since $Z_{\mathbf{x}}(\omega, f)$ and $\mathcal{Z}_{\mathbf{x}}(\omega, f)$ are constant multiples of each other, it is enough to consider the convergence of $\mathcal{Z}_{\mathbf{x}}(\omega, f)$. Since f is locally constant function with compact support U, then

$$|\mathcal{Z}_{\mathbf{x}}(\omega, f)| \leq \int_U |P(x)|^{\frac{3}{2}(\sigma-1)}\, dx.$$

For $\sigma > 1$, the above integral is finite by the continuity of $|P(x)|$ on the compact set U and the finiteness of the Haar measure dx on compact sets. If $U \subseteq V'_{\mathbf{K}}$, then $|P(x)|$ has a nonzero lower bound on U and hence the above integral also conveges for $\sigma \leq 1$. ■

The computation of the next section will show that the abscissa of absolute convergence is $\frac{1}{3}$.

We point out here that the name "*orbital zeta function*" is motivated by the fact that the definition of $\mathcal{Z}_{\mathbf{x}}(\omega, f)$ depends on the $H_{\mathbf{K}}$-orbit of $\mathbf{x}$ and not on $\mathbf{x}$ itself. We also remark here that the number of distinct $\mathcal{Z}_{\mathbf{x}}(\omega, f)$ is finite for any local field $\mathbf{K}$. This follows from the fact that the $H_{\mathbf{K}}$-orbits in $V'_{\mathbf{K}}$ are in one-to-one correspondence with extensions of $\mathbf{K}$ of degree less than or equal to 2, by Corollary 2.1, and the number of such extensions of a local field $\mathbf{K}$ is finite, see [12]. More information about similar orbital zeta functions can be found in [3].

5.2 Computing $Z_{\mathbf{x}}(\omega, f)$

In this section we start evaluating $Z_{\mathbf{x}}(\omega, f)$. The credit for computing $Z_{\mathbf{x}}(\omega, f)$ goes to Datskovsky [2]. For the sake of completeness, we choose to describe his method of computing $Z_{\mathbf{x}}(\omega, f)$.

Recall that for $f = \prod_{v \in M(K)} f_v \in \mathcal{S}(V_{\mathbf{A}})$, all but finitely many f_v are characteristic functions of V_{O_v}. Therefore we need to compute $Z_{\mathbf{x}}(\omega, f)$ for f the characteristic function of $V_{\mathbf{O}}$, where, according to the notation set up in the last section, $\mathbf{O}$ is the ring of integers of the nonarchimedean local field $\mathbf{K}$. To achieve this, we first fix a choice of an orbital representative $\mathbf{x}$. Set

$$F_{\mathbf{x}}(u,v) = \begin{cases} uv & \text{if } \mathbf{K_x} = \mathbf{K} \\ (u+\theta v)(u+\theta' v) & \text{if } [\mathbf{K_x} : \mathbf{K}] = 2, \end{cases}$$

where $\mathbf{O_x} = \mathbf{O}[\theta]$ is the ring of integers in $\mathbf{K_x}$. In fact, if $K_{\mathbf{x}} = K(\theta)$ is a quadratic ramified extension of K, then we may choose θ to be any uniformizer $\pi_{\mathbf{x}}$ of $K_{\mathbf{x}}$. And if $K_{\mathbf{x}}$ is a quadratic unramified extension of K, then we may choose θ to be a unit in $O_{\mathbf{x}}$ not congruent mod $\pi_{\mathbf{x}}$ to any unit in O. In either case, we get $O_{\mathbf{x}} = O[\theta]$. For more information, see [26]. We also point out that the choice of the orbital representative is taken so that $P(\mathbf{x})$ is the relative discriminant of $\mathbf{K_x}$ over $\mathbf{K}$.

Next, we need to describe a Haar measure on $H_{\mathbf{K}}$. This will be done, exactly as we did in Chapter 3, by applying the Iwasawa decomposition to $G_{\mathbf{K}}$. Every $g \in G_{\mathbf{K}}$ can be written in the form $g = \kappa d(t,t_1)n(u)a(\tau)$, where $\kappa \in \mathcal{K} = G_{\mathbf{O}}$, $t, t_1, \tau \in \mathbf{K}^*$, and $u \in \mathbf{K}$. Define a measure dg on $G_{\mathbf{K}}$ by setting $dg = d\kappa d^*t d^*t_1 du d^*\tau$, with the normalization $\int_{\mathcal{K}} d\kappa = 1$, $\int_{\mathbf{O}} du = 1$, and $\int_{\mathbf{O}^*} d^*t = 1$. Since $H_{\mathbf{K}} \cong G_{\mathbf{K}}/(T_\varrho)_{\mathbf{K}}$, where $(T_\varrho)_{\mathbf{K}} = \{d(t_1^{-2}, t_1) : t_1 \in \mathbf{K}^*\}$, we define a Haar measure dh on $H_{\mathbf{K}}$ by setting $dg = d^*t_1 dh$. Thus if we write $h = \varrho(\kappa d(t,1)n(u)a(\tau))$, then $dh = d\kappa d^*t du d^*\tau$.

We also need to define two invariant measures $d'_{\mathbf{x}}h'$ and $d''_{\mathbf{x}}h''$ on $H_{\mathbf{K}}/(H^\circ_{\mathbf{x}})_{\mathbf{K}}$ and $(H^\circ_{\mathbf{x}})_{\mathbf{K}}$, respectively. For the measure $d'_{\mathbf{x}}h'$, it is defined, as in the last section, as follows: Since $H_{\mathbf{K}}/(H^\circ_{\mathbf{x}})_{\mathbf{K}}$ forms a double cover of the open orbit $V_{\mathbf{x}} = H_{\mathbf{K}} \cdot \mathbf{x}$, then there is a unique left $H_{\mathbf{K}}$-invariant measure on $H_{\mathbf{K}}/(H^\circ_{\mathbf{x}})_{\mathbf{K}}$, depending only on the $H_{\mathbf{K}}$-orbit of $\mathbf{x}$, such that for any $\phi \in L^1(V_{\mathbf{x}}, \frac{dx}{|P(x)|^{3/2}})$, we have

$$\int_{V_{\mathbf{x}}} \phi(x) \frac{dx}{|P(x)|^{3/2}} = \int_{H_{\mathbf{K}}/(H^\circ_{\mathbf{x}})_{\mathbf{K}}} \phi(h'\cdot \mathbf{x})\, d'_{\mathbf{x}}h'.$$

We note that with this choice of the measure $d'_{\mathbf{x}}h'$, we have, as in (5.6),

$$Z_{\mathbf{x}}(\omega, f) = \omega(P(\mathbf{x}))^{-3/2} \mathcal{Z}_{\mathbf{x}}(\omega, f). \tag{5.9}$$

As for the measure $d''_{\mathbf{x}}h''$ on $(H^{\circ}_{\mathbf{x}})_{\mathbf{K}}$, choose one normalized in such a way that $\int_{(H^{\circ}_{\mathbf{x}})_{\mathbf{O}}} d''_{\mathbf{x}}h'' = 1$. The relation between the measures dh, $d'_{\mathbf{x}}h'$, and $d''_{\mathbf{x}}h''$ is given by $dh = b_{\mathbf{x}}d'_{\mathbf{x}}h'd''_{\mathbf{x}}h''$, where $b_{\mathbf{x}}$ is some constant depending only on $\mathbf{x}$.

The main idea for computing $Z_{\mathbf{x}}(\omega, f)$ is contained in the following lemma of Datskovsky.

Lemma 5.1 *Let $M_2(\boldsymbol{O})$ denote the set of all 2×2 matrices with entries in $\boldsymbol{O}$. Define the set Ω_{x} as follows:*

1. $\Omega_{\mathrm{x}} = \{\varrho(\kappa d(t,1)n(u)a(\tau)) \in H_{\boldsymbol{K}} : t \in \boldsymbol{O},\ \tau \in \boldsymbol{O}^*,\ tu \in \boldsymbol{O}\}$ if $\boldsymbol{K}_{\mathrm{x}} = \boldsymbol{K}$

2. $\Omega_{\mathrm{x}} = \{\varrho((t,g)) \in H_{\boldsymbol{K}} : t \in \boldsymbol{O},\ |t| = 1 \text{ or } q^{-1},\ g \in Gl_2(\boldsymbol{K}) \cap M_2(\boldsymbol{O})\}$ if $\boldsymbol{K}_{\mathrm{x}}$ is a quadratic unramified extension of $\boldsymbol{K}$

3. $\Omega_{\mathrm{x}} = \{\varrho((t,g)) \in H_{\boldsymbol{K}} : t \in \boldsymbol{O},\ |t| = 1,\ g \in Gl_2(\boldsymbol{K}) \cap M_2(\boldsymbol{O})\}$ if $\boldsymbol{K}_{\mathrm{x}}$ is a quadratic ramified extension of $\boldsymbol{K}$.

Further, let Ψ_{x} be the characteristic function of Ω_{x} and let f be the characteristic function of $V_{\boldsymbol{O}}$. Then we have

$$b_{\mathrm{x}}Z_{\mathrm{x}}(\omega, f) = \int_{H_{\boldsymbol{K}}} \omega(det(h))\Psi_{\mathrm{x}}(h)\, dh. \tag{5.10}$$

Proof : See the proof of Lemma 4.1 in [2]. ■

Lemma 5.1 implies the next proposition.

Proposition 5.2 *Let f be the characteristic function of $V_{\boldsymbol{O}}$ and $\omega = \tilde{\omega}\omega_s \in \Omega(\boldsymbol{K}^*)$. Then*

$$b_{\mathrm{x}}Z_{\mathrm{x}}(\omega, f) = \eta(\omega^3)\begin{cases} \frac{1}{1-q^{-(3s-1)}} & \text{if } \boldsymbol{K}_{\mathrm{x}} = \boldsymbol{K} \\ \frac{1+q^{-3s}}{(1-q^{-3s})(1-q^{-(3s-1)})} & \text{if } [\boldsymbol{K}_{\mathrm{x}} : \boldsymbol{K}] = 2, \text{ unramified} \\ \frac{1}{(1-q^{-3s})(1-q^{-(3s-1)})} & \text{if } [\boldsymbol{K}_{\mathrm{x}} : \boldsymbol{K}] = 2, \text{ ramified} \end{cases}$$

where $\eta(\omega) = 1$ if $\tilde{\omega}$ is trivial on $\boldsymbol{O}^*$ and 0 otherwise.

Proof : Equation (5.10) of Lemma 5.1 implies

$$b_{\mathbf{x}}Z_{\mathbf{x}}(\omega, f) = \int_{\Omega_{\mathbf{x}}} \omega(det(h))\, dh \tag{5.11}$$

If $\mathbf{K_x} = \mathbf{K}$, then $det(h) = (det(\kappa)t\tau)^3$, with $t \in \mathbf{O}$, $\tau \in \mathbf{O}^*$, $tu \in \mathbf{O}$. Integration with respect to $d\kappa$ and $d^*\tau$ gives $\eta(\omega^3)$. So we wind up with

$$\eta(\omega^3) \int_{tu \in \mathbf{O}} \int_{t \in \mathbf{O}} \omega(t^3)\, d^*t\, du$$

Let $\mathrm{v} = tu$, then $d\mathrm{v} = |t|du$. so we end up with

$$\eta(\omega^3) \int_{\mathrm{v} \in \mathbf{O}} \int_{t \in \mathbf{O}} \omega(t^3)|t|^{-1}\, d^*t\, d\mathrm{v} = \frac{\eta(\omega^3)}{1 - \mathrm{q}^{-(3s-1)}}.$$

Asuume that $\mathbf{K_x}$ is a quadratic unramified extension of $\mathbf{K}$. Then $h \in \Omega_{\mathbf{x}}$ can be written in the form $h = \varrho((t,g)) = \varrho(\kappa d(t,t_1)n(u)a(\tau)) = \varrho((t, \kappa \begin{pmatrix} t_1 & 0 \\ t_1 u & t_1 \tau \end{pmatrix}))$, with $|t| = 1$ or q^{-1}, $t_1, t_1 u, t_1 \tau \in \mathbf{O}$, and $\kappa \in Gl_2(\mathbf{O})$. Thus $det(h) = (det(\kappa)tt_1^2\tau)^3$. As before, integration with respect to $d\kappa$ is $\eta(\omega^3)$ and integration with respect to d^*t is equal to

$$\int_{\mathbf{O}} \omega^3(t)\, d^*t = \int_{|t|=1} \omega^3(t)\, d^*t + \int_{|t|=\mathrm{q}^{-1}} \omega^3(t)\, d^*t = (1 + \mathrm{q}^{-3s})\eta(\omega^3).$$

So we get

$$\int_{\Omega_{\mathbf{x}}} \omega(det(h))\, dh = \eta(\omega^3)(1 + \mathrm{q}^{-3s}) \int_{t_1 \in \mathbf{O}} \int_{t_1\tau \in \mathbf{O}} \int_{t_1 u \in \mathbf{O}} \omega^3(t_1^2\tau)\, du d^*\tau d^*t_1.$$

Let $t_2 = t_1\tau$ and $\mathrm{v} = t_1 u$, then $d^*t_2 = d^*t_1$ and $d\mathrm{v} = |t_1|du$. So the last integral becomes

$$\eta(\omega^3)(1 + \mathrm{q}^{-3s}) \int_{t_1 \in \mathbf{O}} \int_{t_2 \in \mathbf{O}} \int_{\mathrm{v} \in \mathbf{O}} \omega^3(t_1)\omega^3(t_2)|t_1|^{-1}\, d\mathrm{v} d^*t_2 d^*t_1$$
$$= \frac{\eta(\omega^3)(1 + \mathrm{q}^{-3s})}{(1 - \mathrm{q}^{-(3s-1)})(1 - \mathrm{q}^{-3s})}.$$

When $\mathbf{K_x}$ is a quadratic ramified extension of $\mathbf{K}$, we will get the same result as in the unramified case except that the quantity $(1 + \mathrm{q}^{-3s})$ will disappear because in this case we have $|t| = 1$ for $h = \varrho((t,g)) \in \Omega_{\mathbf{x}}$. ■

Next we proceed to compute $b_{\mathbf{x}}$. Recall that $\mathcal{K} = G_{\mathbf{O}}$. By the normalization of the measures, we get for $h \in \varrho(\mathcal{K})$

$$\int_{\varrho(\mathcal{K})} dh = \int_{K} \int_{\mathbf{O}^*} \int_{\mathbf{O}} \int_{\mathbf{O}^*} d\kappa d^*t du d^*\tau = 1.$$

On the other hand, we also have

$$1 = \int_{\varrho(\mathcal{K})} dh = b_{\mathbf{x}} \int_{\varrho(\mathcal{K})(H^\circ_{\mathbf{x}})_{\mathbf{K}}/(H^\circ_{\mathbf{x}})_{\mathbf{K}}} \int_{\varrho(\mathcal{K})\cap(H^\circ_{\mathbf{x}})_{\mathbf{K}}} d'_{\mathbf{x}}h' d''_{\mathbf{x}}h''.$$

Since $\varrho(\mathcal{K}) \cap (H^\circ_{\mathbf{x}})_{\mathbf{K}} = (H^\circ_{\mathbf{x}})_{\mathbf{O}}$, then integration with respect to $d''_{\mathbf{x}}h''$ is 1 by normalization. So the last integral becomes

$$1 = b_{\mathbf{x}} \int_{\varrho(\mathcal{K})(H^\circ_{\mathbf{x}})_{\mathbf{K}}/(H^\circ_{\mathbf{x}})_{\mathbf{K}}} d'_{\mathbf{x}}h' = b_{\mathbf{x}} \int_{\varrho(\mathcal{K})\cdot\mathbf{x}} \frac{dx}{|P(x)|^{3/2}}.$$

But for $x \in \varrho(\mathcal{K}) \cdot \mathbf{x}$, we have $|P(x)| = |P(\varrho(\kappa)\cdot\mathbf{x})| = |(det(\kappa))^{2/3}P(\mathbf{x})| = |P(\mathbf{x})|$. So we end up with

$$b_{\mathbf{x}} = \frac{|P(\mathbf{x})|^{3/2}}{\int_{\varrho(\mathcal{K})\cdot\mathbf{x}} dx}. \tag{5.12}$$

Equipped with this formula, we obtain the following proposition.

Proposition 5.3 *Let q be the cardinality of the residue field of $\mathbf{K}$. Then*

$$b_{\mathbf{x}} = \begin{cases} \frac{2}{1-q^{-2}} & \text{if } \mathbf{K}_{\mathbf{x}} = \mathbf{K} \\ \frac{2}{(1-q^{-1})^2} & \text{if } [\mathbf{K}_{\mathbf{x}} : \mathbf{K}] = 2, \text{ unramified} \\ \frac{2q^{-1/2}}{(1-q^{-1})^2(1+q^{-1})} & \text{if } [\mathbf{K}_{\mathbf{x}} : \mathbf{K}] = 2, \text{ ramified}, q \neq 2^n. \end{cases}$$

Proof : Let π be a uniformizer of $\mathbf{K}$. Assume $\mathbf{K}_{\mathbf{x}} = \mathbf{K}$. By our choice of an orbital representative, $\mathbf{x} = uv$. Define the set $D_{\mathbf{x}} = \{x \in V_{\mathbf{K}} : F_x(u,v) \equiv uv \mod \pi\}$. We state some properties of $D_{\mathbf{x}}$. If $x \in D_{\mathbf{x}}$, then clearly $x \in V_{\mathbf{O}}$. For $x = (x_1, x_2, x_3) \in D_{\mathbf{x}}$, $x_2^2 - 4x_1x_3 \equiv 1 \mod \pi$. By the ultrametric inequality, $|x_2^2 - 4x_1x_3| = 1$, and hence $|P(x)| = |P(\mathbf{x})| = 1$. Also Hensel's Lemma implies that all forms in $D_{\mathbf{x}}$ split. Further, $D_{\mathbf{x}} \subset \varrho(\mathcal{K}) \cdot \mathbf{x}$. To prove this, note that if $x \in D_{\mathbf{x}}$, then x is $H_{\mathbf{K}}$-equivalent to $\mathbf{x}$. So there exists $h \in H_{\mathbf{K}}$ such that $x = h \cdot \mathbf{x} \in H_{\mathbf{K}} \cdot \mathbf{x} = V_{\mathbf{x}}$. Since $x \in V_{\mathbf{O}}$, then $x = h \cdot \mathbf{x} \in (V_{\mathbf{x}})_{\mathbf{O}}$. But by Lemma 5.1 (its proof), $\Omega_{\mathbf{x}} \cdot \mathbf{x} = (V_{\mathbf{x}})_{\mathbf{O}}$. So we may assume $h \in \Omega_{\mathbf{x}}$, i.e, $h = \varrho(\kappa d(t,1)n(u)a(\tau))$ with $t \in \mathbf{O}$, $\tau \in \mathbf{O}^*$, and $tu \in \mathbf{O}$. From the identity $P(h \cdot \mathbf{x}) = (det(h))^{2/3}P(\mathbf{x})$, we deduce that $t \in \mathbf{O}^*$, and hence $u \in \mathbf{O}$. Thus $h \in \varrho(\mathcal{K})$. So this shows $D_{\mathbf{x}} \subset \varrho(\mathcal{K}) \cdot \mathbf{x}$. Also it is clear that if $\kappa_1 \equiv \kappa_2 \mod \pi$, then $\varrho(\kappa_1) \cdot D_{\mathbf{x}} = \varrho(\kappa_2) \cdot D_{\mathbf{x}}$.

Now to compute $b_{\mathbf{x}}$, let $\varrho(\mathcal{K}_{\mathbf{O}/\pi\mathbf{O}})$ act on $D_{\mathbf{x}}$. The order of $\varrho(\mathcal{K}_{\mathbf{O}/\pi\mathbf{O}})$ is $(\mathrm{q}-1)(\mathrm{q}^2-1)(\mathrm{q}^2-\mathrm{q})$. The stabilizer group of the form $\mathbf{x} = uv$, by Section 2.2, consists of elements of the forms $\varrho(((t_1t_2)^{-1}, \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}))$ and $\varrho((t_1t_2)^{-1}, \begin{pmatrix} 0 & t_1 \\ t_2 & 0 \end{pmatrix})$; so its order is $2(\mathrm{q}-1)^2$. The orbit of $\mathbf{x}$, $\varrho(\mathcal{K}_{\mathbf{O}/\pi\mathbf{O}})\cdot\mathbf{x}$, thus has order $\frac{\mathrm{q}(\mathrm{q}^2-1)}{2}$. This in turn implies that $\varrho(\mathcal{K})\cdot\mathbf{x}$ consists of $\frac{\mathrm{q}(\mathrm{q}^2-1)}{2}$ disjoint copies of $D_{\mathbf{x}}$. Since $D_{\mathbf{x}} = (0,1,0) + (\pi\mathbf{O})^3$, then the measure of $D_{\mathbf{x}}$ is the same as that of $(\pi\mathbf{O})^3$, which is q^{-3}. Applying formula (5.12), we get the value of $b_{\mathbf{x}}$ stated in the proposition.

Next assume that $\mathbf{K}_{\mathbf{x}}$ is a quadratic unramified extension of $\mathbf{K}$. By the choice of an orbital representative, $\mathbf{x} = (u+\theta v)(u+\theta' v)$. As in the above case, let $\varrho(\mathcal{K}_{\mathbf{O}/\pi\mathbf{O}})$ act on $D_{\mathbf{x}} = \{x \in V_{\mathbf{K}} : F_x(u,v) \equiv (u+\theta v)(u+\theta' v) \mod \pi\}$. The stabilizer group of the form $(u+\theta v)(u+\theta' v)$ can be described, by Section 2.2, as consisting of matrices that act on $(u+\theta v)$ by multiplication by an element of $\mathbf{O}_{\mathbf{x}}/\pi\mathbf{O}_{\mathbf{x}}$ or by multiplication by an element of $\mathbf{O}_{\mathbf{x}}/\pi\mathbf{O}_{\mathbf{x}}$ followed by Galois conjugation. Thus its order is $2(\mathrm{q}^2-1)$. This implies that $\varrho(\mathcal{K})\cdot\mathbf{x}$ consists of $\frac{(\mathrm{q}-1)(\mathrm{q}^2-\mathrm{q})(\mathrm{q}^2-1)}{2(\mathrm{q}^2-1)}$ disjoint copies of $D_{\mathbf{x}}$. Since the measure of $D_{\mathbf{x}}$ is q^{-3} and $|P(\mathbf{x})| = 1$, formula (5.12) again gives the value of $b_{\mathbf{x}}$ in this case.

Finally, we consider the case when $\mathbf{K}_{\mathbf{x}}$ is a quadratic ramified extension of $\mathbf{K}$. Since $\mathrm{q} \neq 2^n$, there are exactly two ramified extensions $\mathbf{K}_{\mathbf{x}_1}$ and $\mathbf{K}_{\mathbf{x}_2}$ of $\mathbf{K}$. Let $D_{1,2} = \{x = (x_1, x_2, x_3) \in V_{\mathbf{K}} : x_1 \in \mathbf{O}^*,\ x_2 \equiv 0 \mod \pi,\ x_3 \equiv 0 \mod \pi,$ *but* $x_3 \not\equiv 0 \mod \pi^2\}$. It follows that $D_{1,2} \subset \varrho(\mathcal{K})\cdot\mathbf{x}_1 \cup \varrho(\mathcal{K})\cdot\mathbf{x}_2$. As above, let $\varrho(\mathcal{K}_{\mathbf{O}/\pi\mathbf{O}})$ act on $D_{1,2}$. The stabilizer group of the set $D_{1,2}$ consists of those κ such that $\kappa \equiv (t, \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}) \mod \pi$. Thus $\varrho(\mathcal{K})\cdot\mathbf{x}_1 \cup \varrho(\mathcal{K})\cdot\mathbf{x}_2$ consists of $\frac{(\mathrm{q}-1)(\mathrm{q}^2-\mathrm{q})(\mathrm{q}^2-1)}{(\mathrm{q}-1)^3\mathrm{q}}$ disjoint copies of $D_{1,2}$. Since the measure of $D_{1,2}$ is $(1-\mathrm{q}^{-1})\mathrm{q}^{-1}(\mathrm{q}^{-1}-\mathrm{q}^{-2})$ and $|P(\mathbf{x})| = \mathrm{q}^{-1}$, then using formula (5.12), bearing in mind that we have two ramified extensions, gives the value of $b_{\mathbf{x}}$ in this case. ■

Combining Proposition 5.2 and Proposition 5.3 gives the following theorem.

Theorem 5.1 *Let f be the characteristic function of V_O and $\omega = \tilde{\omega}\omega_s \in \Omega(K^*)$. Then*

$$Z_x(\omega, f) = \eta(\omega^3)\begin{cases} \frac{1-q^{-2}}{2(1-q^{-(3s-1)})} & \text{if } \boldsymbol{K}_x = \boldsymbol{K} \\ \frac{(1-q^{-1})^2(1+q^{-3s})}{2(1-q^{-3s})(1-q^{-(3s-1)})} & \text{if } [\boldsymbol{K}_x : \boldsymbol{K}] = 2, \text{ unramified} \\ \frac{(1-q^{-1})^2(1+q^{-1})}{2q^{-1/2}(1-q^{-3s})(1-q^{-(3s-1)})} & \text{if } [\boldsymbol{K}_x : \boldsymbol{K}] = 2, \text{ ramified, } q \neq 2^n \end{cases}$$

where $\eta(\omega) = 1$ if $\tilde{\omega}$ is trivial on $\boldsymbol{O}^*$ and 0 otherwise.

CHAPTER 6

A MEAM VALUE THEOREM FOR CLASS NUMBERS OF QUADRATIC EXTENSIONS OF FUNCTION FIELDS

6.1 Constructing Dirichlet Series

We continue our work to obtain a mean value theorem for class numbers of quadratic extensions of the function field K. After studying the adelic zeta function $Z(\omega, f)$ and the accompanying local orbital zeta functions $Z_x(\omega_v, f_v)$ and $\mathcal{Z}_x(\omega_v, f_v)$, our next objective is to construct Dirichlet series that will eventually yield the mean value theorem we are after. This will be done by putting together the global and local information we obtained in the last two chapters.

Recall the decomposition of $Z(\omega, f)$ we obtained in Section 5.1:

$$Z(\omega, f) = \frac{1}{2} \sum_{x \in H_K \backslash V_K''} c_x \mu(x) \int_{H_{\mathbb{A}}/(H_x^\circ)_{\mathbb{A}}} \omega(det(h')) f(h' \cdot x) \, d_x' h', \qquad (6.1)$$

where

$$\mu(x) = \int_{(H_x^\circ)_{\mathbf{A}}/(H_x^\circ)_K} d_x''h'', \tag{6.2}$$

$d_x'h'$ and $d_x''h''$ are Haar measures on $H_{\mathbf{A}}/(H_x^\circ)_{\mathbf{A}}$ and $(H_x^\circ)_{\mathbf{A}}/(H_x^\circ)_K$ respectively, and c_x is a constant given by $dh = c_x d_x'h' d_x''h''$. We will first choose the measures $d_x'h'$ and $d_x''h''$ suitably so that c_x will have the same value for all $x \in H_K \backslash V_K''$.

Let du and d^*t be respectively the additive and multiplicative Haar measures on $\mathbf{A}$ and $\mathbf{A}^*$ normalized as in Chapter 4. Let du_v and d^*t_v be respectively the Haar measures on K_v and K_v^* normalized as in Chapter 5. The relations between the adelic and local measures are given by, see [26],

$$du = q^{1-\mathbf{g}} \prod_{v \in M(K)} du_v, \quad d^*t = \rho_K^{-1} \prod_{v \in M(K)} d^*t_v$$

where $\rho_K = \frac{h_{0,K}}{q-1}$ and $h_{0,K}$ is the divisor class number of K.

In Chapter 4, the adelic measure dh on $H_{\mathbf{A}}$ is given by $dh = d\kappa d^*t du d^*\tau$. While in Chapter 5, the local measure dh_v on H_{K_v} is given by $dh_v = d\kappa_v d^*t_v du_v$ $d^*\tau_v$. Since $d\kappa = \prod_{v \in M(K)} d\kappa_v$, the above relations between the adelic and local measures yield

$$dh = q^{1-\mathbf{g}} \rho_K^{-2} \prod_{v \in M(K)} dh_v.$$

Next we define $d_x'h'$ and $d_x''h''$ for $x \in V_K''$. Suppose $x \in V_K''$. Let $\mathrm{x} = (\mathrm{x}_v)_{v \in M(K)}$ be the standard $H_{\mathbf{A}}$-orbital representative of x, i.e, for every $v \in M(K)$, x_v is a standard representative of the orbit $H_{K_v} \cdot x$ as chosen in Section 5.2. Stated differently, $(K_v)_x = (K_v)_{\mathrm{x}_v}$ for every $v \in M(K)$. This implies $x = h \cdot \mathrm{x}$ for some $h = (h_v) \in H_{\mathbf{A}}$ and hence $(H_x^\circ)_{K_v} = h_v H_{\mathrm{x}_v}^\circ h_v^{-1}$ for every $v \in M(K)$. Because of the last relation, we define the measure $d_x''h_v''$ on $(H_x^\circ)_{K_v}$ to be the measure $d_{\mathrm{x}_v}''h_v''$ on $(H_{\mathrm{x}_v}^\circ)_{K_v}$ as defined in Section 5.2. Recall also that in Section 5.2 we defined the measure $d_{\mathrm{x}_v}'h_v'$ on $H_{K_v}/(H_{\mathrm{x}_v}^\circ)_{K_v}$ by using the double cover map. The relation between the measure dh_v on H_{K_v} and the measures $d_{\mathrm{x}_v}'h_v'$ and $d_{\mathrm{x}_v}''h_v''$ is $dh_v = b_{\mathrm{x}_v} d_{\mathrm{x}_v}'h_v' d_{\mathrm{x}_v}''h_v''$. Now we define $d_x'h'$ and

$d''_x h''$ by setting

$$d'_x h' = \prod_{v \in M(K)} b_{\mathrm{x}_v} d'_{\mathrm{x}_v} h'_v, \quad d''_x h'' = \prod_{v \in M(K)} d''_{\mathrm{x}_v} h''_v.$$

Combining all the measure relations gives $dh = q^{1-\mathrm{g}} \rho_K^{-2} d'_x h' d''_x h''$, and hence $c_x = q^{1-\mathrm{g}} \rho_K^{-2}$ for all $x \in H_K \backslash V''_K$. Thus equation (6.1) becomes

$$Z(\omega, f) = \frac{1}{2} q^{1-\mathrm{g}} \rho_K^{-2} \sum_{[K_x:K]=2} \mu(x) \int_{H_{\mathbb{A}}/(H_x^\circ)_{\mathbb{A}}} \omega(det(h')) f(h' \cdot x)\, d'_x h'. \qquad (6.3)$$

For the measure $d'_x h' = \prod_{v \in M(K)} b_{\mathrm{x}_v} d'_{\mathrm{x}_v} h'$ and for $f = \prod_{v \in M(K)} f_v$, the integral in (6.3) decomposes into the product

$$\prod_{v \in M(K)} b_{\mathrm{x}_v} \int_{H_{K_v}/(H_x^\circ)_{K_v}} \omega_v(det(h'_v)) f_v(h'_v \cdot x)\, d'_{\mathrm{x}_v} h' = \prod_{v \in M(K)} b_{\mathrm{x}_v} Z_x(\omega_v, f_v) \quad (6.4)$$

where ω_v is the restriction of ω to K_v. Also by equation (5.9), we have

$$\begin{aligned} Z_x(\omega_v, f_v) &= \omega_v(P(x))^{-3/2} \mathcal{Z}_x(\omega_v, f_v) \\ &= \omega_v(P(x))^{-3/2} \mathcal{Z}_{\mathrm{x}_v}(\omega_v, f_v) \\ &= (\tfrac{\omega_v(P(\mathrm{x}_v))}{\omega_v(P(x))})^{3/2} Z_{\mathrm{x}_v}(\omega_v, f_v). \end{aligned} \qquad (6.5)$$

Since $P(x) \in K^*$, $\prod_{v \in M(K)} \omega_v(P(x)) = \omega(P(x)) = 1$. Also $\prod_{v \in M(K)} \omega_v(P(\mathrm{x}_v))$ $= \omega(D_{K_x/K})$, where $D_{K_x/K}$ is the relative discriminant of the field extension K_x over K, viewed as an idele. This is because the standard local orbital representative x_v is chosen so that $P(\mathrm{x}_v)$ is the local discriminant of the field extension $(K_x)_v$ over K_v. For more information about the idelic discriminant, see [8]. So we wind up with

$$Z(\omega, f) = \frac{1}{2} q^{1-\mathrm{g}} \rho_K^{-2} \sum_{[K_x:K]=2} \mu(x) \omega(D_{K_x/K})^{3/2} \prod_{v \in M(K)} b_{\mathrm{x}_v} Z_{\mathrm{x}_v}(\omega_v, f_v). \qquad (6.6)$$

Before we continue our analysis, we introduce some notations. Let $\mathrm{X}_v \subset V'_{K_v}$ be a set of standard representatives of all the H_{K_v}-orbits in V'_{K_v}. Since K_v is a local field, X_v is a finite set. Let $S \subset M(K)$ be a finite set. Set

$$\mathrm{X}_S = \prod_{v \in S} \mathrm{X}_v.$$

Then X_S is also a finite set. The elements of X_S are standard representatives of all the H_S-orbits in V'_S where $H_S = \prod_{v \in S} H_{K_v}$ and $V'_S = \prod_{v \in S} V'_{K_v}$.

Notation 6.1 *Let $x \in V_K'$ and $\mathrm{x}_S = (\mathrm{x}_v)_{v\in S} \in \mathrm{X}_S$. We say x is H_S-equivalent to x_S, written*

$$x \sim \mathrm{x}_S,$$

if x is H_{K_v}-equivalent to x_v for every $v \in S$.

We next describe what the notation $x \sim \mathrm{x}_S$ tells us about K_x. Let $v \in M(K)$. How v extends to an absolute value on K_x depends on how the polynomial $F_x(u,1)$ factors over K_v. Since K_x is a quadratic extension of K, v will yield two or one absolute value on K_x. This is usually expressed using tensor products as

$$K_v \otimes_K K_x \cong (K_x)_{w_1} \oplus (K_x)_{w_2}, \quad \text{or} \quad K_v \otimes_K K_x \cong (K_x)_w$$

where $[(K_x)_{w_i} : K_v] = 1$ for $i = 1, 2$ and $[(K_x)_w : K_v] = 2$. So in the first case, w_1 and w_2 are the extensions of v to K_x; and in the second case, w is the extension of v to K_x.

Now the notation $x \sim \mathrm{x}_v$ tells us how $v \in M(K)$ extends to K_x. Suppose $\mathrm{x}_v = uv$. Since $x \sim \mathrm{x}_v$, then $F_x(u,1)$ factors over K_v and hence $K_v \otimes_K K_x \cong (K_x)_{w_1} \oplus (K_x)_{w_2} \cong K_v \oplus K_v$. On the other hand, if $\mathrm{x}_v = (u+\theta v)(u+\theta' v)$, then $x \sim \mathrm{x}_v$ implies that $F_x(u,1)$ does not factor over K_v and hence $K_v \otimes_K K_x \cong (K_x)_w$ with $[(K_x)_w : K_v] = 2$. In addition, x_v determines whether $(K_x)_w$ is a quadratic ramified or a quadratic unramified extension of K_v.

So in general, the notation $x \sim \mathrm{x}_S$ tells us how each $v \in S \subset M(K)$ extends to K_x. In this situation we also say that x_S is an S-signature of K_x over K.

It is worth mentioning at this point that since the characteristic of K is not 2, then there are exactly three quadratic extensions of K_v; one unramified and two ramified. In our analysis we distinguish between these two quadratic ramified extensions. So this implies that the cardinality of X_v, defined above, is 4.

From now on, let S be a fixed finite subset of $M(K)$. We shall partition the elements in the orbit space $H_K\backslash V_K''$ according to their H_S-orbits, or equivalently, according to their S-signatures. For $\mathrm{x}_S = (\mathrm{x}_v)_{v\in S} \in \mathrm{X}_S$, set

$$Z_{\mathrm{x}_S}(\omega|_S, f|_S) = \prod_{v\in S} b_{\mathrm{x}_v} Z_{\mathrm{x}_v}(\omega_v, f_v) \tag{6.7}$$

where ω_v is once again the restriction of ω to K_v, $\omega|_S = \prod_{v\in S}\omega_v$, and $f|_S = \prod_{v\in S} f_v$. Then we can rewrite (6.6) as

$$Z(\omega, f) = \frac{1}{2}q^{1-\mathbf{g}}\rho_K^{-2} \sum_{\mathrm{x}_S\in \mathrm{X}_S} \sum_{x\sim \mathrm{x}_S} \mu(x)\omega(D_{K_x/K})^{\frac{3}{2}} Z_{\mathrm{x}_S}(\omega|_S, f|_S) \prod_{v\notin S} b_{\mathrm{x}_v} Z_{\mathrm{x}_v}(\omega_v, f_v). \tag{6.8}$$

Define $\eta_{x,S}$ by setting

$$\eta_{x,S}(\omega) = \prod_{v\notin S} b_{\mathrm{x}_v} Z_{\mathrm{x}_v}(\omega_v, f_v). \tag{6.9}$$

If we choose S suitably, we can evaluate $\eta_{x,S}$ explicitly. For this, suppose S is also chosen in such a way that ω is unramified outside of S, i.e, ω_v is trivial on O_v^* and hence $\omega_v = \tilde{\omega}_v\omega_{s_v} = \omega_{s_v}$ for every $v \notin S$, and that f_v is the characteristic function of V_{O_v} for every $v \notin S$. Then Proposition 5.2 yields

$$\eta_{x,S}(\omega) = \frac{L_{K,S}(\omega_{-1}\omega^3)L_{K,S}^2(\omega^3)}{L_{K_x,S}(\omega^3)} \tag{6.10}$$

where

$$L_{K,S}(\omega) = \prod_{v\notin S}(1 - q_v^{-s_v})^{-1}$$

and

$$L_{K_x,S}(\omega) = \prod_{\mu\in M(K_x),\mu|v,v\notin S} (1 - q_\mu^{-s_v})^{-1}$$

are the truncated Hecke L-series of K and K_x, respectively.

Set

$$\xi_{\mathrm{x}_S}(\omega) = \sum_{[K_x:K]=2,x\sim \mathrm{x}_S} \mu(x)\omega(D_{K_x/K})^{\frac{3}{2}}\eta_{x,S}(\omega). \tag{6.11}$$

This is the Dirichlet series that will eventually yield the mean value theorem we are after. With all of these notations, we end up with

$$Z(\omega, f) = \frac{1}{2}q^{1-\mathbf{g}}\rho_K^{-2} \sum_{\mathrm{x}_S\in \mathrm{X}_S} Z_{\mathrm{x}_S}(\omega|_S, f|_S)\xi_{\mathrm{x}_S}(\omega). \tag{6.12}$$

To get some analytic information about $\xi_{x_S}(\omega)$ that is enough for our purposes, we have to specialize f in (6.12). We already know that $f = \prod_{v\in M(K)} f_v$ has the property that f_v is the characteristic function of V_{O_v} for every $v \notin S$. If we further choose f so that $f|_S = \prod_{v\in S} f_v$ has compact support in $V'_S = \prod_{v\in S} V'_{K_v}$, then by Proposition 5.1, $Z_{x_S}(\omega|_S, f|_S)$ becomes an entire function of ω. Further, we can choose $f|_S$ so that its support lies in the H_S-orbit of only one $x_S = (x_v)_{v\in S}$, i.e, for every $v \in S$, f_v has support in the orbit $V_{x_v} = H_{K_v} \cdot x_v$; for instance, we can take $f_v(x_v) = \tilde{\omega}_v(P(x_v))^{\frac{-3}{2}}\cdot$ (characteristic function of $\varrho(K_v)\cdot x_v$). Then $Z_{x_v}(\omega_v, f_v)$ is independent of s. So with this choice of f, only one $Z_{x_S}(\omega|_S, f|_S)$ is nonzero and independent of s. So (6.12) reduces to

$$Z(\omega, f) = \frac{1}{2}q^{1-g}\rho_K^{-2} Z_{x_S}(\omega|_S, f|_S)\xi_{x_S}(\omega). \tag{6.13}$$

With all of this at hand, we get the following proposition.

Proposition 6.1 *Let S and f be as described above. Then* (6.13) *implies the following:*

1. $\xi_{x_S}(\omega)$ is an analytic function of ω in the region $\Re(\omega) > 1$. It can be continued to a meromorphic function analytic everywhere except for simple poles at $\omega = \tilde{\omega}\omega_s$, $\tilde{\omega}^3 = 1$, $s = 1 + \frac{2\pi i n}{3\log q}$, and double poles at ω with $\tilde{\omega}^3 = 1$, $s = \frac{2}{3} + \frac{2\pi i n}{3\log q}$ and at the poles of $\Sigma_2^-(\omega^3\omega_{-1}, 0, f)$.

2. For $x_S = (x_v)_{v\in S}$, we have

$$Res_{s=1}\xi_{x_S}(\omega_s) = \sigma_K \rho_K^2 \prod_{v\in S} \frac{|P(x_v)|_v^{3/2}}{b_{x_v}}$$

where

$$\sigma_K = \frac{4q^{1-g}\zeta_K(2)}{3(\log q)Res_q\zeta_K}.$$

Proof : Since $Z(\omega, f)$ is analytic in the region $\Re(\omega) > 1$, then so is $\xi_{x_S}(\omega)$ by (6.13) and the choice of f given above. Also, by the choice of f, it follows that $Z(\omega, f)$ and $\xi_{x_S}(\omega)$ have the same poles. Since f was chosen so that its support lies in $V'_{\mathbb{A}}$, then the poles of $Z(\omega, f)$ at $s = 0 + \frac{2\pi i n}{3\log q}$ will disappear. Also, for $v \in S$, f_v has support in V'_{K_v}. This will imply that $\Sigma_1(w, f) = 0$

and hence $R(f) = \Gamma(f) = 0$. Thus the poles of $Z(\omega, f)$ at $s = \frac{1}{3} + \frac{2\pi i n}{3\log q}$ also disappear. For the poles of $\Sigma_2^-(\omega^3\omega_{-1}, 0, f)$, we point out that since for $v \in S$, support of f_v is in $\mathcal{K}_v \cdot \mathrm{x}_v$, then $\Sigma_2^-(\omega^3\omega_{-1}, 0, f) = 0$ unless each x_v, $v \in S$, has the property that $(K_v)_{\mathrm{x}_v} = K_v$. In the latter case the poles can be found from the expression of Lemma 4.7(2).

As for the second part, it will follow from (6.13) once we calculate the residue $Res_{s=1}Z(\omega_s, f)$. By Theorem 4.1, it is easy to show that $Res_{s=1}Z(\omega_s, f)$ $= \frac{2\zeta_K(2)}{q^{1-\mathbf{g}}Res_q\zeta_K}\frac{\hat{f}(0)}{3\log q}$. So we need to calculate $\hat{f}(0)$. Note that

$$\hat{f}(0) = \int_{V_{\mathbf{A}}} f(x)\,dx = q^{3-3\mathbf{g}} \prod_{v\in M(K)} \int_{V_{K_v}} f_v(x_v)\,dx_v$$

where $dx = dx_1dx_2dx_3$ is the measure on $V_{\mathbf{A}}$ chosen in Chapter 4 and $dx_v = dx_{1,v}dx_{2,v}dx_{3,v}$ is the measure on V_{K_v} chosen in Chapter 5. If $v \notin S$, then f_v is the characteristic function of $V_{\mathcal{O}_v}$ and hence $\int_{V_{K_v}} f_v(x_v)\,dx_v = 1$ by normalization of the measure dx_v. So suppose $v \in S$. Recall that in this case f was chosen so that f_v has compact support in V'_{K_v} (see the discussion before the statement of this proposition.) Also $V'_{K_v} = \bigsqcup_{\mathrm{x}_v\in \mathrm{X}_v} H_{K_v}\cdot \mathrm{x}_v$. Thus we get

$$\begin{aligned}\hat{f}(0) &= q^{3-3\mathbf{g}}\textstyle\prod_{v\in S}(\sum_{\mathrm{x}_v\in\mathrm{X}_v} Z_{\mathrm{x}_v}(\omega_1, f_v))\\ &= q^{3-3\mathbf{g}}\textstyle\prod_{v\in S}(\sum_{\mathrm{x}_v\in\mathrm{X}_v} \frac{\omega_1(P(\mathrm{x}_v))^{3/2}}{b_{\mathrm{x}_v}} b_{\mathrm{x}_v} Z_{\mathrm{x}_v}(\omega_1, f_v))\\ &= q^{3-3\mathbf{g}}\textstyle\sum_{\mathrm{x}_S=(\mathrm{x}_v)_{v\in S}}\prod_{v\in S}\frac{|P(\mathrm{x}_v)|_v^{3/2}}{b_{\mathrm{x}_v}} Z_{\mathrm{x}_S}(\omega_1|_S, f|_S).\end{aligned}$$

Now since f was chosen so that $f|_S$ has support in the H_S-orbit of only one x_S, then the above sum reduces to only one term corresponding to the x_S in (6.13). Taking $\omega = \omega_s$ in (6.13) and taking the residue of both sides at $s = 1$ yield the result. ■

We close this section by calculating $\mu(x)$. Let h_{0,K_x} be the divisor class number of K_x. Then we have the following proposition.

Proposition 6.2

$$\mu(x) = \frac{2h_{0,K_x}}{h_{0,K}}$$

Proof : Recall that

$$\mu(x) = \int_{(H_x^\circ)_{\mathbf{A}}/(H_x^\circ)_K} d''_x h''$$

where the measure $d''_x h''$ is as given at the begining of this section. $(H^\circ_x)_{\mathbf{A}} \cong (G^\circ_x)_{\mathbf{A}}/(T_\varrho)_{\mathbf{A}}$ where $T_\varrho = \{(t^{-2}, \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix})\}$. Let d^*t be a Haar measure on $(T_\varrho)_{\mathbf{A}}$. Define an invariant measure $d''_x g''$ on $(G^\circ_x)_{\mathbf{A}}$ by setting $d''_x g'' = d''_x h'' d^*t$.

Let $(G^\circ_x)^1_{\mathbf{A}} = \{g'' = (t, g) \in (G^\circ_x)_{\mathbf{A}} : |det(g)|_{\mathbf{A}} = 1\}$. Then every element of $(G^\circ_x)_{\mathbf{A}}$ is T_ϱ-equivalent to an element of $(G^\circ_x)^1_{\mathbf{A}} \bigsqcup g(G^\circ_x)^1_{\mathbf{A}}$ for some $g \in (G^\circ_x)_{\mathbf{A}}$ satisfying $|det(g)|_{\mathbf{A}} = q$. Then $(H^\circ_x)_{\mathbf{A}} \cong (G^\circ_x)^1_{\mathbf{A}}/(T_\varrho)^1_{\mathbf{A}} \bigsqcup g(G^\circ_x)^1_{\mathbf{A}}/(T_\varrho)^1_{\mathbf{A}}$. Define an invariant measure $d^1_x g''$ on $(G^\circ_x)^1_{\mathbf{A}}$ by setting $d^1_x g'' = d''_x h'' d^1 t$. Then we get

$$\begin{aligned} 2\int_{(G^\circ_x)^1_{\mathbf{A}}/(G^\circ_x)_K} d^1_x g'' &= \int_{(H^\circ_x)_{\mathbf{A}}/(H^\circ_x)_K} d''_x h'' \int_{(T_\varrho)^1_{\mathbf{A}}/(T_\varrho)_K} d^1 t \\ &= \mu(x) \int_{\mathbf{A}^1/K^*} d^1 t \\ &= \mu(x). \end{aligned} \tag{6.14}$$

Since $G^\circ_x(K) = R_{K_x/K}(K_x) \cong Gl_1(K_x)$, by Proposition 2.2, then $(G^\circ_x)_K \cong K^*_x$ and $(G^\circ_x)_{\mathbf{A}} \cong \mathbf{A}^*_{K_x}$ where $\mathbf{A}^*_{K_x}$ is the idele group of K_x. Thus $(G^\circ_x)_{\mathbf{A}}/(G^\circ_x)_K \cong \mathbf{A}^*_{K_x}/K^*_x$. Next we compare the measure d^*t_x on $\mathbf{A}^*_{K_x}$ with the measure $d''_x g''$ on $(G^\circ_x)_{\mathbf{A}}$.

Since by our choice $d''_x h'' = \prod_{v \in M(K)} d''_{\mathrm{x}_v} h''_v$, then $d''_x g'' = \frac{q-1}{h_{0,K}} \prod_{v \in M(K)} d''_{\mathrm{x}_v} h''_v d^* t_v$ and $d^*t_x = \frac{q-1}{h_{0,K_x}} \prod_{v \in M(K_x)} d^*(t_x)_v$. We show $d''_{\mathrm{x}_v} h''_v d^* t_v = d^*(t_x)_v$ for every $v \in M(K)$.

Recall the map $\phi : G^\circ_x(K) \longrightarrow Gl_1(K_x)$ given by $\phi(g) = a + b\theta$ for $g = (*, \begin{pmatrix} a & b \\ * & * \end{pmatrix}) \in G^\circ_x(K)$. Suppose $x' = \gamma \cdot x$ for some $\gamma \in G_K$. Then $G^\circ_{x'}(K) = \gamma G^\circ_x(K)\gamma^{-1}$. Similarly, consider the map $\phi' : G^\circ_{x'}(K) \longrightarrow Gl_1(K_{x'})$ given by $\phi'(g') = a' + b'\theta$ for $g' = (*, \begin{pmatrix} a' & b' \\ * & * \end{pmatrix}) \in G^\circ_{x'}(K)$. As $g' = \gamma g \gamma^{-1}$, an easy computation shows $\phi'(g') = \phi(g)$. Thus it is enough to consider the map ϕ with x a standard orbital representative.

If $K_x \otimes_K K_v \cong K_v \oplus K_v$, then $d^*(t_x)_v = d^*t_1 d^*t_2$ where d^*t_i is a Haar measure on K^*_v. In this case the orbital representative is $\mathrm{x}_v = uv$ and $G^\circ_{\mathrm{x}_v}(K_v) = \{((t_1t_2)^{-1}, \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\ t_1, t_2 \in K^*_v\}$. The measure on $G^\circ_{\mathrm{x}_v}(K_v)$ is $d''_{\mathrm{x}_v} h''_v d^* t_v = d''_{\mathrm{x}_v} g''_v = d^*t_1 d^*t_2$. So in this case we have $d''_{\mathrm{x}_v} h''_v d^* t_v = d^*(t_x)_v$.

If $K_x \otimes_K K_v \cong (K_x)_v$, with $[(K_x)_v : K_v] = 2$, then let $d^*(t_x)_v$ be the Haar measure on $(K_x)_v^*$ normalized so that $\int_{(O_x)_v^*} d^*(t_x)_v = 1$. In this case to show $d''_{x_v} h''_v d^* t_v = d^*(t_x)_v$, we use the local map $\phi_v : G_x^\circ(K_v) \longrightarrow Gl_1((K_x)_v)$ and find the measure of $(O_x)_v^*$ with respect to $d^*(t_x)_v$ and the measure of $\phi_v^{-1}((O_x)_v^*)$ with respect to $d''_{x_v} h''_v d^* t_v = d''_{x_v} g''_v$. Since $\phi_v^{-1}((O_x)_v^*) = (G_x^\circ)_{O_v}$, $\varrho(\phi_v^{-1}((O_x)_v^*)) = (H_x^\circ)_{O_v}$ and $(T_\varrho)_{K_v} \cap \phi_v^{-1}((O_x)_v^*) = (T_\varrho)_{O_v} \cong O_v^*$, then

$$\int_{(G_x^\circ)_{O_v}} d''_{x_v} g''_v = \int_{(H_x^\circ)_{O_v}} d''_{x_v} h''_v \int_{O_v^*} d^* t_v = 1.$$

So in this case again we have $d''_{x_v} h''_v d^* t_v = d^*(t_x)_v$. Thus we conclude that $d''_x g'' = \frac{h_{0,K_x}}{h_{0,K}} d^* t_x$. Now with (6.14), we get

$$\mu(x) = 2 \int_{(G_x^\circ)_{\mathbf{A}}^1/(G_x^\circ)_K} d_x^1 g'' = \frac{2h_{0,K_x}}{h_{0,K}} \int_{\mathbf{A}_{K_x}^1/K_x^*} d^1 t_x = \frac{2h_{0,K_x}}{h_{0,K}}.$$

This completes the proof. ■

6.2 The Mean Value Theorem

Now we have all the necessary tools that will enable us to obtain the mean value theorem we are after. Let D_x denote the absolute norm of the relative discriminant $D_{K_x/K}$ of K_x over K. Then D_x is a positive integer. In fact D_x is a square. To see this, let, as in Section 6.1, $\mathrm{x} = (\mathrm{x}_v)_{v \in M(K)}$ be the standard $H_{\mathbf{A}}$-orbital representative of x. Then $x = h \cdot \mathrm{x}$ for some $h = (h_v) \in H_{\mathbf{A}}$; i.e, $x = h_v \cdot \mathrm{x}_v$ for every v. Write $h_v = \varrho(g_v)$. Then $P(x) = \chi(g_v)^2 P(\mathrm{x}_v)$. This gives $1 = \prod_{v \in M(K)} |P(x)|_v = \prod_{v \in M(K)} |\chi(g_v)|_v^2 \prod_{v \in M(K)} |P(\mathrm{x}_v)|_v = \prod_{v \in M(K)} |\chi(g_v)|_v^2 \cdot D_x^{-1}$. Thus $D_x = \prod_{v \in M(K)} |\chi(g_v)|_v^2$.

If $\omega = \omega_s$, then $\omega_s(D_{K_x/K}) = D_x^{-s}$. So if we let $\omega = \omega_s$ in $\xi_{\mathrm{x}S}(\omega)$, then we get the following altered Dirichlet series

$$\xi_{\mathrm{x}S}^*(s) := \xi_{\mathrm{x}S}^*(\omega_s) = \sum_{[K_x:K]=2, x \sim \mathrm{x}S} \frac{h_{0,K_x}}{D_x^{\frac{3}{2}s}} \eta_{x,S}(s) \tag{6.15}$$

where

$$\eta_{x,S}(s) = \frac{\zeta_{K,S}(3s-1)\zeta_{K,S}(3s)^2}{\zeta_{K_x,S}(3s)}.$$

Note that $\xi^*_{\mathrm{x}_S}(s) = \frac{h_{0,K}}{2}\xi_{\mathrm{x}_S}(s)$. By Proposition 6.1, $\xi^*_{\mathrm{x}_S}(s)$ is analytic in the region $\Re(s) > 1$ and has a simple pole at $s = 1$ with residue

$$R_{\mathrm{x}_S} = \frac{\sigma_K h_{0,K}^3}{2(q-1)^2} \prod_{v \in S} \frac{|P(\mathrm{x}_v)|_v^{3/2}}{b_{\mathrm{x}_v}} \tag{6.16}$$

where σ_K is as given in Proposition 6.1.

Next we define a sequence of Dirichlet series. Let $T_1 \subset T_2 \subset T_3 \cdots$ be an increasing sequence of finite subsets of $M(K)$ such that $S \subset T_i$ for all $i \geq 1$ and $\lim_{i\to\infty} T_i = M(K)$. Similar to the definition of $\mathrm{x}_S = (\mathrm{x}_v)_{v\in S}$, define $\mathrm{y}_{T_i} = (\mathrm{y}_v)_{v\in T_i}$ for each $i \geq 1$. We say y_{T_i} restricts to x_S and write $\mathrm{y}_{T_i}|_S = \mathrm{x}_S$ if $\mathrm{y}_v = \mathrm{x}_v$ for every $v \in S$.

For each $i \geq 1$, define the sequence of Dirichlet series

$$\xi^*_{\mathrm{x}_S,T_i}(s) = \sum_{\mathrm{y}_{T_i}|_S = \mathrm{x}_S} \xi^*_{\mathrm{y}_{T_i}}(s) \tag{6.17}$$

or equivalently,

$$\xi^*_{\mathrm{x}_S,T_i}(s) = \sum_{[K_x:K]=2, x\sim \mathrm{x}_S} \frac{h_{0,K_x}}{D_x^{\frac{3}{2}s}} \eta_{x,T_i}(s) \tag{6.18}$$

where once again

$$\eta_{x,T_i}(s) = \frac{\zeta_{K,T_i}(3s-1)\zeta_{K,T_i}(3s)^2}{\zeta_{K_x,T_i}(3s)}.$$

As $\xi^*_{\mathrm{x}_S,T_i}(s)$ is a finite sum of Dirichlet series $\xi^*_{\mathrm{y}_{T_i}}(s)$, then again by Proposition 6.1, $\xi^*_{\mathrm{x}_S,T_i}(s)$ is analytic in the region $\Re(s) > 1$ and it has a simple pole at $s = 1$ with residue

$$\begin{aligned}
\textstyle\sum_{\mathrm{y}_{T_i}|_S = \mathrm{x}_S} R_{\mathrm{y}_{T_i}} &= \textstyle\sum_{\mathrm{y}_{T_i}|_S = \mathrm{x}_S} R_{\mathrm{x}_S} \prod_{v\in T_i\setminus S} \frac{|P(\mathrm{y}_v)|_v^{3/2}}{b_{\mathrm{y}_v}} \\
&= R_{\mathrm{x}_S} \textstyle\sum_{(\mathrm{y}_v)_{v\in T_i\setminus S}} \prod_{v\in T_i\setminus S} \frac{|P(\mathrm{y}_v)|_v^{3/2}}{b_{\mathrm{y}_v}} \\
&= R_{\mathrm{x}_S} \textstyle\prod_{v\in T_i\setminus S} \sum_{\mathrm{Y}_v} \frac{|P(\mathrm{y}_v)|_v^{3/2}}{b_{\mathrm{y}_v}}
\end{aligned}$$

where the last sum is over all elements of Y_v (Y_v is the analogue of X_v, see the paragraph before Notation 6.1). Y_v consists of 4 representatives: one for split case, one for unramified quadratic, two for ramified quadratic extensions of K_v. These representatives satisfy $|P(\mathrm{y}_v)|_v = 1,\ 1,\ q_v^{-1}$ respectively. So by

Proposition 5.3, we get

$$\sum_{Y_v} \frac{|P(y_v)|_v^{3/2}}{b_{y_v}} = 1 - q_v^{-2} - q_v^{-3} + q_v^{-4}.$$

Thus the residue of $\xi^*_{x_S,T_i}(s)$ at $s = 1$ is given by

$$R_{x_S,T_i} = R_{x_S} \prod_{v \in T_i \setminus S} (1 - q_v^{-2} - q_v^{-3} + q_v^{-4}). \tag{6.19}$$

This will imlpy the following proposition.

Proposition 6.3 *The following limit exists:*

$$\lim_{i\to\infty} R_{x_S,T_i} = \mathcal{R}_{x_S} := R_{x_S} \prod_{v \notin S} (1 - q_v^{-2} - q_v^{-3} + q_v^{-4}).$$

Moreover, $\mathcal{R}_{x_S} \neq 0$.

Proof : Since $0 < 1 - q_v^{-2} - q_v^{-3} + q_v^{-4} < 1$, then the sequence of residues $\{R_{x_S,T_i}\}_{i=1}^{\infty}$ is a decreasing sequence of positive real numbers and hence it converges. The result now follows since $\lim_{i\to\infty} T_i = M(K)$. ■

Next we give some properties of the weighting factors $\eta_{x,T_i}(s)$ of the Dirichlet series $\xi^*_{x_S,T_i}(s)$. Recall that

$$\eta_{x,T_i}(s) = \frac{\zeta_{K,T_i}(3s-1)\zeta_{K,T_i}(3s)^2}{\zeta_{K_x,T_i}(3s)}$$

where $\zeta_{K,T_i}(s) = \prod_{v \notin T_i} (1 - q_v^{-s})^{-1}$ is the truncated Dedekind zeta function of K. Because of the nature of the geometric series $(1 - q_v^{-s})^{-1} = \sum_{k=0}^{\infty} q_v^{-ks}$, $\eta_{x,T_i}(s)$ is itself a Dirichlet series which we shall write as

$$\eta_{x,T_i}(s) = \sum_{n=1}^{\infty} \frac{a_n(K_x, T_i)}{n^s} \tag{6.20}$$

We gather some of its properties in the following proposition.

Proposition 6.4 *1.* $\eta_{x,T_i}(s)$ *converges absolutely and locally uniformly in the region* $\Re(s) > \frac{2}{3}$. *Moreover, it is nonzero in that region.*

2. For each $n \geq 1$, $a_n(K_x, T_i)$ *is a nonnegative integer and* $a_1(K_x, T_i) = 1$.

3. For any integer $N > 1$, we can choose the set T_i with i sufficiently large so that $a_n(K_x, T_i) = 0$ for all $1 < n < N$. In particular, this implies that $\lim_{i\to\infty} \eta_{x,T_i}(1) = 1$.

4. Let $T \subset M(K)$ be a finite subset such that $T_i \subset T$. Then for all n,

$$a_n(K_x, T) \le a_n(K_x, T_i).$$

5. Let $\eta_{T_i}(s) = \frac{\zeta_{K,T_i}(3s-1)\zeta_{K,T_i}(3s)^2}{\zeta_{K,T_i}(6s)} = \sum_{n=1}^{\infty} \frac{a_n(K,T_i)}{n^s}$. Then for all n,

$$a_n(K_x, T_i) \le a_n(K, T_i).$$

Proof : (1) follows from the fact that the Dedekind zeta function $\zeta_K(s) = \prod_{v\in M(K)}(1-q_v^{-s})^{-1}$ converges absolutely and locally uniformly in the region $\Re(s) > 1$ and it is nonzero in this region. For (2), note that

$$\eta_{x,T_i}(s) = \prod_{v\notin T_i}(1-q_v^{-(3s-1)})^{-1}(1-q_v^{-3s})^{-2}\cdot \begin{cases} (1-q_v^{-3s})^2 & \text{if } v \text{ splits in } K_x \\ (1-q_v^{-6s}) & \text{if } v \text{ is unramified in } K_x \\ (1-q_v^{-3s}) & \text{if } v \text{ is ramified in } K_x \end{cases}$$

Because of the nature of the geometric series $(1-q_v^{-s})^{-1} = \sum_{k=0}^{\infty} q_v^{-sk}$, if we write each factor of the above product as a Dirichlet series $\sum_{n=0}^{\infty} c_n q_v^{-3ns}$, then $c_0 = 1$ and c_n is a nonnegative integer for each $n \ge 1$. Now (2) follows. For (3), note that if T_i is very large, then we will truncate many factors of $\zeta_K(s) = \prod_{v\in M(K)}(1-q_v^{-s})^{-1}$ and hence many of the coefficients of the corresponding Dirichlet series become zeros. (4) follows from (2) and the fact that $\eta_{x,T}(s)$ has more factors truncated. For (5), note that

$$\eta_{T_i}(s) = \prod_{v\notin T_i}(1-q_v^{-(3s-1)})^{-1}(1-q_v^{-3s})^{-2}(1-q_v^{-6s}).$$

For each v, looking at each $v-$factor of $\eta_{x,T_i}(s)$ and $\eta_{T_i}(s)$ as a Dirichlet series, it is clear to see that the coefficients of the Dirichlet series of the $v-$factor of $\eta_{x,T_i}(s)$ are less or equal to the corresponding coefficients of the Dirichlet series of the corresponding $v-$factor of $\eta_{T_i}(s)$. This implies (5). ■

Now we go back to $\xi^*_{xS,T_i}(s)$. With (6.18) and (6.20), we may write

$$\xi^*_{xS,T_i}(s) = \sum_{n=1}^{\infty} \frac{B(n)}{n^s} \tag{6.21}$$

where

$$B(n) = \sum_{[K_x:K]=2, x\sim \mathrm{x}_S, mD_x^{\frac{3}{2}}=n} h_{0,K_x} a_m(K_x, T_i). \tag{6.22}$$

Throughout the remainder of this section, we will drop the condition $[K_x : K] = 2$ in (6.22) as it is clear from the context.

Proposition 6.5 *We have*

$$\lim_{n\to\infty} \frac{B(q^{3n})}{q^{3n}} = 3\log q \cdot R_{\mathrm{x}_S,T_i}$$

Proof : By the definition of $\xi^*_{\mathrm{x}_S,T_i}(s)$ and Theorem 4.1, it follows that $\xi^*_{\mathrm{x}_S,T_i}(s)$ is a rational function in q^{-3s}. Thus we may write its partial fraction decomposition:

$$\begin{aligned}\xi^*_{\mathrm{x}_S,T_i}(s) &= \tfrac{3\log q \cdot R_{\mathrm{x}_S,T_i}}{1-q^{3-3s}} + \textstyle\sum_j \tfrac{r_j}{(1-q^{3s_j-3s})^{m_j}} \\ &= \textstyle\sum_{n=0}^{\infty} 3\log q \cdot R_{\mathrm{x}_S,T_i} q^{3n} q^{-3sn} + \sum_j r_j (\sum_{n=0}^{\infty} p_j(n) q^{3s_j n} q^{-3sn})\end{aligned}$$

where the sum over j is finite, r_j are reals, s_j are rationals less than 1, m_j is the multiplicity of the pole at s_j, and $p_j(n)$ is a polynomial in n. Comparing the coefficient of q^{-3sn} in the above expression with that of (6.21), we get

$$B(q^{3n}) = 3\log q \cdot R_{\mathrm{x}_S,T_i} q^{3n} + \sum_j r_j p_j(n) q^{3s_j n}.$$

Since $s_j < 1$ for all j, the result follows. ■

With Proposition 6.5 at hand, we now get the main theorem of this chapter.

Theorem 6.1 *We have*

$$\lim_{n\to\infty} \frac{1}{q^{3n}} \sum_{x\sim \mathrm{x}_S, D_x^{\frac{3}{2}}=q^{3n}} h_{0,K_x} = 3\log q \cdot \mathcal{R}_{\mathrm{x}_S}$$

where $\mathcal{R}_{\mathrm{x}_S}$ is as given in Proposition 6.3.

Proof : Since $\sum_{x\sim \mathrm{x}_S, D_x^{\frac{3}{2}}=q^{3n}} h_{0,K_x} \le B(q^{3n})$, then Proposition 6.5 implies

$$\limsup_{n\to\infty} \frac{1}{q^{3n}} \sum_{x\sim \mathrm{x}_S, D_x^{\frac{3}{2}}=q^{3n}} h_{0,K_x} \le 3\log q \cdot R_{\mathrm{x}_S,T_i}.$$

Letting $i \to \infty$, Proposition 6.3 yields

$$\limsup_{n\to\infty} \frac{1}{q^{3n}} \sum_{x\sim x_S, D_x^{\frac{3}{2}}=q^{3n}} h_{0,K_x} \le 3\log q \cdot \mathcal{R}_{x_S} \tag{6.23}$$

We note also that (6.23) implies that there exists a real number $M > 0$ such that

$$\sum_{x\sim x_S, D_x^{\frac{3}{2}}=q^{3n}} h_{0,K_x} \le Mq^{3n} \quad \text{for all } n \ge 1 \tag{6.24}$$

Set

$$C(n) = \sum_{x\sim x_S, mD_x^{\frac{3}{2}}=n} h_{0,K_x} a_m(K, T_i)$$

where $a_m(K, T_i)$ is as defined in Proposition 6.4(5). Since, by Proposition 6.4(5), $a_m(K_x, T_i) \le a_m(K, T_i)$, we get

$$B(n) \le C(n) \tag{6.25}$$

for all $n \ge 1$. Note that

$$\begin{aligned} C(q^{3n}) &= \sum_{x\sim x_S, D_x^{\frac{3}{2}}=q^{3n}} h_{0,K_x} + \sum_{x\sim x_S, mD_x^{\frac{3}{2}}=q^{3n}, m\ge 2} h_{0,K_x} a_m(K, T_i) \\ &\le \sum_{x\sim x_S, D_x^{\frac{3}{2}}=q^{3n}} h_{0,K_x} + \sum_{m=2}^{\infty} a_m(K, T_i) \Big(\sum_{x\sim x_S, D_x^{\frac{3}{2}}=\frac{q^{3n}}{m}} h_{0,K_x}\Big) \\ &\le \sum_{x\sim x_S, D_x^{\frac{3}{2}}=q^{3n}} h_{0,K_x} + \sum_{m=2}^{\infty} a_m(K, T_i) M \frac{q^{3n}}{m} \\ &= \sum_{x\sim x_S, D_x^{\frac{3}{2}}=q^{3n}} h_{0,K_x} + Mq^{3n}(\eta_{T_i}(1) - 1) \end{aligned}$$

where we have used (6.24) and Proposition 6.4(2,5). Thus by (6.25), we get

$$\frac{B(q^{3n})}{q^{3n}} - M(\eta_{T_i}(1) - 1) \le \frac{1}{q^{3n}} \sum_{x\sim x_S, D_x^{\frac{3}{2}}=q^{3n}} h_{0,K_x}.$$

Taking lim inf of both sides yields

$$3\log q \cdot R_{x_S, T_i} - M(\eta_{T_i}(1) - 1) \le \liminf_{n\to\infty} \frac{1}{q^{3n}} \sum_{x\sim x_S, D_x^{\frac{3}{2}}=q^{3n}} h_{0,K_x}.$$

By letting $i \to \infty$, Proposition 6.3 and Proposition 6.4(3) imply

$$3\log q \cdot \mathcal{R}_{x_S} \le \liminf_{n\to\infty} \frac{1}{q^{3n}} \sum_{x\sim x_S, D_x^{\frac{3}{2}}=q^{3n}} h_{0,K_x}. \tag{6.26}$$

Now (6.23) and (6.26) give the theorem. ∎

Next we translate Theorem 6.1 into a mean value theorem for ideal class numbers of the quadratic extensions K_x of K. For this we first need to mention some standard notations.

Let L be a finite separable extension of $F_q(T)$. Let O_L be the integral closure of $F_q[T]$ in L. Let h_L denote the ideal class number of O_L. Let $M(L)$ be the set of all places of L. Denote by $P_{\infty,L}$ the finite subset of $M(L)$ consisting of all infinite places. Here we call a place of L infinite if it is an extension of the infinite place $\frac{1}{T}$ of $F_q(T)$. Let $\mathcal{D}(L)$ be the divisor group of L: the free abelian group generated by the elements of $M(L)$. A typical element of $\mathcal{D}(L)$ is given by

$$\mathcal{D} = \sum_{v \in M(L)} n_v v$$

where $n_v \in \mathbb{Z}$ and $n_v = 0$ for all but finitely many v. The degree of such a divisor is defined by

$$deg(\mathcal{D}) = \sum_{v \in M(L)} n_v deg(v)$$

where $deg(v)$ is given by $q^{deg(v)} = q_v$. Let $\mathcal{D}^0(L)$ be the subgroup of $\mathcal{D}(L)$ consisting of divisors of degree zero. For $f \in L$, define the divisor of f by

$$div(f) = \sum_{v \in M(L)} ord_v(f) v.$$

$div(L) = \{div(f) : f \in L\}$ is called the group of principal divisors of L. It is a subgroup of $\mathcal{D}^0(L)$. Let $\mathcal{D}(P_{\infty,L})$ be the free abelian group generated by the elements of $P_{\infty,L}$. Then $\mathcal{D}(P_{\infty,L}) \subset \mathcal{D}(L)$. Set $\mathcal{D}^0(P_{\infty,L}) = \mathcal{D}(P_{\infty,L}) \cap \mathcal{D}^0(L)$ and $div(P_{\infty,L}) = \mathcal{D}(P_{\infty,L}) \cap div(L)$. Then we have

$$h_{0,L} = |\mathcal{D}^0(L)/div(L)|$$

and the regulator r_L of L is given by

$$r_L = |\mathcal{D}^0(P_{\infty,L})/div(P_{\infty,L})|.$$

Finally define the integer $n_L = g.c.d(deg(v) : v \in P_{\infty,L})$. Then we have the following formula:

$$h_{0,L} n_L = h_L r_L. \tag{6.27}$$

This formula is due to K. F. Schmidt (see[16]).

Applying formula (6.27) with $L = K_x$, Theorem 6.1 can be written in the following form.

Theorem 6.2 *We have*

$$\lim_{n\to\infty} \frac{1}{q^{3n}} \sum_{x\sim \mathrm{x}_S, D_x^{\frac{3}{2}} = q^{3n}} h_{K_x} \frac{r_{K_x}}{n_{K_x}} = 3\log q \cdot \mathcal{R}_{\mathrm{x}_S}$$

where $\mathcal{R}_{\mathrm{x}_S}$ *is as given in Proposition* 6.3.

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